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Voting over income taxation

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Abstract

A major problem of the positive theory of income taxation is to explain why statutory income tax schedules in practice are marginal-rate progressive. While it is commonly believed that this is but a simple consequence of the fact that the number of relatively poor voters exceeds that of richer voters in general, putting this contention in a voting equilibrium context is not a trivial task. We do this here in the context of nonlinear taxation and attempt to provide a formal argument in support of this heuristic claim. We first establish the existence of mixed strategy equilibria and identify certain cases in which marginal-rate progressive taxes are chosen almost surely by the political parties. Unfortunately, we also find that if the tax policy space is not artificially constrained, the support of at least one equilibrium cannot be contained within the set of marginal-rate progressive taxes.

JEL classification: H23; H31; D72; D78

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1. Introduction

One of the well-documented empirical regularities concerning income taxation is that all industrialized democracies—all OECD countries, in particular—implement statutory income tax schedules the marginal tax rates of which are increasing in income.[1] Given that his/her views about the income tax policy is one of the most important traits of a political candidate, it is natural to expect that this stylized fact reflects, at least indirectly, the preferences of the majority of the

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[1] Such tax functions are called *marginal-rate progressive* in public finance; they are simply those tax functions that are convex. Concave tax schedules are called *marginal-rate regressive*.

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constituents of these societies. In fact, this viewpoint suggests a straightforward explanation of the empirically observed popularity of marginal-rate progressivity, provided that one subscribes to the "one-man one-vote" rule. Since the income distributions of these countries are globally right-skewed (in the sense that the median income is strictly smaller than the mean income for any right truncation of the income distribution), the number of the poorer voters always exceeds that of the richer voters, regardless of how one defines the cutoff that separates the poor from the rich. Since poorer voters are typically the supporters of progressive policies, so the argument goes, there would then be a natural tendency for the marginal-rate progressive tax policies to be favored by the majority. Even though the actual political processes are far more complex than the scenario in which people vote directly over policies, this argument appears to suggest a convincing reason for why progressive tax policies are so widely adopted.

Put succinctly, the objective of the present paper is to understand what makes and breaks this heuristic argument. There are, of course, a number of published studies on this matter. [2] However, a canonical representative democracy model suitable for an immediate examination of the said argument is not yet available. In this paper, we try to remedy this situation by focusing on a simple (but general) model that is appropriate for this purpose. Our model consists of a two-party voting game in which each party (whose objective is to win the elections) proposes tax functions from a given set $\mathcal{A}$ of admissible tax functions (that raise a preset amount of revenue), and voters vote selfishly for the tax function that taxes them less (Section 3). Assuming that the income distribution is right-skewed, we ask if there is any reason to suspect that only the marginal-rate progressive tax policies would be proposed in the equilibrium of this game. Such a setup seems abundantly natural for the task at hand.

It is of course not sensible to view this simple model as novel—some of the existing literature can be thought of as studying precisely this sort of a voting game. But this literature functions by considering very restrictive choices for the policy space $\mathcal{A}$ (to the extent of confining attention to only linear and/or quadratic tax functions). This is because, for large $\mathcal{A}$, the voting game becomes one of very large (possibly infinite) dimension, and it is long known that such games lack pure strategy equilibria. Our setup is no exception (Section 4.1). In fact, this is perhaps a good argument *against* the aforementioned claim about the popular support of marginal-rate progressivity. Due to potential voting cycles, one may not really be sure to which direction the majority demand may swing. However, this counter argument is not readily convincing, precisely because it ignores the possibility of mixed strategy equilibria. It is indeed surprising that the common game theoretical methodology of looking for mixed strategy equilibria in games that lack pure strategy equilibria seems not at all considered for the type of voting games we study here. [3]

The contribution of the present paper is then two-fold. First, we study here the problem of existence of equilibrium in mixed strategies. This is not a trivial matter, for the payoff functions of the game feature marked discontinuities, and hence the standard argument based on the Glicksberg Existence Theorem does not apply to this setting. [4] Nevertheless, we show here that the recent

[2] See [6,9,19,20,30,5].

[3] While one may find mixed strategies as conceptually more problematic than pure strategies in general, there is no reason for them to be particularly inappropriate in the present context. All the standard interpretations (as limits of pure strategy equilibria of sequences of slightly perturbed games, direct randomization, stability in beliefs, or summary of average behavior) apply to our setup without modification. (See [15] for a novel interpretation of mixed strategies in voting games.) In fact, mixed strategy equilibria of voting games of redistribution are commonly studied in the literature, cf. [25,18,17,16,2].

[4] Working with finite policy spaces would solve the existence problem, but, in turn, would pose serious—and, to us, unsurmountable—difficulties for the characterization of equilibria. (More on this in footnote 11.)

existence results of Reny [28] can be utilized to establish the existence of mixed strategy equilibria (Section 4.2). This shows that the above mentioned heuristic claim about the majority support for marginal-rate progressive taxation cannot be dismissed simply by saying that the associated voting game lacks an equilibrium.

Second, we ask what a mixed strategy equilibrium would look like in this setup. The answer depends on which tax policies are feasible and which are not. In the apparently most reasonable scenario in which one allows for only minimal restrictions for the allowed tax functions, we find that, *generically*, the voting game that is induced by any environment would have a mixed strategy equilibrium that envisages the play of a non-progressive tax with positive probability (Section 5.1). It follows that even the global right-skewedness of income distributions cannot alone account for the prevalence of progressive taxation schemes.

One can escape from this observation by suitably restricting the class of admissible tax functions, and indeed one can obtain non-trivial majority support theorems in this way. In particular, we show that if we run, for example, the marginal-rate progressive policies against the marginal-rate regressive ones—a restriction which is even more general than what is commonly adopted in the related political economy literature—then it is true that the policy that will be implemented in equilibrium is, almost surely, marginal-rate progressive (Section 5.2). Unfortunately, this restriction is unduly ad hoc. Moreover, as we demonstrate in Section 5.3 by means of a simple example, with other types of domain restrictions, marginal-rate progressivity may well lose its privileged position.

Therefore, absent an economic reason for choosing a particular restriction to work with, we are forced to conclude that one should look somewhere else for the "explanation" of why we commonly observe marginal-rate progressive policies in action. On closer inspection, there is a fundamental difficulty in providing formal support for the claim that "there is a natural tendency for the tax policies to be progressive in societies with right-skewed income distributions." At the very least, this argument, which we formalize below for a suitable restriction of the tax policies that can be proposed in the elections, needs to be supplemented with another approach that explains the otherwise artificial constraints on the policy space.

2. Admissible tax schedules

We consider an endowment economy with continuum many individuals. Let $\mathcal{F}$ be the set of all distribution functions $F : \mathbf{R} \to [0, 1]$ with $F(0) = 0$ and $F(1) = 1$. The incomes of the constituents of the population are distributed on $[0, 1]$ according to an element F of $\mathcal{F}$, so, formally, by an *income distribution* we mean a member of $\mathcal{F}$. Throughout this paper, we always view $\mathcal{F}$ as a metric space under the sup-metric, and denote by $\mathbf{p}_F$ the Lebesgue-Stieltjes probability measure induced by F on $[0, 1]$.

Two subspaces of $\mathcal{F}$ will be of interest in what follows. First, our main results will use distributions that belong to the space of all strictly increasing and continuous members of $\mathcal{F}$; we denote this class by $\mathcal{F}^*$. Second, to capture the heuristic statement that "the number of the rich people in the society is strictly less than that of the poor," we will at times work with the right-skewed distribution functions, or more generally, with those $F \in \mathcal{F}$ that envisage that the median income is strictly less than the average income

$$m_F := F^{-1}(\tfrac{1}{2}) < \int_0^1 x\, dF(x) =: \mu_F.$$

We denote the class of all such distribution functions by $\mathcal{F}^+$, and let $\mathcal{F}^{*+} := \mathcal{F}^* \cap \mathcal{F}^+$.

A map $t \in \mathbf{C}[0,1]$ is said to be a *tax function* if it satisfies the following two properties:

- $0 \leqslant t(x) \leqslant x$ for all x,
- $x \mapsto t(x)$ and $x \mapsto x - t(x)$ are increasing maps on $[0,1]$.

The first property is a feasibility condition that disallows the presence of negative taxation.[5] The interpretation of the first part of the second requirement is straightforward, while the second part guarantees—as all real-world statutory tax policies do—that the before-tax and after-tax income ranking of taxpayers are identical.

We denote the class of all tax functions by $\mathcal{T}$, and view this set as a subspace of $\mathbf{C}[0,1]$. The following two subspaces of $\mathcal{T}$ are of great importance in the theory of income taxation:

$$\mathcal{T}^{prog} := \{t \in \mathcal{T} : t \text{ is convex}\} \quad \text{and} \quad \mathcal{T}^{reg} := \{t \in \mathcal{T} : t \text{ is concave}\}.$$

A member of $\mathcal{T}^{prog}$ (resp. $\mathcal{T}^{reg}$) is a tax function that subjects the richer individuals to higher (resp. lower) tax rates; in the literature on public finance, such a tax function is called *marginal-rate progressive* (resp. *regressive*). Needless to say, almost all *statutory* income tax functions implemented in practice belong to $\mathcal{T}^{prog}$.[6]

In what follows, we will be interested in the aggregation of individual preferences about "how" the tax should be collected assuming that the question of "how much" should be collected is somehow answered outside the model. (So, even though we do not allow for negative taxation, our model is, in effect, one of *pure redistribution*.) Specifically, we shall assume here that, given $F \in \mathcal{F}$, tax policies are designed to collect at least an exogenously given amount of tax revenue $0 < r < \mu_F$. Consequently, by a *taxation environment*, we mean an element of the following set:

$$\mathcal{E} := \{(F,r) \in \mathcal{F} \times (0,1) : 0 < r < \mu_F\}.$$

Define

$$\mathcal{E}^* := \{(F,r) \in \mathcal{E} : F \in \mathcal{F}^*\} \quad \text{and} \quad \mathcal{E}^{*+} := \{(F,r) \in \mathcal{E} : F \in \mathcal{F}^{*+}\}.$$

Given a taxation environment $(F,r) \in \mathcal{E}$, we say that the tax function $t \in \mathcal{T}$ is *admissible* if it meets the associated *revenue requirement*. We denote the set of all admissible tax functions by $\mathcal{T}_{(F,r)}$, that is,

$$\mathcal{T}_{(F,r)} := \left\{ t \in \mathcal{T} : \int_0^1 t \, dF \geqslant r \right\}.$$

Throughout this paper, $\mathcal{T}_{(F,r)}$ is viewed as a metric subspace of $\mathbf{C}[0,1]$.

[5] Disallowing negative taxation should be considered only as a simplifying assumption; the main results of this paper remain valid in the presence of negative taxes as well.

[6] There are a few exceptions. For instance, the famous United States Tax Reform Act of 1986 envisaged a non-progressive statutory income tax scheme, but this was corrected in 1991. Since 1992, the US tax scheme has been marginal-rate progressive. This is also the case for all OECD countries in the period 1994–2000.

Finally, we define $\mathcal{T}^{prog}_{(F,r)} := \mathcal{T}_{(F,r)} \cap \mathcal{T}^{prog}$, $\mathcal{T}^{reg}_{(F,r)} := \mathcal{T}_{(F,r)} \cap \mathcal{T}^{reg}$, and

$$\widehat{\mathcal{T}}^{prog}_{(F,r)} := \left\{ t \in \mathcal{T}_{(F,r)} : t|\text{supp}\{\mathbf{p}_F\} \text{ is convex} \right\}.^7$$

While the members of $\widehat{\mathcal{T}}^{prog}_{(F,r)}$ need not be convex on [0, 1], economically speaking, we should still regard them as marginal-rate progressive taxation schemes, for their behavior outside supp$\{\mathbf{p}_F\}$ does not affect their tax incidence. Of course, if $F \in \mathcal{E}^*$, then $\widehat{\mathcal{T}}^{prog}_{(F,r)} = \mathcal{T}^{prog}_{(F,r)}$.

3. The voting game

We now turn to the basic voting game that we shall investigate in what follows. This game is one of the simplest possible models of political competition that takes place in terms of income tax policies. It thus provides a natural framework for examining the validity of the statement "if the majority of a society is relatively poor, then there would be a majority support for progressive policies," the primary objective of the present paper.

Take any taxation environment $(F, r) \in \mathcal{E}$, and consider two political parties who are engaged in competition to hold office. Each party may choose to advocate an income tax policy in $\mathcal{T}_{(F,r)}$ which is to be put in effect in case this party obtains the support of the majority. Citizens evaluate proposals selfishly, that is, an individual with income x regards the tax function t as more desirable than the tax function τ if $t(x) < \tau(x)$. If party 1 proposes the tax policy t and party 2 proposes τ, the population share of individuals that strictly prefer t over τ is determined as

$$w(t, \tau) := \mathbf{p}_F \{x \in [0, 1] : t(x) < \tau(x)\}.$$

Of course, in this case the population share of individuals who strictly prefer the victory of party 2 is $w(\tau, t)$.

We formalize the scenario described above by means of a two-person zero-sum symmetric strategic game

$$\mathcal{G}_{(F,r)} := (\mathcal{T}_{(F,r)}, (u_1, u_2)),$$

where $\mathcal{T}_{(F,r)}$ corresponds to the action space of either party, and $u_i : \mathcal{T}^2_{(F,r)} \to \mathbf{R}$ models the payoff function of player $i = 1, 2$. Parties are not ideological. We posit that their objective is to maximize the net plurality defined as the difference between the vote shares obtained by the candidates.[8] That is, we suppose that

$$u_i(t, \tau) := \begin{cases} w(t, \tau) - w(\tau, t) & \text{if } i = 1, \\ w(\tau, t) - w(t, \tau) & \text{if } i = 2, \end{cases} \tag{1}$$

a formulation which is also adopted in [13] and [16], inter alia.

[7] By $t|\text{supp}\{\mathbf{p}_F\}$ being convex, we mean that $t|\text{supp}\{\mathbf{p}_F\}$ can be extended to a convex function defined on the entire [0,1].

[8] As is usual, we assume that indifferent voters toss a fair coin to determine their vote or abstain.

Alternatively, one could model the parties as maximizing their vote shares with the proviso that indifferent individuals vote by pure randomization, that is,

$$u_1(t,\tau) := w(t,\tau) + \tfrac{1}{2}\,\mathbf{p}_F\{t=\tau\} \quad \text{and} \quad u_2 := 1-u_1. \tag{2}$$

We adopt the formulation given in (1) throughout this paper, yet our entire development would remain unaltered (essentially *verbatim*) if we used instead the formulation in (2).[9]

In passing, we note that, for any $(F,r) \in \mathcal{E}$ and $\emptyset \neq \mathcal{A} \subseteq \mathcal{T}_{(F,r)}$, by $(\mathcal{A}, (u_1, u_2))$, we mean the strategic (sub)game in which the action space and the payoff function of player i is $\mathcal{A}$ and $u_i|_{\mathcal{A}\times\mathcal{A}}$, respectively, $i = 1, 2$. This notational convention is adopted throughout the paper.

4. Existence of equilibrium

A fundamental difficulty about general voting problems is that they often fail to possess an equilibrium. It is presumably for this reason that the games of the form $\mathfrak{g}_{(F,r)}$ have not been studied thoroughly in the related literature. In this section we show that switching attention to mixed strategies provides a way out of this problem.

4.1. Non-existence of pure strategy equilibrium

While there is reason to view $\mathfrak{g}_{(F,r)}$ as a voting game of fundamental importance for the positive theory of income taxation, things get icy when one looks for its Nash equilibria in pure strategies. Indeed, the (potential) multi-dimensionality of the action spaces of the parties makes it impossible to utilize single dimensional voting equilibrium theorems like the median voter theorem. While this is no guarantee that $\mathfrak{g}_{(F,r)}$ does not possess equilibria, this is unfortunately the case. The existence of Condorcet-type cycles leads to the non-existence of a pure strategy Nash equilibrium for this game in most cases.

Proposition 1. *$\mathfrak{g}_{(F,r)}$ does not have a pure strategy Nash equilibrium for any taxation environment $(F,r) \in \mathcal{E}^*$.*

While its formal proof is somewhat tedious, this result is clearly a folk theorem the intuition of which is quite simple. Given any admissible tax function t with $t(0+) > 0$ (the no tax exemption case), one can always find another tax function τ which is below t over an interval of $\mathbf{p}_F$-measure greater than $\frac{1}{2}$. A similar trick applies to those tax functions with exemption as well, and hence the result. (We omit formalizing this elementary argument here for brevity.) As noted by Marhuenda and Ortuño-Ortín [20], the situation is analogous to the problem of dividing a cake of a fixed size among three agents. The core of the induced (coalitional) game is empty, since for any division of the cake, there is another division which is preferred by exactly two of the individuals. This observation is the main culprit behind Proposition 1.

[9] The reader may wonder about the possibility that candidates maximize their respective probabilities of winning rather than vote shares. This would be equivalent to maximizing a function that takes value 1 for positive net pluralities and 0 otherwise. By an easy application of Urysohn's Lemma, one can show that this objective, say for party 1, can be approximated pointwise by a map of the form $(t,\tau) \mapsto g(w(t,\tau)-w(\tau,t))$, where g is a strictly increasing, continuous and even function on $[-1,1]$. The treatment of the present work is easily adapted to accommodate this alternative formulation.

4.2. Existence of mixed strategy equilibrium

Given the importance of $\mathfrak{g}_{(F,r)}$ for the positive theory of income taxation, a natural next question is if the mixed strategy equilibria for such a game exist. This section is devoted to answering this question. [10,11]

Technically speaking, the difficulty with establishing the existence of equilibria for $\mathfrak{g}_{(F,r)}$ is the discontinuity of the objective functions u_i. In principle, these can be vast enough to yield even the non-measurability of the objective functions, which would in turn disallow the use of mixed strategies for this game. Fortunately, however, this is not the case here due to the following useful observation.

Lemma 1. *For any taxation environment $(F, r) \in \mathcal{E}$, the maps $(t, \tau) \mapsto w(t, \tau)$ and $(t, \tau) \mapsto w(\tau, t)$ are lower semicontinuous on $\mathcal{T}^2_{(F,r)}$.*

Proof. Fix any $(F, r) \in \mathcal{E}$, and take any sequence (t_n, τ_n) in $\mathcal{T}^2_{(F,r)}$ such that $t_n \to t$ and $\tau_n \to \tau$. By Fatou's Lemma,

$$\liminf w(t_n, \tau_n) = \liminf \int_0^1 \mathbf{1}_{\{t_n<\tau_n\}}\, dF \geqslant \int_0^1 \liminf \mathbf{1}_{\{t_n<\tau_n\}}\, dF.$$

But we have

$$\liminf \mathbf{1}_{\{t_n<\tau_n\}} \geqslant \mathbf{1}_{\{\lim t_n<\lim \tau_n\}}.$$

For, if the left-hand side takes value 0 at some $y \in [0, 1]$, then $t_n(y) \geqslant \tau_n(y)$ for infinitely many n, and this means that the right-hand side cannot take value 1 at y. Combining this observation with the previous inequality, we get

$$\liminf w(t_n, \tau_n) \geqslant \int_0^1 \mathbf{1}_{\{\lim t_n<\lim \tau_n\}}\, dF = w(t, \tau),$$

proving that w is lower semicontinuous on $\mathcal{T}_{(F,r)}$. The second claim is proved similarly. □

For any $(F, r) \in \mathcal{E}$, it follows from Lemma 1 that the map $(t, \tau) \mapsto w(t, \tau) - w(\tau, t)$ is Borel measurable on $\mathcal{T}^2_{(F,r)}$, and hence, both of the objective functions of the game $\mathfrak{g}_{(F,r)}$ are Borel measurable. This allows us to well-define the mixed strategy extension of our voting game.

A *mixed strategy* for the game $\mathfrak{g}_{(F,r)}$ is defined as any Borel probability measure on $\mathcal{T}_{(F,r)}$. We extend the payoff functions of the players to the domain of mixed strategy profiles in the

[10] There are other instances in the literature where a basic strategic model lacks an equilibrium in pure strategies, and yet the mixed strategy extension of the model admits an equilibrium, and reveals quite a bit about the structure of the game at hand. For instance, it is well-known that the Bertrand duopoly model with constant marginal costs and suitable capacity constraints does not have a pure strategy equilibrium—this is called the *Edgeworth paradox*. Maskin [21] and Osborne and Pitchik [26], however, have shown that this model possesses interesting mixed strategy equilibria. The definitive work of Kreps and Sheinkman [14] on this issue also requires the use of mixed strategies.

[11] Studying finitistic formulations of $\mathfrak{g}_{(F,r)}$ would solve the existence problem. However, any choice of a finite subset of $\mathcal{T}_{(F,r)}$ may be regarded as arbitrary, and any characterization of the equilibria of finite formulations of the electoral game at hand will be unconvincing, unless, perhaps, one provides results that are valid *for any* "sufficiently" rich family of tax schemes. Providing results that are robust to the addition of tax schemes outside the chosen finite domain seems difficult, for one can show that *any* equilibrium of the game whose strategy space is *any* finite domain of tax schemes ceases to be an equilibrium after some expansion of the chosen finite domain. One could argue that this should not be a problem if the new equilibrium were similar to the original one. But this requires a proof—by no means trivial—that the original equilibrium is essentially unaltered. The alternative of focusing on the infinite game $\mathfrak{g}_{(F,r)}$ has proven more fruitful.

usual way

$$U_i(\mu_1, \mu_2) := \int_{\mathcal{T}_{(F,r)} \times \mathcal{T}_{(F,r)}} u_i d(\mu_1 \times \mu_2), \quad \mu_i \in \mathbb{P}(\mathcal{T}_{(F,r)}), \ \ i = 1, 2,$$

where $\mathbb{P}(\mathcal{T}_{(F,r)})$ stands for the set of all Borel probability measures on $\mathcal{T}_{(F,r)}$. Each $U_i : \mathbb{P}(\mathcal{T}_{(F,r)})^2 \to \mathbf{R}$ is well-defined since any Borel measurable function on $\mathcal{T}^2_{(F,r)}$ is measurable in the associated product measure space. [12] As usual, by a *mixed strategy equilibrium* of $\mathfrak{g}_{(F,r)}$ we mean a Nash equilibrium of the strategic game

$$\mathfrak{G}_{(F,r)} := (\mathbb{P}(\mathcal{T}_{(F,r)}), (U_1, U_2)).$$

The problem that we now pose is the existence of mixed strategy equilibria for $\mathfrak{g}_{(F,r)}$. Unfortunately, this is not a trivial matter, for $\mathfrak{g}_{(F,r)}$ is a game with continuum many actions and discontinuous payoff functions.

To illustrate how badly behaved a game like $\mathfrak{g}_{(F,r)}$ may be, consider the tax functions τ, τ', and t depicted in Fig. 1. Define the sequence (t_n) in $\mathcal{T}_{(F,r)}$ by $t_n := \left(1 - \frac{1}{n}\right)\tau + \frac{1}{n}\tau'$ for each n. Observe that $w(t_n, t) - w(t, t_n)$ is positive and bounded away from zero for all n, yet we have $\|t_n - \tau\|_\infty \to 0$ and $w(\tau, t) - w(t, \tau) = -0.1$. Thus, it is possible that every member of a uniformly convergent sequence of tax policies yields a positive (and bounded away from zero) net plurality against a feasible tax function, whereas the limit of the sequence does a relatively bad job (in terms of net plurality) against this tax function. This example explains why the expected payoff function U_i is not lower semicontinuous, thereby demonstrating that the standard mixed strategy equilibrium existence results (such as those in [8] and [33]) do not apply to a game like $\mathfrak{g}_{(F,r)}$. [13]

Despite these difficulties, it turns out that any $\mathfrak{g}_{(F,r)}$ possesses a mixed strategy Nash equilibrium for a large class of distribution functions. This is the first main result of the present paper.

Theorem 1. *For any taxation environment $(F, r) \in \mathcal{E}^*$, the game $\mathfrak{g}_{(F,r)}$ has at least one mixed strategy Nash equilibrium.*

The proof of Theorem 1 is given in Section 6.1, and is based on a recent existence theorem of Reny [28]. For the record, we note that a similar proof could be furnished using instead the main existence theorem of [3]. [14]

[12] $\mathcal{T}_{(F,r)}$ is a compact, and hence separable, metric space (see Lemma 2, Section 6.1). Consequently, the Borel σ-algebra of $\mathcal{T}^2_{(F,r)}$ is identical to the product Borel σ-algebra on $\mathcal{T}^2_{(F,r)}$.

[13] The techniques used elsewhere to study mixed electoral equilibria are not useful to establish the existence of equilibrium in the present setting. On the one hand, [25,18,17,16] resort to constructive methods that do not apply to the game $\mathfrak{g}_{(F,r)}$. On the other hand, [13] provides a mixed strategy equilibrium existence result for an electoral game in which policies can be represented as points in $\mathbf{R}^n$. Since $\mathcal{T}_{(F,r)}$ has the cardinality of the continuum, Kramer's result does not apply to our framework either.

[14] In passing, we would like to point out that the existence theorem of Simon and Zame [32] is not useful for our analysis. The problem is that natural formulations of the game with an endogenous sharing rule corresponding to $\mathfrak{g}_{(F,r)}$ give a payoff correspondence that is not upper hemicontinuous. Thus, our game provides an example that violates Simon and Zame's conditions (for standard tie-breaking rules) but, nonetheless, satisfies the payoff security condition of Reny [28]. By contrast, the auction setting in [11] (which features, as our setting does, games with marked discontinuities in the associated payoff functions) violates better reply security (again, for standard tie-breaking rules) and satisfies the conditions of an extension of the Simon-Zame theorem to Bayesian games.

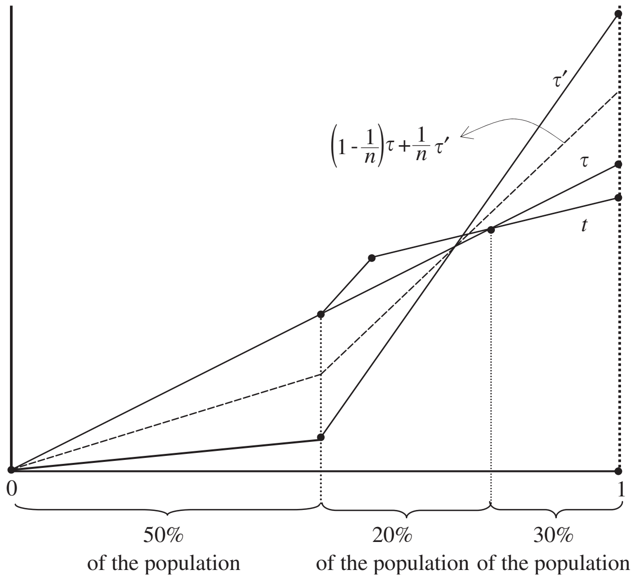

Fig. 1.

In sum, we may conclude that the problem of existence of equilibrium can be resolved in terms of mixed strategies. Thus, it is not a futile exercise to ask qualitative questions about the nature of these equilibria, especially with regards to the majority support of marginal-rate progressive tax schedules. This issue is addressed in the next section.

5. Popular support for progressive taxation

This section is devoted to the analysis of the support of an arbitrary mixed strategy equilibrium of the game $\mathfrak{g}_{(F,r)}$ for any taxation environment $(F, r) \in \mathcal{E}^*$ with $m_F < \mu_F$. The main question is if the right-skewedness of F warrants that such a support consists *only* of progressive taxes. [15] Our main result is that the answer to this question is, *generically* speaking, no. We also show that one can obtain positive answers to this query by suitably restricting the class of admissible tax functions. Yet, absent an economic reason for choosing a particular restriction to work with, the general contention that emanates from our analysis is that the right-skewedness of income distributions cannot alone account for the prevalence of progressive taxation schemes.

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[15] In related contexts, this sort of questions were studied in [7,2]. These works, however, determine supersets of the supports of the mixed strategy equilibria of certain voting games. For example, the main result of [2] entails in the present setup that the support of any mixed strategy equilibrium of $\mathfrak{g}_{(F,r)}$ is contained in the uncovered set of $\mathcal{T}_{(F,r)}$ [22]. Unfortunately, this result does not reveal much here, for the uncovered set of $\mathcal{T}_{(F,r)}$ equals $\mathcal{T}_{(F,r)}$. In this regard, the present query is rather different than that of [2]. In contrast to these authors, here we prove the existence of mixed strategy equilibria of the voting games under consideration, and are interested in what is contained *within* the supports of these equilibria.

5.1. Unrestricted domain of tax functions

5.1.1. The main result

For any taxation environment $(F, r) \in \mathcal{E}^*$, we define the number $0 < x_{(F,r)} < 1$ through the following equation

$$\int_{x_{(F,r)}}^{1} (x - x_{(F,r)})\, dF(x) = r.$$

(Since F is strictly increasing and continuous here, $x_{(F,r)}$ is well-defined.) In words, $x_{(F,r)}$ is the cut-off income level such that if everybody with income less than $x_{(F,r)}$ is exempt from taxation, while everybody else pays all earnings in excess of $x_{(F,r)}$, then the total tax revenue equals exactly r.

Our second main result advances a formal argument to the effect that, at least in taxation environments (F, r) for which the median income level exceeds $x_{(F,r)}$, the supposition that the relative poverty of the majority of a society should warrant a majority support for progressive taxation schemes would be suspect. Informally speaking, in any *generic* such environment, the probability that all income tax functions (that are part of some mixed strategy equilibrium) are convex is zero.

This point is formalized in our second main result.

Theorem 2. *Let $(F, r) \in \mathcal{E}^*$ be a taxation environment with $x_{(F,r)} < m_F$. Every open neighborhood of F contains a distribution function $G \in \mathcal{F}$ such that there exists a mixed strategy Nash equilibrium (μ_1, μ_2) of the game $\mathfrak{g}_{(G,r)}$ with $\mu_i(\widehat{\mathcal{T}}^{prog}_{(F,r)}) < 1$ for each $i = 1, 2$.*

This is a curious observation which launches a major attack on the simplistic explanation of the prevalence of marginal-rate progressive income tax functions. Its validity, however, depends on the assumption that it is not possible to collect the target revenue by taxing at rate 0 up to the median income level. That is, the theorem applies to taxation environments with a relatively large tax revenue requirement. Indeed, it is readily checked that

$$x_{(F,r)} < m_F \quad \text{iff} \quad \int_{m_F}^{1} x\, dF(x) - \tfrac{m_F}{2} < r$$

for any $(F, r) \in \mathcal{E}^*$.

We do not know if this condition can be weakened or completely relaxed in the statement of Theorem 2. However, we note that even extremely right-skewed income distributions may satisfy this condition, even for quite reasonable levels of target tax revenues. The following example illustrates this point.

Example 1. For any $0 \leqslant \alpha \leqslant 1$, consider a taxation environment (F_α, r), where $F_\alpha \in \mathcal{F}^*$ has the following density:

$$f_\alpha(x) = (1 + \alpha) - 2\alpha x, \quad 0 \leqslant x \leqslant 1.$$

It is readily checked that $F_\alpha \in \mathcal{F}^{*+}$ for all $0 < \alpha \leqslant 1$, and that F_α gets more right-skewed as α gets larger. [16] The minimum level of r that guarantees $x_{(F_0,r)} \leqslant m_{F_0}$ here is 0.125 (as the percentage

[16] The coefficient of skewness for the distribution functions F_α range continuously from 0 to 0.565.

of total income). As α increases (and hence F_α gets more right-skewed), this level increases. The worst case scenario obtains in the case of the most right-skewed distribution F_1—we have $x_{(F_1,r)} < m_{F_1}$ iff $r > 0.353$ (as the percentage of total income).

5.1.2. A primer on the proof of Theorem 2

In this subsection, we present an informal discussion on Theorem 2, outlining the key ingredients of its proof and providing a rough idea as to why the result is true. The formal proof is somewhat lengthy, and is given in Section 6.2.

Fix a taxation environment $(F, r) \in \mathcal{E}^*$. Define $\mathcal{G}_{(F,r)}$ to be the family of all $G \in \mathcal{F}$ such that (1) G is discrete on $[0, x_{(F,r)})$; (2) $G = F$ on $[x_{(F,r)}, 1]$; (3) G is sufficiently close to F so that $0 < r < \mu_G$ and $\mathbf{p}_G\{x\} < \mathbf{p}_G\{(x, 1]\}$ for every $x \in \text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$. To bring to the fore the main ingredients behind the proof of Theorem 2, let us assume that $G \in \mathcal{G}_{(F,r)}$ and (G, r) is such that there exists an equilibrium (μ_1, μ_2) of $\mathfrak{g}_{(G,r)}$ with the following properties:

($*$) For each $i = 1, 2$ and $t, f \in \text{supp}\{\mu_i\}$, we have

$$t(x) = f(x) \text{ iff } t(x) = 0 = f(x) \quad \text{whenever } x \in \text{supp}\{\mathbf{p}_G\}. \tag{3}$$

($\diamond$) Define the map $\bar{t} : [0, 1] \to \mathbf{R}$ by

$$\bar{t}(x) := \max\left\{t(x) : t \in \text{supp}\{\mu_2\}\right\}.$$

($\bar{t}$ is the upper envelope of $\text{supp}\{\mu_2\}$.) If there exists $y \in [0, x_{(F,r)})$ such that $\bar{t}(y) > 0$, then there exists an open interval around y on which G is strictly increasing.

Assumption ($*$) says that any two tax functions in the support of the equilibrium mixed strategy μ_i cross only (on $\text{supp}\{\mathbf{p}_G\}$) when they both vanish, $i = 1, 2$. If (G, r) satisfies these two assumptions, we can show that $\mu_i(\widehat{\mathcal{T}}^{prog}_{(F,r)}) < 1$ for each i relatively easily.[17]

Since $\mathfrak{g}_{(G,r)}$ is a symmetric zero-sum game, and (μ_1, μ_2) is an equilibrium of $\mathfrak{g}_{(G,r)}$, a well-known theorem of game theory maintains that (μ_2, μ_2) is also an equilibrium of $\mathfrak{g}_{(G,r)}$. It is without loss of generality that this equilibrium is tight in the sense that

$$U_2(t, \mu_2) = U_2(\mu_2, \mu_2) \quad \text{for all } t \in \text{supp}\{\mu_2\}.^{18} \tag{4}$$

Suppose, to derive a contradiction, we have $\mu_2(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$. It is then without loss of generality to assume $\text{supp}\{\mu_2\} \subseteq \widehat{\mathcal{T}}^{prog}_{(G,r)}$. Let us distinguish between two cases.

Case 1: $\bar{t}|_{[0,x_{(G,r)})} = 0$.

[17] Note that, since G has mass points, t and f may satisfy (3) and at the same time raise the same amount of tax revenue, namely, r.

[18] For, if (μ_2, μ_2) is an equilibrium of $\mathfrak{g}_{(G,r)}$, then $\mathcal{B} := \left\{t \in \mathcal{T}_{(F,r)} : U_2(t, \mu_2) < U_2(\mu_2, \mu_2)\right\}$ is of μ_2-measure zero, and hence $(\tilde{\mu}_2, \tilde{\mu}_2)$ is also an equilibrium of $\mathfrak{g}_{(G,r)}$ when $\tilde{\mu}_2(\mathcal{A}) := \mu_2(\mathcal{A} \setminus \mathcal{B})$ for any Borel subset $\mathcal{A}$ of $\mathcal{T}_{(G,r)}$.

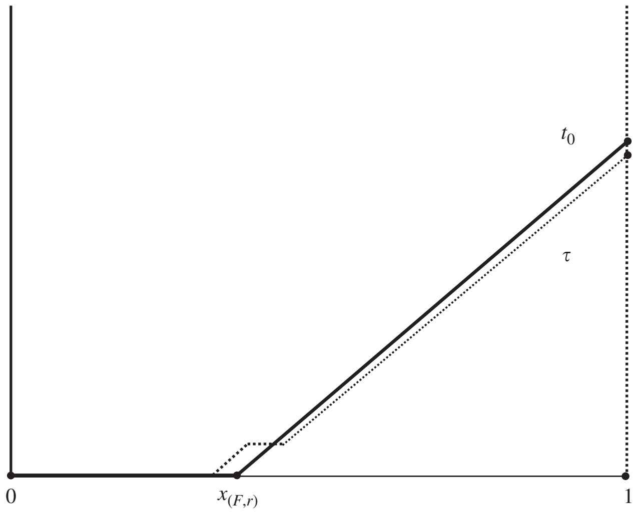

Fig. 2.

In this case, by definition of $x_{(F,r)}$, μ_2 must put probability 1 on the pure strategy $t_0 \in \mathcal{T}_{(G,r)}$ with

$$t_0(x) := \begin{cases} 0 & \text{if } 0 \leqslant x < x_{(F,r)}, \\ x - x_{(F,r)} & \text{if } x_{(F,r)} \leqslant x < 1.^{19} \end{cases}$$

This policy is depicted in Fig. 2. As is clear from that figure, some τ can easily be found in $\mathcal{T}_{(G,r)}$ that defeats t_0.[20] Thus, in this case, μ_2 cannot be part of an equilibrium for the game $\mathfrak{g}_{(G,r)}$, a contradiction.

Case 2: $\bar{t}(y) > 0$ *for some* $y \in [0, x_{(F,r)})$.

In this case, there exists an $f : [0,1] \to \mathbf{R}$ such that

$$f \in \arg\max \left\{ t(y) : t \in \text{supp}\{\mu_2\} \right\} \quad \text{and} \quad f(y) > 0.$$

(f is well-defined, because supp$\{\mu_2\}$ is a closed subset of the compact metric space $\mathcal{T}_{(G,r)}$.) Since $\mu_2(\mathcal{T}^{prog}_{(G,r)}) = 1$ and $f \in \text{supp}\{\mu_2\}$, $f|_{\text{supp}\{\mathbf{p}_G\}}$ must be convex. This, along with the fact that $f(y) > 0$, allows us to find a tax function $\tau \in \mathcal{T}_{(G,r)}$ such that (i) $\tau = 0$ whenever $f = 0$ and (ii) $0 < \tau < f$ everywhere else except on an open interval I that contains y. (See Fig. 3.) By virtue of Assumption ($\diamond$), we may assume, without loss of generality, that G is strictly increasing on I.

When does τ perform worse than f (in terms of u_2) against the policies in the support of μ_2? That is, for which policies t in supp$\{\mu_2\}$ do we have $u_2(t,\tau) \leqslant u_2(t,f)$? By the choice of f and τ, any such t must satisfy $f(x) \leqslant t(x) \leqslant \tau(x)$ for some $x \in I$ (Fig. 3). But, by the choice of f, we must have $f(y) \geqslant t(y)$. By the Intermediate Value Theorem, therefore, there exists a $z \in I$ such that $t(z) = f(z)$. Yet, since $f|_I > 0$ and G is strictly increasing on I, this violates (3). That is, this situation cannot obtain due to the choice of (μ_1, μ_2) (via ($*$) and ($\diamond$)). It follows that

[19] More precisely, μ_2 must assign probability 1 to the set of all pure strategies whose restriction to supp$\{\mathbf{p}_G\}$ coincides with t_0.

[20] Strictly speaking, the validity of this argument depends on the location of the atoms of G in $[0, x_{(F,r)})$. But the argument will work for a large class of distributions. For example, if the atoms of G are sufficiently dispersed over the interval $[0, x_{(F,r)})$, then it is possible to find some τ with the above properties.

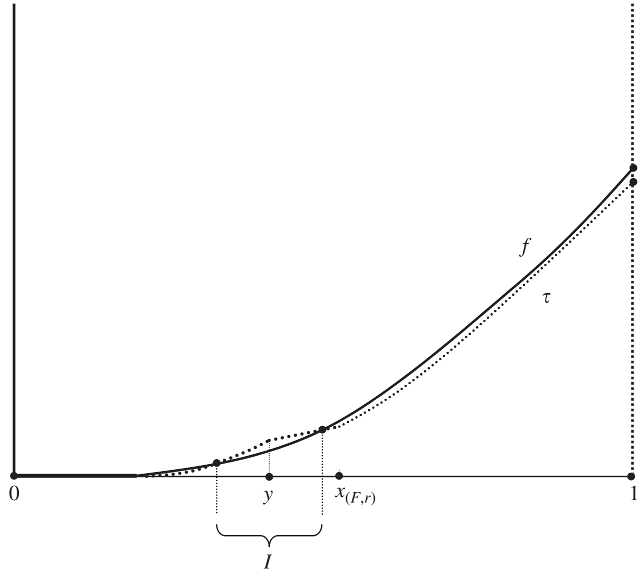

Fig. 3.

$u_2(t,\tau) > u_2(t,f)$ for all $t \in \text{supp}\{\mu_2\}$, and hence, by (4),

$$U_2(\mu_2,\tau) > U_2(\mu_2,f) = U_2(\mu_2,\mu_2),$$

which contradicts (μ_2,μ_2) being an equilibrium of $\mathfrak{g}_{(G,r)}$.

To sum up, if (G,r) is such that there exists an equilibrium (μ_1,μ_2) that satisfies $(*)$ and $(\diamond)$, then there exists an equilibrium (μ_1,μ_2) of $\mathfrak{g}_{(G,r)}$ with $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) < 1$ for each $i = 1, 2$. Now let us indicate how we disburden Assumptions $(*)$ and $(\diamond)$. Given $(F,r) \in \mathcal{E}^*$, let $G \in \mathcal{G}_{(F,r)}$ and consider the game $\mathfrak{g}_{(G,r)}$. By using an argument similar to the one that underlies Theorem 1, we can show that $\mathfrak{g}_{(G,r)}$ has a mixed strategy equilibrium. In fact, we can say more: There exists an equilibrium (μ_1,μ_2) of $\mathfrak{g}_{(G,r)}$ and a sequence $(\mathcal{A}_n)$ of finite subsets of $\mathcal{T}_{(G,r)}$ such that (a) $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$; (b) $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots$ is dense in $\mathcal{T}_{(G,r)}$; (c) there is an equilibrium (μ_1^n,μ_2^n) of each $(\mathcal{A}_n,(u_1,u_2))$ such that (μ_1^n,μ_2^n) converges (weakly) to an equilibrium (μ_1,μ_2) of $\mathfrak{g}_{(G,r)}$; and (d) (3) holds for all $t, f \in \text{supp}\{\mu_i^n\}$, $i = 1, 2$, and $n = 1, 2, \ldots$.

Loosely speaking, if $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$ for some i, (c) enables us to find a sequence $(\mathcal{A}_n,(u_1,u_2))$ of finite subgames of $\mathfrak{g}_{(G,r)}$ whose corresponding sequence of equilibria (μ_1^n,μ_2^n) is such that each $\text{supp}\{\mu_i^n\}$ lies almost entirely within $\widehat{\mathcal{T}}^{prog}_{(G,r)}$. This and properties (d) and (b) allow us to construct an admissible tax policy τ from $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots$ such that $U_2(\mu_2^n,\tau) > 0$ for every n—the argument for this has the same flavor as the one we gave in the first part of the discussion.[21] Finally,

[21] However, there are some differences. To be precise, (d) should be replaced by the following: for all $t, f \in \text{supp}\{\mu_i^n\}$, $i = 1, 2$, and $n = 1, 2, \ldots,$

$$t(x) = f(x) \text{ iff } t(x) = 0 = f(x) \quad \text{whenever } x \in \text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)}),$$

that is, whenever x is an atom of G. In this regard, in the formal proof we cannot use the fact that this property holds on an interval I on which G is strictly increasing (recall Assumption $(\diamond)$), and therefore we cannot appeal to the Intermediate Value Theorem to obtain a contradiction. Indeed, the formal proof proceeds by means of a different argument, and it is at this point that we make use of the hypothesis $x_{(F,r)} < m_F$.

$\tau \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots$ and $U_2(\mu_2^n, \tau) > 0$, together with property (a), gives $U_2(\mu_2^m, \tau) > 0$ for some m with $\tau \in \mathcal{A}_m$, contradicting that (μ_2^m, μ_2^m) is an equilibrium of $(\mathcal{A}_m, (u_1, u_2))$. This implies that the support of each μ_i cannot be contained within $\widehat{\mathcal{T}}^{prog}_{(G,r)}$.

While the formal details of the construction are rather involved, this basic strategic approximation argument conveys, we hope, the main idea behind the proof of Theorem 2.

5.1.3. *On the majority support for progressive taxation*

We have not yet shown if there exists a taxation environment with a right-skewed (pretax) income distribution for which, in equilibrium, political parties do *not* propose progressive schemes almost surely. Giving an explicit example of this is not an easy task. However, as an immediate corollary of Theorem 2, we can establish the existence of such an environment.

Corollary 1. *There exists a taxation environment* $(G, r) \in \mathcal{E}^+$ *with a mixed strategy Nash equilibrium* (μ_1, μ_2) *of the game* $\mathfrak{g}_{(G,r)}$ *such that* $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) < 1$ *for each* $i = 1, 2$.

Proof. Take an arbitrary $F \in \mathcal{F}^{*+}$ with $x_{(F,r)} < m_F$ (say, as in Example 1), note that $\mathcal{F}^+$ is an open neighborhood of F, and apply Theorem 2. □

We could even allow for G to be *globally* right-skewed in this result (in the sense that the median income is strictly smaller than the mean income for any right truncation of G). However, since it is not clear if the set of all globally right-skewed income distributions is open in $\mathcal{F}$, the proof of this fact does not follow readily from Theorem 2. We need instead to notice that our proof of Theorem 2, in fact, delivers more than Theorem 2. It shows that, for any $(F, r) \in \mathcal{E}^*$ with $x_{(F,r)} < m_F$, there is an open neighborhood O of F such that there exists an equilibrium (μ_1, μ_2) of $\mathfrak{g}_{(G,r)}$ with $\mu_i(\widehat{\mathcal{T}}^{prog}_{(F,r)}) < 1$, $i = 1, 2$, for *every* $G \in \mathcal{G}_{(F,r)} \cap O$. (The set $\mathcal{G}_{(F,r)}$ was defined in the second paragraph of the previous subsection.) But, for any open neighborhood O of F, we can easily find a globally right-skewed G in $\mathcal{G}_{(F,r)} \cap O$, provided that F is globally right-skewed. It follows that, in the statement of Corollary 1, G can be taken to be globally right-skewed.

5.2. *Restricted domain of tax functions*

The literature on voting over income taxes is for the most part couched in terms of small subclasses of tax functions. For instance, to be able to make use of the median voter theorem, the seminal papers of Romer [31], Roberts [29], and Meltzer and Richard [23], along with a large fraction of the recent literature on the relation between income inequality and growth (see [4] for a survey), consider only linear tax schemes. Since the linearity assumption is obviously overly restrictive, many authors have tried to study the basic voting problem in terms of larger classes of tax functions. For instance, [9] examines the issue for two-bracket piecewise linear tax functions, and [6] and [30] study quadratic tax functions. More generally, [19,20] allow for the class of all concave (*marginal-rate regressive*) or convex (*marginal-rate progressive*) tax functions, and [5] considers a representative democracy model that allows for the class of all piecewise linear tax functions.

It may then well be asked if there is a nontrivial way of restricting the action spaces of the game $\mathfrak{g}_{(F,r)}$ in a way to obtain marginal-rate progressive taxes, and only such taxes, in any mixed strategy equilibrium of this game. While an economically meaningful way of restricting the admissible set of income tax functions escapes the authors of this paper at present, it may nevertheless be worth noting that the popular support of progressivity can indeed be obtained in mixed strategy equilibrium even with certain infinite dimensional subclasses of $\mathcal{T}_{(F,r)}$.

For an arbitrarily given taxation environment (F, r), we demonstrate this point by restricting the class $\mathcal{T}_{(F,r)}$ to the set of all admissible progressive or regressive taxes:

$$\mathcal{C}_{(F,r)} := \mathcal{T}^{prog}_{(F,r)} \cup \mathcal{T}^{reg}_{(F,r)}.$$

This class contains all the other sets of tax functions mentioned in the first paragraph of this subsection (with the exception of those considered in [5]). It is thus somewhat interesting to study the strategic subgame of $\mathfrak{g}_{(F,r)}$ in which the action space of each political party is taken to be $\mathcal{C}_{(F,r)}$ and the objective functions are restricted to the resulting outcome space. We denote this strategic game by $\mathfrak{h}_{(F,r)}$, that is, $\mathfrak{h}_{(F,r)} := (\mathcal{C}_{(F,r)}, (u_1, u_2))$.

For any taxation environment $(F, r) \in \mathcal{E}$ with $m_F < \mu_F$, Marhuenda and Ortuño-Ortín [19] have observed that

$$w(t, \tau) > \tfrac{1}{2} \quad \text{whenever } (t, \tau) \in \mathcal{N}^{prog}_{(F,r)} \times \mathcal{T}^{reg}_{(F,r)} \text{ and } \int_0^1 t\,dF = \int_0^1 \tau\,dF, \tag{5}$$

where $\mathcal{N}^{prog}_{(F,r)}$ is the set of all *nonlinear* members of $\mathcal{T}^{prog}_{(F,r)}$. In turn, some authors have interpreted this result as "formalizing" the intuition that "in a society in which the numbers of poorer people exceed those of the richer, there would be a majority support for marginal-rate progressive taxation."

Even if one finds the restriction of action spaces to $\mathcal{C}_{(F,r)}$ meaningful, however, this conclusion is suspect, for it is not at all obvious if and how the observation (5) can be interpreted as an *equilibrium* result. After all, like $\mathfrak{g}_{(F,r)}$, the game $\mathfrak{h}_{(F,r)}$ too lacks a pure strategy equilibrium (see [12]). Yet, for what is its worth, this contention turns out to be tenable within the confines of mixed strategy equilibria. First, like $\mathfrak{g}_{(F,r)}$, the game $\mathfrak{h}_{(F,r)}$ is found to possess mixed strategy equilibria for any $(F, r) \in \mathcal{E}^*$. Second, in any equilibrium of $\mathfrak{h}_{(F,r)}$, the regressive taxes (linear or not) are chosen with probability zero.

Theorem 3. *For any taxation environment $(F, r) \in \mathcal{E}^{*+}$, there exists at least one mixed strategy equilibrium of $\mathfrak{h}_{(F,r)}$, and for any equilibrium (μ_1, μ_2) of this game, we have*

$$\mu_1(\mathcal{N}^{prog}_{(F,r)}) = 1 = \mu_2(\mathcal{N}^{prog}_{(F,r)}).$$

This result is the equilibrium version of the popular support theorem in [19]. It says that the parties may have to randomize over their choices (or, what is equivalent, they may have multiple best responses to their equilibrium beliefs about the proposals of their opponents), but when it comes to the observable outcomes, we are bound to see that all proposed tax policies are marginal-rate progressive. Put this way, this result sounds like the formalization of the claim that "there is a natural tendency for the tax policies to be marginal-rate progressive in societies with right-skewed income distributions." (This, of course, provided that one deems reasonable the exclusive focus put on the tax functions in $\mathcal{C}_{(F,r)}$.)

It may be useful to outline the simple intuition behind Theorem 3. Assume that, at an equilibrium of the game $\mathfrak{h}_{(F,r)}$, both players play some regressive taxes with positive probability. It is easily seen that the fact that the players are best responding to each other's strategy implies that only tax functions that collect exactly r are assigned positive probability at the equilibrium. But then at least one of these players may improve her payoff by imitating her opponent's strategy restricted to the part of the support in $\mathcal{N}^{prog}_{(F,r)}$. (If no tax function in $\mathcal{N}^{prog}_{(F,r)}$ lies in the support of the opponent's strategy, any pure strategy from $\mathcal{N}^{prog}_{(F,r)}$ improves the payoffs of the player.) By doing so, this player obtains, on the one hand, a positive net plurality against those regressive tax policies that are assigned positive probability by her opponent (this is guaranteed by (5)). On the other hand, her strategy ties against all progressive taxes that are played with positive probability by the other candidate. Thus, the deviation from the original equilibrium yields a positive payoff. Since the game is symmetric and zero-sum, no player can obtain a positive payoff at an equilibrium, and so the deviation is profitable. [22]

5.3. A caveat

The difference between the basic messages of Theorems 1–3 are striking. The former results tell us clearly that the prevalence of progressivity that we observe in actuality cannot be attributed solely to the right-skewedness of the income distributions. Theorem 3, on the other hand, suggests that this may well be the case, if the political parties, per force, focus only on certain types of tax functions, those that are either marginal-rate progressive or regressive.

We find the strength of the first message here superior to that of the latter. This is mainly because, while it allows to derive pleasing conclusions such as Theorem 3, the restriction of the feasible tax policies to be either marginal-rate progressive or marginal-rate regressive is unduly ad hoc. It is difficult to think of a reason why a tax designer would not consider those tax schedules that are neither progressive nor regressive. [23] Moreover, inclusion of tax functions that "wiggle," in the sense of being regressive on certain regions and progressive over others, may alter the message of Theorem 3 significantly. While Theorem 2 makes this clear in a global sense, the following simple example illustrates the nature of the difficulty in a local sense. It shows that, in a restricted domain, the admissibility of "wiggling" taxes might even result in the emergence of regressive taxes in equilibrium.

Example 2. Let $r = 0.15$, and consider a taxation environment (F, r), where $F \in \mathcal{F}^{*+}$ has the following density:

$$f_\alpha(x) = 2 - 2x, \quad 0 \leqslant x \leqslant 1.$$

Consider the tax functions t_1 and t_2 defined as

$$t_1(x) := \begin{cases} \frac{x}{4} & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \alpha(x - \frac{1}{4}) + \frac{1}{16} & \text{if } \frac{1}{4} < x \leqslant 1, \end{cases}$$

[22] [10] restricts the set of feasible tax functions to quadratic ones and shows that for any feasible tax function there exists a more regressive feasible policy that is preferred by a majority of voters. Notice the difference between this result and the popular support theorem in [19]. Take some measure of progressivity and fix a (progressive) tax policy t. Suppose that a more regressive admissible policy may be found that defeats t. In general, this new policy will *not* defeat *all* other tax functions which are at least as progressive as t. By contrast, a marginal-rate progressive tax defeats *any* marginal-rate regressive tax under pairwise majority voting. This fact is crucial in the argument outlined here (and in the proof of Theorem 3).

[23] In fact, after the 1986 Tax Reform Act, there was a period of three years when the statutory federal income tax schedule in the US was neither convex nor concave due to the non-monotonicity of the top three brackets (see [24]). So, it is fair to say that such tax policies are considered by politicians/tax designers, but only rarely see the daylight.

and

$$t_2(x) := \begin{cases} \frac{x}{2} & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \beta(x - \frac{1}{4}) + \frac{1}{8} & \text{if } \frac{1}{4} < x \leqslant 1, \end{cases}$$

where we set $\alpha = \frac{391}{540}$ and $\beta = \frac{103}{270}$. Both of these tax functions belong to $\mathcal{T}_{(F,r)}$ while t_1 is marginal-rate progressive and t_2 regressive. An easy computation shows that $w(t_1, t_2) = 0.67$, and hence if the game $\mathfrak{g}_{(F,r)}$ was played with the action spaces of both parties being restricted to $\{t_1, t_2\}$, then the unique mixed strategy-equilibrium would be the one in which both parties choose the progressive tax t_1 with probability 1.

Now consider the situation in which a third, "wiggling," tax function is also allowed to be proposed. That is, the game $\mathfrak{g}_{(F,r)}$ is played with the action spaces of both parties being restricted to $\{t_1, t_2, t_3\}$, where

$$t_3(x) := \begin{cases} \frac{(4x)^\beta}{8} & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ \frac{x}{2} & \text{if } \frac{1}{4} < x \leqslant 1, \end{cases}$$

with $\beta := \frac{4\sqrt{6}}{3} + 1$. (As Fig. 4 makes it transparent, t_3 can be taken to be a three-bracket piecewise linear tax function; this tax function is hardly contrived.) It is easily checked that $t_3 \in \mathcal{T}_{(F,r)}$, and we have $w(t_1, t_3) = 0.41$ and $w(t_2, t_3) = 0.57$. The resulting strategic game is thus represented by the following payoff bimatrix:

	t_1	t_2	t_3
t_1	0, 0	0.34, −0.34	−0.18, 0.18
t_2	−0.34, 0.34	0, 0	0.14, −0.14
t_3	0.18, −0.18	−0.14, 0.14	0, 0

While this game has no pure strategy equilibrium, it possesses a unique mixed strategy equilibrium in which both candidates randomize over the three tax policies according to the vector of probabilities $\left(\frac{7}{33}, \frac{3}{11}, \frac{17}{33}\right)$. Curiously, in equilibrium, the lowest probability is placed on the marginal-rate progressive tax function. In particular, the probability that the regressive tax would be proposed by a party in equilibrium is found larger than that of the progressive tax function.

While it is only suggestive, this example points clearly how different restrictions of policy spaces may result in qualitatively different conclusions about the majority support of progressive taxation. Given that there is no obvious way of choosing a particular such space from an economic point of view, then, we are led to put more emphasis on the analysis of the situation in the absence of any such restrictions. Once this is done, Theorem 2 becomes operational, and says that the majority support of progressive taxation cannot be only due to the majority of a population being relatively poor. We simply need to look elsewhere for a fuller explanation.

6. Proofs

6.1. Proof of Theorem 1

Lemma 2. *$\mathcal{T}_{(F,r)}$ is a compact subset of* $\mathbf{C}[0, 1]$ *for any $(F, r) \in \mathcal{E}$.*

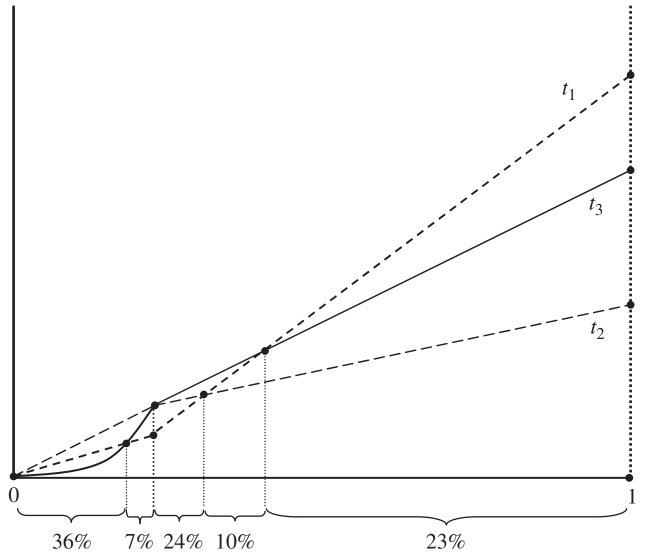

Fig. 4.

Proof. Fix any $(F, r) \in \mathcal{E}$. Let us first show that $\mathcal{T}_{(F,r)}$ is bounded and closed in $\mathbf{C}[0, 1]$. For any $t, \tau \in \mathcal{T}_{(F,r)}$, we have $\|t - \tau\|_\infty \leqslant 1$ since the ranges of both t and τ are contained in $[0, 1]$. Thus $\mathrm{diam}(\mathcal{T}_{(F,r)}) \leqslant 1$ and hence $\mathcal{T}_{(F,r)}$ is bounded. To prove the closedness claim, take any sequence (t_n) in $\mathcal{T}_{(F,r)}$ and assume that $\|t - t_n\|_\infty \to 0$ for some $t \in \mathbf{C}[0, 1]$. Then (t_n) converges uniformly to t, thereby guaranteeing that t is a tax function. Due to uniform convergence, we also have

$$\int_0^1 t\, dF = \int_0^1 \lim t_n\, dF = \lim \int_0^1 t_n\, dF \geqslant r,$$

and hence we may conclude that $t \in \mathcal{T}_{(F,r)}$. Thus $\mathcal{T}_{(F,r)}$ is closed in $\mathbf{C}[0, 1]$.

We next claim that $\mathcal{T}_{(F,r)}$ is equicontinuous. To see this, pick an arbitrary $t \in \mathcal{T}_{(F,r)}$ and take any $0 \leqslant y < x \leqslant 1$. By monotonicity of the post-tax function, we have $x - t(x) \geqslant y - t(y)$ so that $t(x) - t(y) \leqslant x - y$. Interchanging the roles of x and y, we may thus conclude that $|t(x) - t(y)| \leqslant |x - y|$ for all $0 \leqslant x, y \leqslant 1$. So, for any $x \in [0, 1]$ and any $\varepsilon > 0$, we have $|t(x) - t(y)| < \varepsilon$ whenever $|x - y| < \varepsilon$. This proves that $\mathcal{T}_{(F,r)}$ is equicontinuous.

Given the observations noted in the previous two paragraphs, the Arzelà-Ascoli Theorem entails that $\mathcal{T}_{(F,r)}$ is a compact subset of $\mathbf{C}[0, 1]$. □

Lemma 3A. *Let $(F, r) \in \mathcal{E}^*$. For any given $\varepsilon > 0$ and $\tau \in \mathcal{T}_{(F,r)}$ with $\int_0^1 \tau\, dF = r$, there exists a $\tau^* \in \mathcal{T}_{(F,r)}$ with the following property: for every $t \in \mathcal{T}_{(F,r)}$, there exists a $\delta > 0$ such that*

$$w(\tau^*, f) - w(f, \tau^*) > w(\tau, t) - w(t, \tau) - \varepsilon \quad \text{for all } f \in N_\delta(t). \tag{6}$$

(Here $N_\delta(t)$ stands for the open δ-neighhborhood of t in $\mathcal{T}_{(F,r)}$.)

To keep the logic of the proof of Theorem 2 transparent, we relegate the technical proof of this result to the end of this section.

Lemma 3B. *Let $(F, r) \in \mathcal{E}^*$, $p \in \mathbb{P}(\mathcal{T}_{(F,r)})$, and $\varepsilon > 0$. Then, for each $\tau \in \mathcal{T}_{(F,r)}$ there is a $\tau^* \in \mathcal{T}_{(F,r)}$ such that*

$$\liminf \int_{\mathcal{T}_{(F,r)}} \left(w(\tau^*, \cdot) - w(\cdot, \tau^*)\right) dp_n \geqslant \int_{\mathcal{T}_{(F,r)}} (w(\tau, \cdot) - w(\cdot, \tau))\, dp - \varepsilon \tag{7}$$

for every sequence (p_n) in $\mathbb{P}(\mathcal{T}_{(F,r)})$ that converges weakly to p.

Proof. Fix an arbitrary sequence (p_n) in $\mathbb{P}(\mathcal{T}_{(F,r)})$ that converges weakly to p. Pick any $\tau \in \mathcal{T}_{(F,r)}$ with $\int_0^1 \tau\, dF = r$. By Lemma 3A, there is $\tau^* \in \mathcal{T}_{(F,r)}$ such that, for every $t \in \mathcal{T}_{(F,r)}$, there exists a $\delta > 0$ with

$$w(\tau^*, f) - w(f, \tau^*) > w(\tau, t) - w(t, \tau) - \varepsilon \quad \text{for all } f \in N_\delta(t).$$

Define $\varphi : \mathcal{T}_{(F,r)} \to \mathbf{R}$ by

$$\varphi(t) := \lim_{\delta \searrow 0} \inf \left\{ w(\tau^*, f) - w(f, \tau^*) : f \in N_\delta(t) \right\}.$$

Then φ is lower semicontinuous, and hence, so is the map $\sigma \mapsto \int_{\mathcal{T}_{(F,r)}} \varphi\, d\sigma$ on $\mathbb{P}(\mathcal{T}_{(F,r)})$. (See Theorem 14.5 of [1].) Consequently,

$$\liminf \int_{\mathcal{T}_{(F,r)}} \varphi\, dp_n \geqslant \int_{\mathcal{T}_{(F,r)}} \varphi\, dp. \tag{8}$$

Moreover, by the choice of τ^*, we have $\varphi(t) \geqslant w(\tau, t) - w(t, \tau) - \varepsilon$ for every $t \in \mathcal{T}_{(F,r)}$. It follows that

$$w(\tau^*, t) - w(t, \tau^*) \geqslant \varphi(t) \geqslant w(\tau, t) - w(t, \tau) - \varepsilon \quad \text{for all } t \in \mathcal{T}_{(F,r)}.$$

These inequalities, along with (8), imply (7), as desired.

Finally, take any $\tau \in \mathcal{T}_{(F,r)}$ with $\int_0^1 \tau\, dF > r$, and define $a := r / \int_0^1 \tau\, dF$. By what we have established in the previous paragraph, there is a $\tau^* \in \mathcal{T}_{(F,r)}$ such that

$$\begin{aligned} \liminf \int_{\mathcal{T}_{(F,r)}} \left(w(\tau^*, \cdot) - w(\cdot, \tau^*)\right) dp_n &\geqslant \int_{\mathcal{T}_{(F,r)}} (w(a\tau, \cdot) - w(\cdot, a\tau))\, dp - \varepsilon \\ &\geqslant \int_{\mathcal{T}_{(F,r)}} (w(\tau, \cdot) - w(\cdot, \tau))\, dp - \varepsilon, \end{aligned}$$

where the second inequality follows from the fact that $a\tau < \tau$. $\square$

Definition 1 (*Reny, 1999*). The mixed extension $\mathfrak{G}_{(F,r)}$ of $\mathfrak{g}_{(F,r)}$ is *payoff secure* if for every $i \in \{1, 2\}$, $(\mu_1, \mu_2) \in \mathbb{P}(\mathcal{T}_{(F,r)})^2$, and $\varepsilon > 0$, there exists $v_i \in \mathbb{P}(\mathcal{T}_{(F,r)})$ such that $U_i(v_i, \tilde{\mu}_{-i}) \geqslant U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

Lemma 4. *$\mathfrak{G}_{(F,r)}$ is payoff secure for any $(F, r) \in \mathcal{E}^*$.*

Proof. Fix any $\mu = (\mu_1, \mu_2) \in \mathbb{P}(\mathcal{T}_{(F,r)})^2$ and $\varepsilon > 0$, and take any $i \in \{1, 2\}$, say, $i = 1$. It is clear that there exists $\tau \in \mathcal{T}_{(F,r)}$ such that

$$U_1(\tau, \mu_2) > U_1(\mu) - \tfrac{\varepsilon}{4}. \tag{9}$$

By Lemma 3B, there exists $\tau^* \in \mathcal{T}_{(F,r)}$ such that every sequence (μ^n) in $\mathbb{P}(\mathcal{T}_{(F,r)})$ converging weakly to μ_2 satisfies

$$\liminf \int_{\mathcal{T}_{(F,r)}} (w(\tau^*, \cdot) - w(\cdot, \tau^*))\, d\mu^n \geqslant \int_{\mathcal{T}_{(F,r)}} (w(\tau, \cdot) - w(\cdot, \tau))\, d\mu_2 - \tfrac{\varepsilon}{2}.$$

Hence, $U_1(\tau^*, \tilde{\mu}_2) \geqslant U_1(\tau, \mu_2) - \frac{3\varepsilon}{4}$ for every $\tilde{\mu}_2$ in some open neighborhood of μ_2. From these inequalities and (9), we see that $U_1(\tau^*, \sigma_2) \geqslant U_1(\mu) - \varepsilon$ for every σ_2 in some open neighborhood of μ_2, as we sought. $\square$

Proof of Theorem 1. Fix $(F, r) \in \mathcal{E}^*$. The game $\mathfrak{g}_{(F,r)}$ is compact (Lemma 2). Further, its mixed extension $\mathfrak{G}_{(F,r)}$ is payoff secure (Lemma 4) and $u_1 + u_2$ is continuous. It follows from Corollary 5.2 and Proposition 5.1 of [28] that $\mathfrak{g}_{(F,r)}$ possesses a mixed strategy Nash equilibrium. $\square$

The final task at hand is to prove Lemma 3A.

Proof of Lemma 3A. Pick any $\varepsilon > 0$ and $\tau \in \mathcal{T}_{(F,r)}$ with $\int_0^1 \tau\, dF = r$. Define

$$x^* := \inf \{x \in [0, 1] : \tau(y) - \tau(x) = y - x \text{ whenever } x \leqslant y \leqslant 1\}.$$

Since $r < \mu_F$, we cannot have $x^* = 0$, so $0 < x^* \leqslant 1$.

Suppose first that $0 < x^* < 1$. For $\varepsilon > 0$ with $[x^* - \varepsilon, x^* + \varepsilon] \subseteq [0, 1]$, let

$$\underline{t}_\varepsilon(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant x^* - \varepsilon - \tau(x^* - \varepsilon), \\ \tau(x^* + \varepsilon) & \text{if } x^* - \varepsilon + \tau(x^* + \varepsilon) - \tau(x^* - \varepsilon) < x \leqslant 1, \\ \tau(x^* - \varepsilon) - (x^* - \varepsilon) + x & \text{elsewhere,} \end{cases}$$

and put $\bar{t}_\varepsilon := \max\{\tau, \underline{t}_\varepsilon\}$. Then, there exists $\varepsilon^\circ > 0$ such that $\int_0^1 \underline{t}_\varepsilon\, dF < r$ and $\int_0^1 \bar{t}_\varepsilon\, dF > r$ for any $0 < \varepsilon < \varepsilon^\circ$. It follows from the Intermediate Value Theorem that, for each $0 < \varepsilon < \varepsilon^\circ$, there is an $0 < \alpha_\varepsilon < 1$ with

$$\int_0^1 \left(\alpha_\varepsilon \bar{t}_\varepsilon + (1 - \alpha_\varepsilon)\underline{t}_\varepsilon\right) dF = r.$$

For any $0 < \varepsilon < \varepsilon^\circ$, define

$$t_\varepsilon := \begin{cases} \alpha_\varepsilon \bar{t}_\varepsilon + (1 - \alpha_\varepsilon)\underline{t}_\varepsilon & \text{if } 0 < x^* < 1, \\ \beta_\varepsilon \bar{\tau}_\varepsilon + (1 - \beta_\varepsilon)\underline{\tau}_\varepsilon & \text{if } x^* = 1 \end{cases}$$

which is an admissible tax function. There is a sequence (ε_n) of positive real numbers converging to 0 such that $t_{\varepsilon_n} < \tau$ on $\{\tau > 0\} \setminus [x^* - \varepsilon_n, x^* + \varepsilon_n]$ for every n.[24] Thus, we may choose an $\varepsilon^* > 0$ small enough to ensure that

$$\varepsilon^* < \min \left\{x^*, 1 - x^*, \tfrac{\varepsilon}{12}\right\}$$

[24] To see this, it suffices to show that there is a sequence $(\varepsilon_n) \in \mathbf{R}^\infty_{++}$ that converges to 0 such that the left-hand derivative τ'_- of τ at $x^* - \varepsilon_n$ is less than 1 for each n. If τ is constant on (a, x^*) for some a, there is nothing to prove, so let τ be nonconstant on any (a, x^*). For each ε with $x^* \geqslant \varepsilon > 0$, consider the function $f_\varepsilon(x) := \tau(x^*) - \tau(x) - \frac{x^* - x}{\varepsilon}[\tau(x^*) - \tau(x^* - \varepsilon)]$, which vanishes when $x = x^* - \varepsilon$ and $x = x^*$. Because f_ε is continuous and not nil on $[x^* - \varepsilon, x^*]$, there exists $y \in (x^* - \varepsilon, x^*)$ such that the left-hand derivative of f_ε at y is positive, whereby $\frac{\tau(x^*) - \tau(x^* - \varepsilon)}{\varepsilon} > \tau'_-(y)$. Since $\tau \in \mathcal{T}_{(F,r)}$, the left-hand side of this inequality is less than 1, whence $\tau'_-(y) < 1$, as desired.

and

$$t_{\varepsilon^*} < \tau \text{ on } \{\tau > 0\} \setminus [x^* - \varepsilon^*, x^* + \varepsilon^*].$$

In what follows, we let $\tau^* := t_{\varepsilon^*}$. Take any $t \in \mathcal{T}_{(F,r)}$. We wish to find a $\delta > 0$ such that (6) holds. Begin with choosing a number η with

$$\varepsilon^* < \eta < \min\left\{x^*, 1 - x^*, \tfrac{\varepsilon}{12}\right\}.$$

Then define

$$\delta := \begin{cases} \min_{x\in[s_o+\eta,x^*-\eta]\cup[x^*+\eta,1]} |\tau(x) - \tau^*(x)| & \text{if } s_o + \eta \leqslant x^* - \eta, \\ \min_{x\in[x^*+\eta,1]} |\tau(x) - \tau^*(x)| & \text{otherwise,} \end{cases}$$

where $s_o := \sup \tau^{-1}(0)$. Now take an arbitrary $f \in N_\delta(t)$. Let S^c be the complement of

$$S := \left[s_o, s_o + \eta\right] \cup \left[x^* - \eta, x^* + \eta\right]$$

in [0, 1]. Then,

$$\begin{aligned} w(\tau, t) - w(t, \tau) &= 2w(\tau, t) + \mathbf{p}_F\{\tau = t\} - 1 \\ &= 2\left[\mathbf{p}_F(\{\tau < t\} \cap S) + \mathbf{p}_F(\{\tau < t\} \cap S^c)\right] \\ &\quad + \mathbf{p}_F(\{\tau = t\} \cap S) + \mathbf{p}_F(\{\tau = t\} \cap S^c) - 1. \end{aligned}$$

Because the length of S is at most 4η, and, by definition, $\eta < \frac{\varepsilon}{12}$, the length of S is less than $\frac{\varepsilon}{3}$. Therefore,

$$2\left(\mathbf{p}_F(\{\tau < t\} \cap S)\right) + \mathbf{p}_F(\{\tau = t\} \cap S) < \varepsilon.$$

Combining this inequality with the previous equality gives

$$w(\tau, t) - w(t, \tau) < \varepsilon + 2\mathbf{p}_F(\{\tau < t\} \cap S^c) + \mathbf{p}_F(\{\tau = t\} \cap S^c) - 1.$$

By means of a straightforward decomposition, it follows that

$$\begin{aligned} & w(\tau, t) - w(t, \tau) \\ &\quad < \varepsilon + 2\left[\mathbf{p}_F(\{\tau^* < \tau < t\} \cap S^c) + \mathbf{p}_F(\{\tau^* \geqslant \tau < t\} \cap S^c)\right] \\ &\qquad + \mathbf{p}_F(\{\tau^* < \tau = t\} \cap S^c) + \mathbf{p}_F(\{\tau^* \geqslant \tau = t\} \cap S^c) - 1 \\ &\quad = \varepsilon + 2[\mathbf{p}_F(\{f \leqslant \tau^* < \tau < t\} \cap S^c) + \mathbf{p}_F(\{f > \tau^* < \tau < t\} \cap S^c) \\ &\qquad + \mathbf{p}_F(\{f \leqslant \tau^* \geqslant \tau < t\} \cap S^c) + \mathbf{p}_F(\{f > \tau^* \geqslant \tau < t\} \cap S^c)] \\ &\qquad + \mathbf{p}_F(\{f \leqslant \tau^* < \tau = t\} \cap S^c) + \mathbf{p}_F(\{f > \tau^* < \tau = t\} \cap S^c) \\ &\qquad + \mathbf{p}_F(\{f \leqslant \tau^* \geqslant \tau = t\} \cap S^c) + \mathbf{p}_F(\{f > \tau^* \geqslant \tau = t\} \cap S^c) - 1. \end{aligned}$$

Therefore, if

$$\mathbf{p}_F(\{f \leqslant \tau^* < \tau < t\} \cap S^c) = 0, \tag{10}$$

$$\mathbf{p}_F(\{f \leqslant \tau^* < \tau = t\} \cap S^c) = 0, \tag{11}$$

and

$$\mathbf{p}_F(\{f \leqslant \tau^* \geqslant \tau \leqslant t\} \cap S^c) \leqslant \mathbf{p}_F\{f = \tau^*\}, \tag{12}$$

then

$$\begin{aligned} w(\tau,t)-w(t,\tau) &< \varepsilon+2\mathbf{p}_F(\{f>\tau^*\geqslant\tau<t\}\cap S^c)+\mathbf{p}_F(\{f>\tau^*\geqslant\tau<t\}\cap S^c)\\ &\quad+\mathbf{p}_F(\{f>\tau^*<\tau=t\}\cap S^c)+\mathbf{p}_F\{f=\tau^*\}\\ &\quad+\mathbf{p}_F(\{f>\tau^*\geqslant\tau=t\}\cap S^c)-1\\ &\leqslant \varepsilon+2w(\tau^*,f)+\mathbf{p}_F\{\tau^*=f\}-1\\ &= \varepsilon+w(\tau^*,f)-w(f,\tau^*), \end{aligned}$$

as desired. Thus, the proof for the case where $0<x^*<1$ will be complete if we can show that (10)–(12) hold. Observe that, by construction of τ^*, we have $\tau^*=\tau$ on $[0,s_o]$, and hence

$$\mathbf{p}_F(\{f\leqslant\tau^*<\tau<t\}\cap[0,s_o])=0. \tag{13}$$

On the other hand, because $f\in N_\delta(t)$, we have

$$|f-t|<\delta\leqslant|\tau-t| \text{ on } [s_o+\eta,1]\cap S^c.$$

Combining this with (13) yields (10) and (11). To see why (12) must hold, it suffices to observe that the construction of τ^* entails

$$\tau^*=\tau=0 \text{ on } [0,s_o] \quad\text{and}\quad \tau^*<\tau \text{ on } [s_o,1]\cap S^c,$$

whence

$$\begin{aligned} \mathbf{p}_F(\{f\leqslant\tau^*\geqslant\tau\leqslant t\}\cap S^c) &\leqslant \mathbf{p}_F(\{f\leqslant\tau^*\geqslant\tau\}\cap S^c)\\ &\leqslant \mathbf{p}_F(\{f\leqslant\tau^*=\tau=0\}\cap S^c)\\ &\leqslant \mathbf{p}_F\{f=\tau^*\}. \end{aligned}$$

It remains to consider the case where $x^*=1$. But this case is analyzed analogously by redefining $\underline{t}_\varepsilon$ as

$$\underline{t}_\varepsilon(x):=\begin{cases}0 & \text{if } 0\leqslant x\leqslant 1-\varepsilon-\tau(1-\varepsilon),\\ \tau(1-\varepsilon)-(1-\varepsilon)+x & \text{if } 1-\varepsilon-\tau(1-\varepsilon)<x\leqslant 1,\end{cases}$$

and $\overline{t}_\varepsilon:=\max\{\tau,\underline{t}_\varepsilon\}$. □

6.2. *Proof of Theorem 2*

6.2.1. *Construction of a strategic approximation to $\mathfrak{g}_{(G,r)}$*

Definition 2 (*Reny, 1996*). Suppose that $(G,r)\in\mathcal{E}$. A *strategic approximation* to the game $\mathfrak{g}_{(G,r)}$ is a countable set of pure strategies $\mathcal{Q}\subseteq\mathcal{T}_{(G,r)}$ such that

(i) $\sup\{U_i(t,\mu_{-i}):t\in\mathcal{Q}\}=\sup\{U_i(t,\mu_{-i}):t\in\mathcal{T}_{(G,r)}\}$ for all $\mu_{-i}\in\mathbb{P}(\mathcal{T}_{(G,r)})$, $i=1,2$; and
(ii) whenever $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\mathcal{Q}$, any limit of equilibria of the sequence of finite games $(\mathcal{A}_n,(u_1,u_2))$ is an equilibrium of $\mathfrak{g}_{(G,r)}$.

In the remainder of this subsection, we construct a particular approximation to the game $\mathfrak{g}_{(G,r)}$ for a suitably chosen (G,r).

Given $(F,r)\in\mathcal{E}^*$, define $\mathcal{G}_{(F,r)}$ as the family of all $G\in\mathcal{F}$ such that G is discrete on $[0,x_{(F,r)})$, $G=F$ on $[x_{(F,r)},1]$, G is sufficiently close to F to ensure that $0<r<\mu_G$, and $\mathbf{p}_G\{x\}<\mathbf{p}_G\{(x,1]\}$ for every $x\in\operatorname{supp}\{\mathbf{p}_G\}\cap[0,x_{(F,r)})$.

For any $(F, r) \in \mathcal{E}^*$ and $G \in \mathcal{G}_{(F,r)}$, $\mathcal{T}_{(G,r)}$ is a compact, hence separable, metric subspace of $\mathbf{C}[0, 1]$. So, there exists a countable dense subset $\tilde{\mathcal{T}}_{(G,r)}$ of $\mathcal{T}_{(G,r)}$. Let $\{q_1, q_2, \ldots\}$ be an enumeration of the rational numbers in [0, 1], and define

$$\mathcal{T}^{q_n}_{(G,r)} := \left\{t \in \mathcal{T}_{(G,r)} : t|_{[0,q_n]} = 0 \quad \text{and} \quad t|_{(q_n,1]} > 0\right\}, \quad n = 1, 2, \ldots .$$

Each $\mathcal{T}^{q_n}_{(G,r)}$ is a subset of a separable metric space, and is thus a separable metric space itself. So, for each $\mathcal{T}^{q_n}_{(G,r)}$, there is a countable dense subset $\tilde{\mathcal{T}}^{q_n}_{(G,r)}$ of $\mathcal{T}^{q_n}_{(G,r)}$. Put

$$\mathcal{T}^*_{(G,r)} := \tilde{\mathcal{T}}_{(G,r)} \cup \left(\bigcup_{n=1}^{\infty} \tilde{\mathcal{T}}^{q_n}_{(G,r)}\right).$$

$\mathcal{T}^*_{(G,r)}$ is obviously countable; we enumerate this set as $\{t_1(G, r), t_2(G, r), \ldots\}$. Take a sequence (ε_n) of positive real numbers that converges to 0. Define

$$\mathcal{T}_1(G, r) := \{\tau_1^{\varepsilon_1}(G, r)\}, \quad \mathcal{T}_2(G, r) := \{\tau_1^{\varepsilon_2}(G, r), \tau_2^{\varepsilon_1}(G, r)\}, \ldots$$

as follows:

- $\tau_1^{\varepsilon_1}(G, r)$ is some element of $N_{\varepsilon_1}(t_1(G, r))$ such that $\tau_1^{\varepsilon_1}(G, r) = 0$ on $[0, a - \varepsilon_1]$ whenever $t_1(G, r) = 0$ on $[0, a]$ [25];
- for $n \in \{2, 3, \ldots\}$ and $k \in \{1, \ldots, n\}$, $\tau_k^{\varepsilon_{n-k+1}}(G, r)$ is some element of $N_{\varepsilon_{n-k+1}}(t_k(G, r))$ such that
  - $\tau_k^{\varepsilon_{n-k+1}}(G, r) = 0$ on $[0, a - \varepsilon_{n-k+1}]$ whenever $t_k(G, r) = 0$ on $[0, a]$;
  - $\tau_k^{\varepsilon_{n-k+1}}(G, r)(x) \neq \tau_l^{\varepsilon_{m-l+1}}(G, r)(x)$ if $\tau_k^{\varepsilon_{n-k+1}}(G, r)(x) > 0$ for each $x \in \text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$, every $l \in \{1, \ldots, k-1\}$, and any $m \in \{1, \ldots, n-1\}$.

  Set

$$\mathcal{Q}_{(G,r)} := \bigcup_{n=1}^{\infty} \mathcal{T}_n(G, r).$$

6.2.2. Showing that $\mathcal{Q}_{(G,r)}$ is a strategic approximation to $\mathfrak{g}_{(G,r)}$

Lemma 5. *Let $(F, r) \in \mathcal{E}^*$ and $G \in \mathcal{G}_{(F,r)}$. For each $t \in \mathcal{T}_{(G,r)}$, $\varepsilon > 0$, and every neighborhood $\mathcal{K}$ of t, there exists a $\tau \in \mathcal{Q}_{(G,r)} \cap \mathcal{K}$ such that, for any $0 \leqslant a \leqslant 1$, $t|_{[0,a]} = 0$ implies $\tau|_{[0,a-\varepsilon]} = 0$.*

Proof. Take any $t \in \mathcal{T}_{(G,r)}$, $\varepsilon > 0$, and a neighborhood $\mathcal{K}$ of t. Pick any $0 \leqslant a \leqslant 1$ such that $t|_{[0,a]} = 0$.

Observe that we may choose a rational number q in $(a-\varepsilon, a)$ such that $\mathcal{T}^q_{(G,r)} \cap \mathcal{K} \neq \emptyset$. Because $\mathcal{T}^q_{(G,r)} \cap \mathcal{K} \neq \emptyset$, and $\tilde{\mathcal{T}}^q_{(G,r)}$ is dense in $\mathcal{T}^q_{(G,r)}$, we can pick an f from $\tilde{\mathcal{T}}^q_{(G,r)} \cap \mathcal{K}$. Since $f \in \mathcal{K}$, there is an open neighborhood $\mathcal{O}$ of f with $\mathcal{O} \subseteq \mathcal{K}$. Because $f \in \{t_1(G, r), t_2(G, r), \ldots\}$ (recall that we enumerated the set $\mathcal{T}^*_{(G,r)}$ as $\{t_1(G, r), t_2(G, r), \ldots\}$ in the previous subsection), there is some integer k with $t_k(G, r) = f$. Let η be the diameter of $\mathcal{O}$. Then, for n sufficiently large,

$$0 < \varepsilon_n < \min\{\tfrac{\eta}{2}, q - (a - \varepsilon)\}$$

(where (ε_n) is the sequence of positive integers defined in the previous subsection, after the definition of $\mathcal{T}^*_{(G,r)}$), and $\tau_k^{\varepsilon_n}(G, r)$ is an element of $N_{\varepsilon_n}(t_k(G, r))$ with $\tau_k^{\varepsilon_n}(G, r) = 0$ on $[0, q-\varepsilon_n]$.

[25] In what follows, given any $(F, r) \in \mathcal{E}$, $t \in \mathcal{T}_{(F,r)}$, and $\delta > 0$, $N_\delta(t)$ stands for the open δ-neighborhood of t in $\mathcal{T}_{(F,r)}$.

It follows that $\tau_k^{\varepsilon_n}(G,r) \in \mathcal{K}$ and $\tau_k^{\varepsilon_n}(G,r) = 0$ on $[0, a-\varepsilon]$. Since $\tau_k^{\varepsilon_n}(G,r) \in \mathcal{Q}_{(G,r)}$, the proof is complete. $\square$

Lemma 6. *Let $(F,r) \in \mathcal{E}^*$, $G \in \mathcal{G}_{(F,r)}$, and $p \in \mathbb{P}(\mathcal{T}_{(G,r)})$. Then, for any $\varepsilon > 0$ and $\tau \in \mathcal{T}_{(G,r)}$, there exists a $\tau^* \in \mathcal{Q}_{(G,r)}$ such that*

$$\liminf \int_{\mathcal{T}_{(G,r)}} (w(\tau^*,\cdot) - w(\cdot,\tau^*))\, dp_n \geqslant \int_{\mathcal{T}_{(G,r)}} (w(\tau,\cdot) - w(\cdot,\tau))\, dp - \varepsilon$$

for every sequence (p_n) in $\mathbb{P}(\mathcal{T}_{(G,r)})$ that converges weakly to p.

Proof. The argument given in the proof of Lemma 3 can be used to prove this lemma. One needs only to observe that the analogue of x^* must lie in $[x_{(F,r)}, 1]$, and that the analogue of t_{ε^*} may actually be chosen from $\mathcal{Q}_{(G,r)}$ by means of Lemma 5. $\square$

Definition 3 (*Reny, 1996*). A subset $\mathcal{A}$ of $\mathcal{T}_{(G,r)}$ *ensures local payoff security of U_i on $\mathbb{P}(\mathcal{T}_{(G,r)})^2$* if, for each $(\mu_1,\mu_2) \in \mathbb{P}(\mathcal{T}_{(G,r)})^2$ and every $\varepsilon > 0$, there exists $v_i \in \mathbb{P}(\mathcal{T}_{(G,r)})$ with $v_i(\mathcal{A}) = 1$ such that $U_i(v_i, \tilde{\mu}_{-i}) \geqslant U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

Lemma 7. *Let $(F,r) \in \mathcal{E}^*$ and $G \in \mathcal{G}_{(F,r)}$. Then, the set $\mathcal{Q}_{(G,r)}$ ensures local payoff security of U_i on $\mathbb{P}(\mathcal{T}_{(G,r)})^2$ for each $i = 1, 2$.*

Proof. Fix any $i \in \{1,2\}$, $\mu = (\mu_1,\mu_2) \in \mathbb{P}(\mathcal{T}_{(G,r)})^2$, and $\varepsilon > 0$. We have to show that there exists a $v_i \in \mathbb{P}(\mathcal{T}_{(G,r)})$ such that $v_i(\mathcal{Q}_{(G,r)}) = 1$ and $U_i(v_i, \tilde{\mu}_{-i}) \geqslant U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

It is clear that there exists $\tau \in \mathcal{T}_{(G,r)}$ such that

$$U_i(\tau, \mu_{-i}) > U_i(\mu) - \tfrac{\varepsilon}{4}. \tag{14}$$

By Lemma 6, there exists a $\tau^* \in \mathcal{Q}_{(G,r)}$ such that every sequence (μ_{-i}^n) in $\mathbb{P}(\mathcal{T}_{(G,r)})$ that converges weakly to μ_{-i} satisfies

$$\liminf \int_{\mathcal{T}_{(G,r)}} \left(w(\tau^*,\cdot) - w(\cdot,\tau^*)\right) d\mu_{-i}^n \geqslant \int_{\mathcal{T}_{(G,r)}} (w(\tau,\cdot) - w(\cdot,\tau))\, d\mu_{-i} - \tfrac{\varepsilon}{2}.$$

Hence, $U_i(\tau^*, \tilde{\mu}_{-i}) \geqslant U_i(\tau, \mu_{-i}) - \frac{3\varepsilon}{4}$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}. From these inequalities and (14), we see that $U_i(\tau^*, \tilde{\mu}_{-i}) \geqslant U_i(\mu) - \varepsilon$ for every $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}. $\square$

Lemma 8. *Let $(F,r) \in \mathcal{E}^*$ and $G \in \mathcal{G}_{(F,r)}$. $\mathcal{Q}$ is a strategic approximation to $\mathfrak{g}_{(G,r)}$ if $\mathcal{Q}$ is a countable subset of $\mathcal{T}_{(G,r)}$ that ensures local payoff security of U_i on $\mathbb{P}(\mathcal{T}_{(G,r)})^2$ for each i.*

Proof. Suppose that $\mathcal{Q}$ is a countable subset of $\mathcal{T}_{(G,r)}$ that ensures local payoff security of U_i on $\mathbb{P}(\mathcal{T}_{(G,r)})^2$ for each $i = 1, 2$. The set $\mathcal{T}_{(G,r)}$ is a compact metric space (Lemma 2). Further, $u_1 + u_2$ is obviously continuous—it is constant—on $\mathbb{P}(\mathcal{T}_{(G,r)})^2$. It follows from Theorem 4 of [27] that $\mathcal{Q}$ is a strategic approximation to $\mathfrak{g}_{(G,r)}$. $\square$

Lemma 9. *For any $(F,r) \in \mathcal{E}^*$ and $G \in \mathcal{G}_{(F,r)}$, $\mathcal{Q}_{(G,r)}$ is a strategic approximation to $\mathfrak{g}_{(G,r)}$.*

Proof. Apply Lemmata 7 and 8. $\square$

6.2.3. Characterization of an equilibrium of $\mathfrak{g}_{(G,r)}$ via $\mathcal{Q}_{(G,r)}$

Lemma 10. *Let $(F, r) \in \mathcal{E}^*$. Then, any $G \in \mathcal{G}_{(F,r)}$ sufficiently close to F satisfies $u_i(\tau, t) > 0$, $i = 1, 2$, for some $\tau \in \mathcal{T}_{(G,r)}$ and all $t \in \mathcal{T}_{(G,r)}$ with $t = 0$ on* $\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$.

Proof. Because the set of probability measures on [0, 1] with finite support is dense in $\mathcal{F}$, any $G \in \mathcal{G}_{(F,r)}$ sufficiently close to *F* has

$$b := \max\left\{\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})\right\}$$

and

$$a := \max\left\{\text{supp}\{\mathbf{p}_G\} \cap [0, b)\right\}$$

sufficiently close to $x_{(F,r)}$, and $\mathbf{p}_G\{b\} < \frac{1}{2}\mathbf{p}_G\{[\frac{1}{2}x_{(F,r)} + \frac{1}{2}, 1]\}$. For *G* with *a* and *b* close to $x_{(F,r)}$, $u_i(\tau, t_0) > 0$, $i = 1, 2$, for some $\tau \in \mathcal{T}_{(G,r)}$, where $t_0|_{[0,x_{(F,r)})} := 0$, and $t_0|_{[x_{(F,r)},1]} := x - x_{(F,r)}$.[26] But the definition of $x_{(F,r)}$ implies that every $t \in \mathcal{T}_{(G,r)}$ with $t = 0$ on $\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$ has $t = t_0$ on $[x_{(F,r)}, 1]$, and so $u_i(\tau, t) > 0$, $i = 1, 2$, for some $\tau \in \mathcal{T}_{(G,r)}$ and all $t \in \mathcal{T}_{(G,r)}$ with $t = 0$ on $\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$. □

Lemma 11. *Take any $(F, r) \in \mathcal{E}^*$ with $x_{(F,r)} < m_F$. Let $G \in \mathcal{G}_{(F,r)}$ be sufficiently close to F in the sense of Lemma 10. If $\mu = (\mu_1, \mu_2)$ is a Nash equilibrium of $\mathfrak{g}_{(G,r)}$* with $\mu^n = (\mu_1^n, \mu_2^n) \to \mu$ *for some sequence (μ^n), where each μ^n is a Nash equilibrium of $(\mathcal{A}_n, (u_1, u_2))$ and $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\mathcal{Q}_{(G,r)}$, then $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) < 1$ for each $i = 1, 2$.*

The proof of this result is fairly technical. To draw the big picture clearly, therefore, we relegate its proof to Section 6.2.5, and proceed directly to the proof of Theorem 2.

6.2.4. Completion of the proof of Theorem 2

Fix any $(F, r) \in \mathcal{E}^*$ with $x_{(F,r)} < m_F$. Take any $G \in \mathcal{G}_{(F,r)}$ sufficiently close to *F* in the sense of Lemma 10. Let $(\mathcal{A}_n)$ be an increasing sequence of finite sets whose union is $\mathcal{Q}_{(G,r)}$. For each n, there exists a Nash equilibrium (μ_1^n, μ_2^n) of $\mathfrak{g}_{(G,r)}(\mathcal{A}_n)$.

The set $\mathcal{Q}_{(G,r)}$ is a strategic approximation to $\mathfrak{g}_{(G,r)}$ (Lemma 9), and hence, any limit of equilibria of the sequence of finite games $(\mathcal{A}_n, (u_1, u_2))$ is an equilibrium of $\mathfrak{g}_{(G,r)}$. Since $\mathbb{P}(\mathcal{T}_{(G,r)})^2$ is a compact metric space, there exists a subsequence of (μ_1^n, μ_2^n) that converges to some (μ_1, μ_2) in $\mathbb{P}(\mathcal{T}_{(G,r)})^2$. Consequently, (μ_1, μ_2) is a Nash equilibrium of $\mathfrak{g}_{(G,r)}$. Applying Lemma 11, therefore, completes the proof of Theorem 2.

[26] In fact, one can take

$$\tau(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant a, \\ x - a & \text{if } a < x \leqslant b, \\ b - a & \text{if } b < x \leqslant x_{(F,r)} + \varepsilon, \\ x - (x_{(F,r)} + \varepsilon) & \text{if } x_{(F,r)} + \varepsilon < x, \end{cases}$$

where is $\varepsilon > 0$ is small enough to ensure that $\int_0^1 \tau\, dF \geqslant r$. This function ties with t_0 on $[0, a]$, loses against t_0 on $(a, x_{(F,r)} + \varepsilon]$, and defeats t_0 on $(x_{(F,r)} + \varepsilon, 1]$. When *a* is close enough to $x_{(F,r)}$, τ defeats t_0 because $\mathbf{p}_G\{b\} < \frac{1}{2}\mathbf{p}_G\{[\frac{1}{2}x_{(F,r)} + \frac{1}{2}, 1]\}$.

6.2.5. Proof of Lemma 11

Let $\mu = (\mu_1, \mu_2)$ be a Nash equilibrium of $\mathfrak{g}_{(G,r)}$ with $\mu^n = (\mu_1^n, \mu_2^n) \to \mu$ for some sequence (μ^n), where each μ^n is a Nash equilibrium of $(\mathcal{A}_n, (u_1, u_2))$, and $(\mathcal{A}_n)$ is an increasing sequence of finite sets whose union is $\mathcal{Q}_{(G,r)}$. Suppose that $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$ for some i, say, for $i = 2$. It is then without loss of generality to assume $\text{supp}\{\mu_2\} \subseteq \widehat{\mathcal{T}}^{prog}_{(G,r)}$. We wish to derive a contradiction.

Define the map $\bar{t} : [0, 1] \to \mathbf{R}$ by

$$\bar{t}(x) := \sup \left\{ t(x) : t \in \text{supp}\{\mu_2\} \right\}.$$

A contradiction is easily obtained if $\bar{t} = 0$ on $\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$, for then Lemma 10 gives a $\tau \in \mathcal{T}_{(G,r)}$ such that $u_1(\tau, t) > 0$ for every $t \in \text{supp}\{\mu_2\}$, while this contradicts μ being a Nash equilibrium of $\mathfrak{g}_{(G,r)}$.

Throughout the rest of the argument, we assume that $\bar{t} > 0$ for some point in $\text{supp}\{\mathbf{p}_G\} \cap [0, x_{(F,r)})$. Let

$$y := \min\{x : x \in \text{supp}\{\mathbf{p}_G\} \text{ and } \bar{t}(x) > 0\}.$$

In the sequel we use the following notation:

$$Y^- := [0, y) \cap \text{supp}\{\mathbf{p}_G\},$$
$$Y^+ := (y, 1] \cap \text{supp}\{\mathbf{p}_G\}.$$

We proceed by means of three auxiliary assertions.

Claim 1. *For each $\delta > 0$ and $\varepsilon > 0$, there exists an integer N such that*

$$\mu_2^n(\text{cl}(N_\delta(\text{supp}\{\mu_2\})) > 1 - \varepsilon \quad \textit{for all } n \geqslant N.$$[27]

Proof. Fix any $\delta, \varepsilon > 0$. Because $\mu_2^n \to \mu_2$ and $N_\delta(\text{supp}\{\mu_2\})$ is open, we have

$$\liminf \mu_2^n(N_\delta(\text{supp}\{\mu_2\})) \geqslant \mu_2(N_\delta(\text{supp}\{\mu_2\})) = 1.$$

It follows that there exists an N such that

$$\mu_2^n(\text{cl}\, N_\delta(\text{supp}\{\mu_2\})) \geqslant \mu_2^n(N_\delta(\text{supp}\{\mu_2\})) > 1 - \varepsilon$$

for all $n \geqslant N$. $\square$

Claim 2. *There exists an $\alpha > 0$ such that, for any sequence (f_n) with*

$$f_n \in \arg\max \left\{ t(y) : t \in \text{supp}\{\mu_2^n\} \cap \text{cl}\, N_\alpha \left(\arg\max \left\{ h(y) : h \in \text{supp}\{\mu_2\} \right\} \right) \right\},$$[28]

we can find (1) $\varepsilon, \gamma > 0$, (2) *a subsequence* (n_k) *of* (n), *and* (3) *a natural number K such that the following is true*: *For every $k \geqslant K$, there exist $f_{n_k}^* \in \mathcal{T}_{(G,r)}$ and $\mathcal{B}_{n_k} \subseteq \mathcal{T}_{(G,r)}$ such that*

(i) $\mu_2^{n_k}(\mathcal{B}_{n_k}) > \gamma$,
(ii) $f_{n_k}^*|_{Y^+} < f_{n_k}|_{Y^+} - \varepsilon$,
(iii) $f_{n_k}^*|_{Y^-} = 0$, *and*

[27] $N_\delta(\text{supp}\{\mu_2\}) := \bigcup_{t \in \text{supp}\{\mu_2\}} N_\delta(t)$.

[28] Since $\mu_2^n \to \mu_2$, $\text{supp}\{\mu_2^n\} \cap \text{cl}\, N_\alpha \left(\arg\max \left\{ t(y) : t \in \text{supp}\{\mu_2\} \right\} \right)$ is non-empty for sufficiently large n.

(iv) $u_1(\tilde{f}_{n_k}, t) > u_1(f_{n_k}, t) + \varepsilon$ *for all* $\tilde{f}_{n_k} \in N_\beta(f^*_{n_k})$ *that vanish on* Y^-, *all* $t \in \mathcal{B}_{n_k}$, *and some* $\beta > 0$.

Proof. Consider the set of tax functions

$$\arg\max \left\{t(y) : t \in \operatorname{supp}\{\mu_2\}\right\},$$

and denote by $\overline{\tau}$ and $\underline{\tau}$ its upper and lower envelopes:

$$\overline{\tau} := \sup \left(\arg\max \left\{t(y) : t \in \operatorname{supp}\{\mu_2\}\right\}\right)$$

and

$$\underline{\tau} := \inf \left(\arg\max \left\{t(y) : t \in \operatorname{supp}\{\mu_2\}\right\}\right).$$

Set $y^+ := \min\left\{x : x \in Y^+\right\}$ and $y^- := \max\left\{x : x \in Y^-\right\}$.

We must have $\overline{\tau}(y) < y - y^-$ and

$$\overline{\tau}(y) < \underline{\tau}(y^+) \leqslant \overline{\tau}(y^+) < \overline{\tau}(y) + y^+ - y.$$

This is true because $\mu_2(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$ and $y < x_{(F,r)}$ hold. In fact, if $\overline{\tau}(y) \geqslant y - y^-$ were true, a policy $g \in \operatorname{supp}\{\mu_2\}$ with $g(y^-) = 0$ and $g(y) = \overline{\tau}(y)$, and therefore with $g(y) \geqslant y - y^-$, would exist. But g, being a tax function, could only satisfy $g(y) \geqslant y - y^-$ if $g(y) = y - y^-$. On the other hand, g would need to be convex (when restricted to $\operatorname{supp}\{\mathbf{p}_G\}$), because $\mu_i(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$. This, along with the fact that $g(y) = y - y^-$, would imply that a lower envelope for g would be the map

$$\underline{g}(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant y^-, \\ x - y^- & \text{otherwise.} \end{cases}$$

But since $y < x_{(F,r)}$, this would imply $\int_0^1 g\,dF > r$, thereby contradicting that a player will never randomize over a subset of (admissible) tax functions collecting more than r. Hence $\overline{\tau}(y) < y - y^-$.

To see that $\overline{\tau}(y) < \underline{\tau}(y^+)$, observe that $\overline{\tau}(y) = \underline{\tau}(y)$. This implies $\overline{\tau}(y) \leqslant \underline{\tau}(y^+)$ (because $\underline{\tau}(y) \leqslant \underline{\tau}(y^+)$, which follows from $y \leqslant y^+$ and the fact that admissible tax policies are nondecreasing). But if $\overline{\tau}(y) = \underline{\tau}(y^+)$ were true then the set $\arg\max\left\{t(y) : t \in \operatorname{supp}\{\mu_2\}\right\}$ would contain a nonconvex function, thereby contradicting $\mu_2(\widehat{\mathcal{T}}^{prog}_{(G,r)}) = 1$. Therefore, $\overline{\tau}(y) < \underline{\tau}(y^+)$. A similar argument demonstrates that $\overline{\tau}(y^+) < \overline{\tau}(y) + y^+ - y$ as well. The inequality $\underline{\tau}(y^+) \leqslant \overline{\tau}(y^+)$ is obvious.

Take two numbers m and α with $m > \max\left\{2, \frac{1+\mathbf{p}_G\{[0,y)\}}{\mathbf{p}_G\{y\}}\right\}$ and

$$0 < \alpha < \frac{\min\{y - y^- - \overline{\tau}(y), \underline{\tau}(y^+) - \overline{\tau}(y), \overline{\tau}(y) + y^+ - y - \overline{\tau}(y^+), \overline{\tau}(y)\}}{m+2}. \tag{15}$$

For each n, take any

$$f_n \in \arg\max\{t(y) : t \in \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha(\arg\max\{h(y) : h \in \operatorname{supp}\{\mu_2\}\})\} \tag{16}$$

(see footnote 28). Now pick a number η with $0 < \eta < \min\left\{\alpha, \frac{1}{2}(1 - x_{(F,r)})\right\}$, and let

$$\varepsilon_{\eta/4} := \min\left\{G\left(a + \tfrac{\eta}{4}\right) - G(a) : x_{(F,r)} \leqslant a \leqslant 1 - \tfrac{\eta}{4}\right\}. \tag{17}$$

Since G is strictly increasing on $[x_{(F,r)}, 1]$, we have $\varepsilon_{\eta/4} > 0$. Now take any number ε with

$$0 < \varepsilon < \tfrac{1}{2} \min \left\{ \eta, \varepsilon_{\eta/4} \right\}. \tag{18}$$

For each n, define the map $f_n^* : [0, 1] \to \mathbf{R}$ by

$$f_n^*(x) := \begin{cases} 0 & \text{if } 0 \leqslant x < y^-, \\ \frac{\overline{\tau}(y)+m\alpha}{y-y^-}(x - y^-) & \text{if } y^- \leqslant x < y, \\ \overline{\tau}(y) + m\alpha + \frac{f_n(y^+)-\eta-\overline{\tau}(y)-m\alpha}{y^+-y}(x - y) & \text{if } y \leqslant x < y^+, \\ f_n(x) - \eta & \text{if } y^+ \leqslant x \leqslant 1, \end{cases}$$

and the sets

$$\begin{aligned} \mathcal{S}_n &:= \left\{ t \in \operatorname{supp}\{\mu_2^n\} : t \text{ intersects with } \tfrac{1}{2} f_n^* + \tfrac{1}{2} f_n \text{ in } [x_{(F,r)}, 1] \right\}, \\ \overline{\mathcal{S}}_n &:= \left\{ t \in \operatorname{supp}\{\mu_2^n\} : t > \tfrac{1}{2} f_n^* + \tfrac{1}{2} f_n \text{ on } [x_{(F,r)}, 1] \right\}, \\ \underline{\mathcal{S}}_n &:= \left\{ t \in \operatorname{supp}\{\mu_2^n\} : t < \tfrac{1}{2} f_n^* + \tfrac{1}{2} f_n \text{ on } [x_{(F,r)}, 1] \right\}, \\ \mathcal{O}_n &:= \left\{ t \in \operatorname{supp}\{\mu_2^n\} : t(y) > f_n(y) \right\}. \end{aligned}$$

Observe that $f_n^* \in \mathcal{T}_{(G,r)}$ for all n. That f_n^* is a tax function follows from the following facts. First, the definition of α entails $\frac{\overline{\tau}(y)+m\alpha}{y-y^-} \leqslant 1$. Second,

$$f_n(y^+) - \eta - \overline{\tau}(y) - m\alpha \geqslant \underline{\tau}(y^+) - \alpha - \eta - \overline{\tau}(y) - m\alpha \geqslant 0$$

if $\underline{\tau}(y^+) - \overline{\tau}(y) \geqslant (2 + m)\alpha$, which holds true by the definition of α. Third, $\frac{f_n(y^+)-\eta-\overline{\tau}(y)-m\alpha}{y^+-y} \leqslant 1$ if

$$(1 - m)\alpha - \eta \leqslant y^+ - y + \overline{\tau}(y) - \overline{\tau}(y^+),$$

which is true by the choice of α and η. To see that f_n^* is an admissible tax function, observe that

$$\int_0^1 f_n^* \, dG \geqslant \int_0^1 f_n \, dG \geqslant r$$

if $\alpha \left(\mathbf{p}_G\{y\}(m - 1) - \mathbf{p}_G\{[0, y)\} \right) \geqslant \eta \left(1 - \mathbf{p}_G\{y\} \right)$, while the latter holds true by the choice of m, α, and η. Since $f_n^* \in \mathcal{T}_{(G,r)}$ for all n and $\mathcal{T}_{(G,r)}$ is compact (Lemma 2), we may assume, passing to a subsequence if necessary, $f_n^* \to f^*$ for some $f^* \in \mathcal{T}_{(G,r)}$.

We next show that the sequence $(\mu_2^n(\mathcal{O}_n))$ contains a subsequence, which we denote again by $(\mu_2^n(\mathcal{O}_n))$, such that $\mu_2^n(\mathcal{O}_n) \to 0$. We consider two cases. First, suppose that it is possible to write, passing to a subsequence if necessary, $f_n(y) \leqslant \overline{\tau}(y)$ for all n. This, along with the definition of f_n in (16), implies $f_n \in \arg\max\{t(y) : t \in \operatorname{supp}\{\mu_2^n\}\}$ for all n. But, in this case, $\mathcal{O}_n = \emptyset$ for all n, so $\mu_2^n(\mathcal{O}_n) = 0$ for all n. Next, suppose that no subsequence (f_{n_k}) of (f_n) has $f_{n_k}(y) \leqslant \overline{\tau}(y)$ for all k. In this case, we may write, passing to a subsequence if necessary, $f_n(y) > \overline{\tau}(y)$ for all n. Pick a sequence (δ_n) with $\delta_n \in (0, f_n(y) - \overline{\tau}(y))$ for all n. Claim 1 gives, passing to a subsequence if necessary, $\mu_2^n(\operatorname{cl} N_{\delta_n}(\operatorname{supp}\{\mu_2\})) \to 1$, so

$$\mu_2^n(\mathcal{T}_{(G,r)} \setminus \operatorname{cl} N_{\delta_n}(\operatorname{supp}\{\mu_2\})) \to 0. \tag{19}$$

For any $h \in \mathrm{cl}\, N_{\delta_n}(\mathrm{supp}\{\mu_2\})$, we must have

$$h(y) \leqslant \overline{\tau}(y) + \delta_n < f_n(y)$$

(where the last inequality follows from the definition of δ_n). But then, for any h, $h \in \mathrm{cl}\, N_{\delta_n}(\mathrm{supp}\{\mu_2\})$ implies $h \notin \mathcal{O}_n$, so $\mathcal{O}_n \subseteq \mathcal{T}_{(G,r)} \setminus \mathrm{cl}\, N_{\delta_n}(\mathrm{supp}\{\mu_2\})$. Combine this inclusion with (19) to obtain $\mu_2^n(\mathcal{O}_n) \to 0$, as desired.

Because $x_{(F,r)} < m_F$, and $t < f_n$ on $[x_{(F,r)}, 1]$ for all $t \in \underline{\mathcal{S}}_n$ and n, there exists a $\theta_1 > 0$ such that $u_1(f_n, t) < -\theta_1$ for all $t \in \underline{\mathcal{S}}_n$ and n. On the other hand, since $x_{(F,r)} < m_F$ and $\tau < t$ on $[x_{(F,r)}, 1]$ whenever $(\tau, t) \in \underline{\mathcal{S}}_n \times \overline{\mathcal{S}}_n$, there exists a $\theta_2 > 0$ such that $u_1(\tau, t) > \theta_2$ for all $(\tau, t) \in \underline{\mathcal{S}}_n \times \overline{\mathcal{S}}_n$ and n. Moreover, since $x_{(F,r)} < m_F$ and $f_n^* < \frac{1}{2} f_n^* + \frac{1}{2} f_n < t$ on $[x_{(F,r)}, 1]$ whenever $t \in \overline{\mathcal{S}}_n$, there exists a $\theta_3 > 0$ such that $u_1(\tau, t) > \theta_3$ for all

$$(\tau, t) \in \{h \in N_{\theta_3}(f_n^*) : h|_{Y^-} = 0\} \times \overline{\mathcal{S}}_n$$

and n.

Taking $0 < \theta < \min\{\theta_1, \theta_2, \theta_3\}$, we conclude that $u_1(f_n, t) < -\theta$ for all $t \in \underline{\mathcal{S}}_n$ and all n, and $u_1(\tau, t) > \theta$ for all

$$(\tau, t) \in \left(\left\{h \in N_\theta(f_n^*) : h|_{Y^-} = 0\right\} \cup \underline{\mathcal{S}}_n\right) \times \overline{\mathcal{S}}_n,$$

$n = 1, 2, \ldots$.

Since $f_n^* \to f^*$, there exists an integer L such that $f_n^* \in N_{\theta/2}(f^*)$ for all $n \geqslant L$. Therefore, $u_1(\tau, t) > \theta$ for each

$$(\tau, t) \in \{\tilde{t} \in N_{\theta/2}(f^*) : \tilde{t}|_{Y^-} = 0\} \times \overline{\mathcal{S}}_n$$

and $n \geqslant L$. Now Lemma 5 gives an $f^\circ \in \mathcal{Q}_{(G,r)}$ with $u_1(f^\circ, t) > \theta$ for all $t \in \overline{\mathcal{S}}_n$ and $n \geqslant L$.

We will now consider three cases, the first two ones leading to a contradiction: (a) $\mu_2^n(\mathcal{S}_n) \to 0$ and $\mu_2^n(\overline{\mathcal{S}}_n) \to 0$; (b) $\mu_2^n(\mathcal{S}_n) \to 0$ and $\mu_2^n(\overline{\mathcal{S}}_n)$ does not converge to 0; (c) $\mu_2^n(\mathcal{S}_n)$ does not converge to 0. Since the list is exhaustive, if (a) and (b) are not possible, we must have (c).

If $\mu_2^n(\mathcal{S}_n) \to 0$ and $\mu_2^n(\overline{\mathcal{S}}_n) \to 0$, then $\mu_2^n(\underline{\mathcal{S}}_n) \to 1$, and hence

$$\begin{aligned}
U_1(f_n, \mu_2^n) &= \sum_{t \in \underline{\mathcal{S}}_n} \mu_2^n(t) u_1(f_n, t) + \sum_{t \in \mathcal{S}_n \cup \overline{\mathcal{S}}_n} \mu_2^n(t) u_1(f_n, t) \\
&\leqslant \mu_2^n(\underline{\mathcal{S}}_n)\left(\max_{t \in \underline{\mathcal{S}}_n} u_1(f_n, t)\right) + \mu_2^n\left(\mathcal{S}_n \cup \overline{\mathcal{S}}_n\right)\left(\max_{t \in \mathcal{S}_n \cup \overline{\mathcal{S}}_n} u_1(f_n, t)\right) \\
&< -\theta \mu_2^n(\underline{\mathcal{S}}_n) + \mu_2^n\left(\mathcal{S}_n \cup \overline{\mathcal{S}}_n\right)\left(\max_{t \in \mathcal{S}_n \cup \overline{\mathcal{S}}_n} u_1(f_n, t)\right) \\
&\to -\theta \\
&< 0,
\end{aligned}$$

thereby contradicting that (μ_2^n, μ_2^n) is a Nash equilibrium of $(\mathcal{A}_n, (u_1, u_2))$.

If $\mu_2^n(\mathcal{S}_n) \to 0$ but $\mu_2^n(\overline{\mathcal{S}}_n)$ does not converge to 0, then there exists a subsequence $\left(\mu_2^{n_k}(\overline{\mathcal{S}}_{n_k})\right)$ with $\mu_2^{n_k}(\overline{\mathcal{S}}_{n_k}) \to \rho$ for some $\rho > 0$. Without loss of generality, let $\left(\mu_2^n(\overline{\mathcal{S}}_n)\right)$ be one such subsequence.

If $\rho = 1$, then

$$\begin{aligned} U_1(f^\circ, \mu_2^n) &= \sum_{t \in \overline{\mathcal{S}}_n} \mu_2^n(t) u_1(f^\circ, t) + \sum_{t \in \mathcal{A}_n \setminus \overline{\mathcal{S}}_n} \mu_2^n(t) u_1(f^\circ, t) \\ &\to \theta \\ &> 0. \end{aligned}$$

But since $(\mathcal{A}_n)$ is an increasing sequence of sets whose union is $\mathcal{Q}_{(G,r)}$, and $f^\circ \in \mathcal{Q}_{(G,r)}$, there exists an integer l with $f^\circ \in \mathcal{A}_l$ and $U_1(f^\circ, \mu_2^l) > 0$, contradicting that (μ_2^l, μ_2^l) is a Nash equilibrium of $(\mathcal{A}_l, (u_1, u_2))$.

If, on the other hand, $0 < \rho < 1$, then we may write (passing to a subsequence if necessary) $\mu_2^n(\underline{\mathcal{S}}_n) \to \varrho$ for some $\varrho > 0$. Define, for each n, the probability measure $\tilde{\mu}_2^n$ on $\mathcal{A}_n$ as follows:

$$\tilde{\mu}_2^n(\mathcal{B}) := \tfrac{\mu_2^n(\mathcal{B} \cap \underline{\mathcal{S}}_n)}{\mu_2^n(\underline{\mathcal{S}}_n)}.$$

Then,

$$\begin{aligned} & U_1(\tilde{\mu}_2^n, \mu_2^n) \\ &= \sum_{\tau \in \mathcal{A}_n} \tilde{\mu}_2^n(\tau) \left(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \overline{\mathcal{S}}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n} \mu_2^n(t) u_1(\tau, t) \right) \\ &= \sum_{\tau \in \underline{\mathcal{S}}_n} \tilde{\mu}_2^n(\tau) \left(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \overline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \right. \\ &\quad \left. + \sum_{t \in \overline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \right) \\ &= \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \sum_{t \in \mathcal{A}_n} \mu_2^n(\tau) \mu_2^n(t) u_1(\tau, t) \\ &\quad + \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(\tau) \left(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \overline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \right. \\ &\quad \left. + \sum_{t \in \overline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \right) \\ &> \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \sum_{t \in \mathcal{A}_n} \mu_2^n(\tau) \mu_2^n(t) u_1(\tau, t) \\ &\quad + \frac{\theta}{\mu_2^n(\underline{\mathcal{S}}_n)} \mu_2^n \left(\underline{\mathcal{S}}_n \setminus \mathcal{O}_n \right) \mu_2^n \left(\overline{\mathcal{S}}_n \setminus \mathcal{O}_n \right) \\ &\quad + \frac{1}{\mu_2^n(\underline{\mathcal{S}}_n)} \sum_{\tau \in \underline{\mathcal{S}}_n \setminus \mathcal{O}_n} \mu_2^n(\tau) \left(\sum_{t \in \mathcal{S}_n} \mu_2^n(t) u_1(\tau, t) \right. \\ &\quad \left. + \sum_{t \in \overline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) + \sum_{t \in \underline{\mathcal{S}}_n \cap \mathcal{O}_n} \mu_2^n(t) u_1(\tau, t) \right) \\ &\to \frac{\theta \varrho \rho}{\varrho} \\ &> 0, \end{aligned}$$

because $\mu_2^n(\mathcal{S}_n) \to 0$, $\mu_2^n(\mathcal{O}_n) \to 0$, $\mu_2^n(\underline{\mathcal{S}}_n \setminus \mathcal{O}_n) \to \varrho$, and $\mu_2^n(\overline{\mathcal{S}}_n \setminus \mathcal{O}_n) \to \rho$. Thus, a contradiction obtains in this case as well.

We conclude that $\mu_2^n(\mathcal{S}_n)$ cannot converge to 0. Thus, there exists a subsequence $\left(\mu_2^{n_k}(\mathcal{S}_{n_k})\right)$ such that $\mu_2^{n_k}(\mathcal{S}_{n_k}) \to \lambda$ for some $\lambda > 0$. Without loss of generality, let $\left(\mu_2^n(\mathcal{S}_n)\right)$ be one such subsequence. Because $\mu_2^n(\mathcal{S}_n) \to \lambda$ and $\mu_2^n(\mathcal{O}_n) \to 0$, there exists an integer K and $\gamma > 0$ such that $\mu_2^n(\mathcal{S}_n \setminus \mathcal{O}_n) > \gamma$ for all $n \geqslant K$. Define $\mathcal{B}_n := \mathcal{S}_n \setminus \mathcal{O}_n$ for each n, and choose any $0 < \beta < \frac{\eta}{8}$.

Fix any $n \geqslant K$. We have $f_n^* \in \mathcal{T}_{(G,r)}$, $\mathcal{B}_n = \mathcal{S}_n \setminus \mathcal{O}_n \subseteq \mathcal{T}_{(G,r)}$, and, since $n \geqslant K$, $\mu_2^n(\mathcal{B}_n) > \gamma$. On the other hand, the definitions of ε and f_n^* entail $f_n^*|_{Y^-} = 0$ and $f_n^*|_{Y^+} < f_n|_{Y^+} - \varepsilon$. We have therefore established the statements (i)–(iii) (see Claim 2). It remains to show (iv), i.e., that $u_1(\tilde{f}_n, t) > u_1(f_n, t) + \varepsilon$ for all $\tilde{f}_n \in N_\beta(f_n^*)$ that vanish on Y^-, and all $t \in \mathcal{B}_n$ (β was chosen at the end of the previous paragraph).

Fix any $t \in \mathcal{B}_n$. The map t intersects with $\frac{1}{2} f_n^* + \frac{1}{2} f_n$ in $[x_{(F,r)}, 1]$ (because $t \in \mathcal{S}_n$), say, at z. We consider the case where $[z, z + \frac{\eta}{4}] \subseteq [x_{(F,r)}, 1]$ (if this were not the case, we would have $[z - \frac{\eta}{4}, z] \subseteq [x_{(F,r)}, 1]$ and a similar argument would work).

Let $I := [z, z + \frac{\eta}{4}]$ and fix any $\tilde{f}_n \in N_\beta(f_n^*)$ that vanishes on Y^-. Since $\tilde{f}_n \in N_\beta(f_n^*)$ and $0 < \beta < \frac{\eta}{8}$, we have $\tilde{f}_n < f_n^* + \frac{\eta}{8}$. Because $f_n^* = f_n - \eta$ on I, we have $f_n^* + \frac{\eta}{2} = \frac{1}{2} f_n^* + \frac{1}{2} f_n$ on I. On the other hand, the definition of z implies

$$t(z) = \left(\frac{1}{2} f_n^* + \frac{1}{2} f_n\right)(z). \tag{20}$$

Since t is nondecreasing, $t(z) \leqslant t|_I$, and finally, $f_n^*|_I \leqslant f_n^*(z) + \frac{\eta}{4}$ because the right-derivative of f_n^* is at most 1. So, we have, on I,

$$\begin{aligned}
\tilde{f}_n &< f_n^* + \tfrac{\eta}{8} \\
&\leqslant f_n^*(z) + \tfrac{\eta}{4} + \tfrac{\eta}{8} \\
&< f_n^*(z) + \tfrac{\eta}{2} \\
&= \tfrac{1}{2} f_n^*(z) + \tfrac{1}{2} f_n(z) \\
&= t(z) \\
&\leqslant t.
\end{aligned}$$

Therefore, $\tilde{f}_n|_I < t|_I$, and hence

$$\mathbf{p}_G\left\{\{\tilde{f}_n < t\} \cap I\right\} = \mathbf{p}_G\{I\} = G\left(z + \tfrac{\eta}{4}\right) - G(z).$$

Combine this with (17) and (18) to obtain

$$\mathbf{p}_G\left\{\{\tilde{f}_n < t\} \cap I\right\} \geqslant \varepsilon_{\eta/4} > \varepsilon. \tag{21}$$

Next, observe that, because $\tilde{f}_n|_{(y,1]} < f_n^*|_{(y,1]} + \frac{\eta}{8}$ and $f_n^*|_{(y,1]} = f_n|_{(y,1]} - \eta$, we have $\tilde{f}_n|_{(y,1]} < f_n|_{(y,1]}$. This, along with $\tilde{f}_n|_{Y^-} = 0$, implies

$$\tilde{f}_n \leqslant f_n \text{ on } \operatorname{supp}\{\mathbf{p}_G\} \setminus \{y\}. \tag{22}$$

Moreover, $t(y) \leqslant f_n(y)$ (since $t \in \mathcal{B}_n$ and so $t \notin \mathcal{O}_n$), and either $t(y) = f_n(y) = 0$ or $t(y) \neq f_n(y)$ (for $t, f_n \in \operatorname{supp}\{\mu_2^n\}$ and therefore $t, f_n \in \mathcal{Q}_{(G,r)}$, which implies, by the definition of $\mathcal{Q}_{(G,r)}$, $t(y) \neq f_n(y)$ whenever $t(y) \neq 0$ or $f_n(y) \neq 0$). It follows that

$$\text{either } t(y) = f_n(y) = 0 \quad \text{or} \quad t(y) < f_n(y). \tag{23}$$

Now, by the choice of f_n in (16),

$$\begin{aligned} f_n(y) &\geqslant \arg\max\{t(y) : t \in \operatorname{supp}\{\mu_2\}\} - \alpha \\ &= \overline{\tau}(y) - \alpha. \end{aligned}$$

But since the choice of α in (15) entails $\alpha < \overline{\tau}(y)$, we must have $f_n(y) > 0$. This, together with (23), gives $t(y) < f_n(y)$. Combining this inequality and (22), we find

$$\mathbf{p}_G\{\{\tilde{f}_n < t\} \cap I^c\} - \mathbf{p}_G\{t < \tilde{f}_n\} \geqslant \mathbf{p}_G\left\{\{f_n < t\} \cap I^c\right\} - \mathbf{p}_G\{t < f_n\}, \tag{24}$$

where I^c represents the complement of I in [0, 1].

Finally, we have

$$\begin{aligned} t|_I &\leqslant t(z) + \tfrac{\eta}{4} \\ &= \tfrac{1}{2} f_n^*(z) + \tfrac{1}{2} f_n(z) + \tfrac{\eta}{4} \\ &= \tfrac{1}{2}(f_n(z) - \eta) + \tfrac{1}{2} f_n(z) + \tfrac{\eta}{4} \\ &= f_n(z) - \tfrac{\eta}{4} \\ &\leqslant f_n(z) \\ &\leqslant f_n|_I, \end{aligned}$$

where the first inequality follows from the fact that the right-derivative of t is at most 1, the first equality follows from (20), the second equality follows from $f_n^*|_I = f_n|_I - \eta$, and the last inequality holds because f_n is nondecreasing. Hence $t|_I \leqslant f_n|_I$, and

$$\mathbf{p}_G\,\{\{f_n < t\} \cap I\} = 0. \tag{25}$$

Using (21), (24), and (25), we obtain

$$\begin{aligned} u_1(\tilde{f}_n, t) &= \mathbf{p}_G\{\tilde{f}_n < t\} - \mathbf{p}_G\{t < \tilde{f}_n\} \\ &= \mathbf{p}_G\left\{\{\tilde{f}_n < t\} \cap I\right\} + \mathbf{p}_G\left\{\{\tilde{f}_n < t\} \cap I^c\right\} - \mathbf{p}_G\{t < \tilde{f}_n\} \\ &> \varepsilon + \mathbf{p}_G\left\{\{\tilde{f}_n < t\} \cap I^c\right\} - \mathbf{p}_G\{t < \tilde{f}_n\} \\ &\geqslant \varepsilon + \mathbf{p}_G\left\{\{f_n < t\} \cap I^c\right\} - \mathbf{p}_G\{t < f_n\} \\ &> \varepsilon + \mathbf{p}_G\,\{\{f_n < t\} \cap I\} + \mathbf{p}_G\left\{\{f_n < t\} \cap I^c\right\} - \mathbf{p}_G\{t < f_n\} \\ &> \varepsilon + u_1(f_n, t), \end{aligned}$$

as we sought. $\square$

Claim 3. *There exist $f^\bullet \in \mathcal{Q}_{(G,r)}$ and a subsequence $(\mu_2^{n_k})$ of (μ_2^n) such that $U_1(f^\bullet, \mu_2^{n_k}) > 0$ for every $k = 1, 2, \ldots$.*

Proof. Let $\alpha > 0$ as found in Claim 2. For each natural number n, choose

$$f_n \in \arg\max\left\{t(y) : t \in \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha\left(\arg\max\left\{h(y) : h \in \operatorname{supp}\{\mu_2\}\right\}\right)\right\}$$

(see footnote 28). Now pick $\varepsilon, \gamma, \beta > 0$, a subsequence (n_k) of strictly increasing positive integers, an integer K, along with $f_{n_k}^*$ and $\mathcal{B}_{n_k}$ (for each $k \geqslant K$), as in Claim 2. To simplify notation, we assume $(n_k) = (1, 2, \ldots)$. Because the sequence (f_n^*) lies in the compact metric space $\mathcal{T}_{(G,r)}$ (Lemma 2), it contains a subsequence that converges in $\mathcal{T}_{(G,r)}$. We denote this subsequence again by (f_n^*), and write f^* for its limit.

Since $f_n^* \to f^*$, there exists an M such that $f_n^* \in N_{\min\{\beta/2,\varepsilon/2\}}(f^*)$ for every $n \geqslant M$. Moreover, since $f_n^* \to f^*$ and $f_n^*|_{Y^-} = 0$ for every n, we have $f^*|_{Y^-} = 0$. Now, Claim 2 gives, for $n \geqslant K$, $f_n^*|_{Y^+} < f_n|_{Y^+} - \varepsilon$, $f_n^*|_{Y^-} = 0$, and $u_1(\tilde{f}_n, t) > u_1(f_n, t)$ for all $\tilde{f}_n \in N_\beta(f_n^*)$ that vanish on Y^-, and all $t \in \mathcal{B}_n$. It follows that, for every $n \geqslant \max\{K, M\}$, $f^*|_{Y^+} < f_n|_{Y^+} - \frac{\varepsilon}{2}$, $f^*|_{Y^-} = 0$, and

$$u_1(\tilde{f}, t) > u_1(f_n, t) + \varepsilon$$

for each $t \in \mathcal{B}_n$ and every $\tilde{f} \in N_{\beta/2}(f^*)$ that vanishes on Y^-. This and Lemma 5 give us an $f^\bullet \in \mathcal{Q}_{(G,r)}$ such that, for each $n \geqslant \max\{K, M\}$,

$$f^\bullet|_{Y^+} < f_n|_{Y^+}, \tag{26}$$

$$f^\bullet|_{Y^-} = 0, \tag{27}$$

and

$$u_1(f^\bullet, t) > u_1(f_n, t) + \varepsilon \quad \text{for all } t \in \mathcal{B}_n. \tag{28}$$

By the definition of f_n, along with (26), (27), and because $f^\bullet$, $f_n \in \mathcal{Q}_{(G,r)}$ for every n (so that $h(y) \neq f_n(y)$ for each $h \in \mathcal{Q}_{(G,r)}$ and every n), and $G \in \mathcal{G}_{(F,r)}$ (so that $\mathbf{p}_G\{y\} < \mathbf{p}_G\{(y, 1]\}$), we have

$$u_1(f^\bullet, t) \geqslant u_1(f_n, t) \text{ for all } t \in \mathcal{T}_n \text{ and } n \geqslant \max\{K, M\}, \tag{29}$$

where

$$\mathcal{T}_n := \operatorname{supp}\{\mu_2^n\} \cap \operatorname{cl} N_\alpha(\operatorname{supp}\{\mu_2\}).$$

By Claim 1, there exists an integer N such that

$$\mu_2^n(\mathcal{T}_n) > 1 - \tfrac{\gamma\varepsilon}{4} \quad \text{for all } n \geqslant N.$$

From this, (28), (29), and the fact that $\mu_2^n(\mathcal{B}_n) > \gamma$ for every $n \geqslant K$, we obtain, for $n \geqslant \max \{K, M, N\}$ (recall that $\mathcal{A}_n$ stands for the strategy space for the finite game $(\mathcal{A}_n, (u_1, u_2))$),

$$\begin{aligned}
U_1(f^\bullet, \mu_2^n) &= \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&\geqslant \sum_{t \in \mathcal{B}_n} \mu_2^n(t) \Big(u_1(f_n, t) + \varepsilon \Big) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&> \gamma\varepsilon + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) + \sum_{t \in \mathcal{A}_n \setminus \mathcal{B}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&= \gamma\varepsilon + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) \\
&\quad + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \setminus \mathcal{T}_n} \mu_2^n(t) u_1(f^\bullet, t) + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \cap \mathcal{T}_n} \mu_2^n(t) u_1(f^\bullet, t) \\
&\geqslant \gamma\varepsilon + \sum_{t \in \mathcal{B}_n} \mu_2^n(t) u_1(f_n, t) \\
&\quad + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \setminus \mathcal{T}_n} \mu_2^n(t) u_1(f^\bullet, t) + \sum_{t \in (\mathcal{A}_n \setminus \mathcal{B}_n) \cap \mathcal{T}_n} \mu_2^n(t) u_1(f_n, t)
\end{aligned}$$

$$
\begin{aligned}
&> \tfrac{3\gamma\varepsilon}{4} + \sum_{t\in\mathcal{B}_n} \mu_2^n(t)u_1(f_n,t) + \sum_{t\in(\mathcal{A}_n\backslash\mathcal{B}_n)\cap\mathcal{T}_n} \mu_2^n(t)u_1(f_n,t) \\
&\geqslant \tfrac{\gamma\varepsilon}{2} + \sum_{t\in\mathcal{B}_n} \mu_2^n(t)u_1(f_n,t) \\
&\quad + \sum_{t\in(\mathcal{A}_n\backslash\mathcal{B}_n)\backslash\mathcal{T}_n} \mu_2^n(t)u_1(f_n,t) + \sum_{t\in(\mathcal{A}_n\backslash\mathcal{B}_n)\cap\mathcal{T}_n} \mu_2^n(t)u_1(f_n,t) \\
&= \tfrac{\gamma\varepsilon}{2} + \sum_{t\in\mathcal{A}_n} \mu_2^n(t)u_1(f_n,t) \\
&= \tfrac{\gamma\varepsilon}{2} \\
&> 0,
\end{aligned}
$$

where the last equality holds true because $f_n \in \operatorname{supp}\{\mu_2^n\}$ and (μ_2^n, μ_2^n) is a Nash equilibrium of $(\mathcal{A}_n, (u_1, u_2))$. □

Let $(\mathcal{A}_{n_k})$ be the subsequence of $(\mathcal{A}_n)$ that corresponds to $(\mu_2^{n_k})$ found in Claim 3. Since $f^\bullet \in \mathcal{Q}_{(G,r)}$ and $(\mathcal{A}_n)$ is an increasing sequence whose union is $\mathcal{Q}_{(G,r)}$, $f^\bullet \in \mathcal{A}_{n_k}$ for some k. This and Claim 3 yield $f^\bullet \in \mathcal{A}_{n_k}$ and $U_1(f^\bullet, \mu_2^{n_k}) > 0$ for some k. But, since $(\mathcal{A}_{n_k}, (u_1, u_2))$ is a symmetric zero-sum game, and $(\mu_1^{n_k}, \mu_2^{n_k})$ is a Nash equilibrium of this game, $(\mu_2^{n_k}, \mu_2^{n_k})$ is also a Nash equilibrium of $(\mathcal{A}_{n_k}, (u_1, u_2))$. But this contradicts $U_1(f^\bullet, \mu_2^{n_k}) > 0$, and completes the proof.

6.3. Proof of Theorem 3

Lemma 12. *$\mathcal{T}_{(F,r)}^{prog}$ is a compact subset of* $\mathbf{C}[0,1]$ *for any $(F,r) \in \mathcal{E}$.*

Proof. The proof is analogous to that of Lemma 2 with $\mathcal{T}_{(F,r)}^{prog}$ playing the role of $\mathcal{T}_{(F,r)}$. □

Lemma 13. *Let $(F,r) \in \mathcal{E}^*$, $p \in \mathbb{P}(\mathcal{T}_{(F,r)}^{prog})$, and $\varepsilon > 0$. Then, for every $\tau \in \mathcal{T}_{(F,r)}^{prog}$, there exists a $\tau^* \in \mathcal{T}_{(F,r)}^{prog}$ such that*

$$
\liminf \int_{\mathcal{C}_{(F,r)}} (w(\tau^*,\cdot) - w(\cdot,\tau^*))\, d\mu_n \geqslant \int_{\mathcal{T}_{(F,r)}^{prog}} (w(\tau,\cdot) - w(\cdot,\tau))\, d\mu - \varepsilon
$$

for every sequence (μ_n) in $\mathbb{P}(\mathcal{T}_{(F,r)}^{prog})$ that converges weakly to μ.

Proof. The proof is analogous to that of Lemmas 3A-B with $\mathcal{T}_{(F,r)}^{prog}$ playing the role of $\mathcal{T}_{(F,r)}$ and $\underline{t}_\varepsilon$ being redefined as follows:

$$
\underline{t}_\varepsilon(x) := \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant x_\varepsilon, \\ \frac{[\tau(x^*+\varepsilon)-\tau(x^*-\varepsilon)](x-x^*+\varepsilon)}{2\varepsilon} + \tau(x^*-\varepsilon) & \text{if } x_\varepsilon < x \leqslant 1, \end{cases}
$$

where $x_\varepsilon := \frac{(x^*-\varepsilon)\tau(x^*+\varepsilon)-(x^*+\varepsilon)\tau(x^*-\varepsilon)}{\tau(x^*+\varepsilon)-\tau(x^*-\varepsilon)}$. □

Definition 4. Let $\mathfrak{v}_{(F,r)} := \left(\mathcal{T}^{prog}_{(F,r)}, (u_1, u_2)\right)$. By the mixed extension of $\mathfrak{v}_{(F,r)}$, we understand the strategic game

$$\mathfrak{V}_{(F,r)} := (\mathbb{P}(\mathcal{T}^{prog}_{(F,r)}), (U_1, U_2)),$$

with each U_i being the restriction of the corresponding payoff function of $\mathfrak{G}_{(F,r)}$ to $\mathcal{T}^{prog}_{(F,r)} \times \mathcal{T}^{prog}_{(F,r)}$. We say that $\mathfrak{V}_{(F,r)}$ is *payoff secure* if, for every $i \in \{1, 2\}$, $(\mu_1, \mu_2) \in \mathbb{P}(\mathcal{T}^{prog}_{(F,r)})^2$, and $\varepsilon > 0$, there exists $v_i \in \mathbb{P}(\mathcal{T}^{prog}_{(F,r)})$ such that $U_i(v_i, \tilde{\mu}_{-i}) \geqslant U_i(\mu) - \varepsilon$ for all $\tilde{\mu}_{-i}$ in some open neighborhood of μ_{-i}.

Lemma 14. *$\mathfrak{V}_{(F,r)}$ is payoff secure for any $(F, r) \in \mathcal{E}^*$.*

Proof. The proof is analogous to that of Lemma 4 with Lemma 13 playing the role of Lemma 3, and $\mathcal{T}^{prog}_{(F,r)}$ that of $\mathcal{T}_{(F,r)}$. $\square$

Lemma 15. *Let $(F, r) \in \mathcal{E}^{*+}$. We have $w(t, \tau) > \frac{1}{2}$ for any $(t, \tau) \in \mathcal{N}^{prog}_{(F,r)} \times \mathcal{T}^{reg}_{(F,r)}$ with $\int_0^1 t\, dF \leqslant \int_0^1 \tau\, dF$.*

Proof. Take any $(t, \tau) \in \mathcal{N}^{prog}_{(F,r)} \times \mathcal{T}^{reg}_{(F,r)}$ with $\int_0^1 t\, dF \leqslant \int_0^1 \tau\, dF$. Since t is convex (but not affine) and τ concave, either $t < \tau$ everywhere on (0, 1], or $t(x^*) = \tau(x^*)$ for some (unique) $0 < x^* \leqslant 1$. In the former case, we have $w(t, \tau) = 1$. In the latter case, $t|_{(0,x^*)} < \tau|_{(0,x^*)}$ and $t|_{(x^*,1]} > \tau|_{(x^*,1]}$. Moreover, since $t - \tau$ is a non-affine convex function on [0, 1], the Jensen Inequality yields

$$(t - \tau)(\mu_F) < \int_0^1 (t - \tau)\, dF \leqslant 0.$$

Therefore, $m_F < \mu_F < x^*$, and hence $w(t, \tau) > F(m_F) = \frac{1}{2}$. $\square$

Lemma 16. *Let $(F, r) \in \mathcal{E}^*$. If $\mathfrak{v}_{(F,r)}$ has a mixed strategy Nash equilibrium, then so does $\mathfrak{h}_{(F,r)}$.*

Proof. (To simplify the notation, we write $\mathcal{T}^{prog}$ and $\mathcal{T}^{reg}$ for $\mathcal{T}^{prog}_{(F,r)}$ and $\mathcal{T}^{reg}_{(F,r)}$, respectively.) Let $v = (v_1, v_2)$ be a mixed strategy Nash equilibrium for $\mathfrak{v}_{(F,r)}$. We wish to show that v is also a mixed strategy Nash equilibrium for $\mathfrak{h}_{(F,r)}$. Pick any $\mu_1 \in \mathbb{P}(\mathcal{C}_{(F,r)})$ such that $\int_0^1 t\, dF = r$ for any $t \in \text{supp}\{\mu_1\}$.

If $\mu_1(\mathcal{T}^{prog}) = 0$, we have

$$U_1(\mu_1, v_2) = \int_{\mathcal{T}^{prog}} \int_{\mathcal{N}^{reg}} u_1\, d\mu_1\, dv_2,$$

where $\mathcal{N}^{reg}$ is the set of all nonlinear members of $\mathcal{T}^{reg}$. It is easy to see that, because v is a Nash equilibrium of $\mathfrak{v}_{(F,r)}$, any $t \in \text{supp}\{v_2\}$ satisfies $\int_0^1 t\, dF = r$. Since, by Lemma 15, $u_1(t, \tau) \leqslant 0$ for any $(t, \tau) \in \mathcal{N}^{reg} \times \mathcal{T}^{prog}$ with $\int_0^1 t\, dF = \int_0^1 \tau\, dF$, we have $U_1(\mu_1, v_2) \leqslant 0$. But $U_1(v) = 0$ because v is a Nash equilibrium, so $U_1(\mu_1, v_2) \leqslant U_1(v)$.

If, on the other hand, $\mu_1(\mathcal{T}^{prog}_{(F,r)}) > 0$, we define the Borel probability measure σ_1 on $\mathcal{T}^{prog}_{(F,r)}$ by

$$\sigma_1(\mathcal{B}) := \mu_1(\mathcal{B}) \left(1 + \tfrac{\mu_1(\mathcal{N}^{reg})}{\mu_1(\mathcal{T}^{prog})}\right).$$

Then,

$$
\begin{aligned}
U_1(\mu_1, \nu_2) &= \int_{\mathcal{T}^{prog}} \int_{\mathcal{N}^{reg}} u_1 \, d\mu_1 \, d\nu_2 + \int_{\mathcal{T}^{prog}} \int_{\mathcal{T}^{prog}} u_1 \, d\mu_1 \, d\nu_2 \\
&\leqslant \int_{\mathcal{T}^{prog}} \int_{\mathcal{T}^{prog}} u_1 \, d\mu_1 \, d\nu_2 \\
&\leqslant \left(1 + \tfrac{\mu_1(\mathcal{N}^{reg})}{\mu_1(\mathcal{T}^{prog})}\right) \int_{\mathcal{T}^{prog}} \int_{\mathcal{T}^{prog}} u_1 \, d\mu_1 \, d\nu_2 \\
&= \int_{\mathcal{T}^{prog}} \int_{\mathcal{T}^{prog}} u_1 \, d\sigma_1 \, d\nu_2 \\
&= U_1(\sigma_1, \nu_2) \\
&\leqslant U_1(\nu),
\end{aligned}
$$

where the first inequality follows from Lemma 15 and the last one from the fact that ν is a mixed strategy Nash equilibrium for $\mathfrak{v}_{(F,r)}$. We conclude that $U_1(\mu_1, \nu_2) \leqslant U_1(\nu)$ for every $\mu_1 \in \mathbb{P}(\mathcal{C}_{(F,r)})$ such that $\int_0^1 t \, dF = r$ for any $t \in \operatorname{supp}\{\mu_1\}$. Obviously, it follows from this observation that $U_1(\mu_1, \nu_2) \leqslant U_1(\nu)$ for every $\mu_1 \in \mathbb{P}(\mathcal{C}_{(F,r)})$. Thus, ν_1 is a best response to ν_2 in the game $\mathfrak{h}_{(F,r)}$. Interchanging the roles of the players 1 and 2 in this argument completes the proof. □

Lemma 17. *Let $(F, r) \in \mathcal{E}^{*+}$ and take any $(\mu_1, \mu_2) \in \mathbb{P}(\mathcal{T}_{(F,r)})^2$ such that $\int_0^1 t \, dF = r$ for every $t \in \operatorname{supp}\{\mu_1\} \cup \operatorname{supp}\{\mu_2\}$. If $\mu_1(\mathcal{N}^{prog}_{(F,r)}) > 0$ and $\mu_2(\mathcal{T}^{reg}_{(F,r)}) > 0$, then*

$$\int_{\mathcal{T}^{reg}_{(F,r)}} \int_{\mathcal{N}^{prog}_{(F,r)}} u_1(t, \tau) \, d\mu_1 \, d\mu_2 > 0.$$

Proof. (To simplify the notation, we write $\mathcal{N}^{prog}$ and $\mathcal{T}^{reg}$ for $\mathcal{N}^{prog}_{(F,r)}$ and $\mathcal{T}^{reg}_{(F,r)}$, respectively.) By Lemma 15, we have $w(\mathcal{N}^{prog} \times \mathcal{T}^{reg}) > \frac{1}{2}$. Pick any $(t, \tau) \in \mathcal{N}^{prog} \times \mathcal{T}^{reg}$ in the support of $\mu_1 \times \mu_2$ (which exists by hypothesis). Since w is lower semicontinuous (Lemma 1), there exist an $\varepsilon > 0$ and an open neighborhood O of (t, τ) such that $\inf w(O) \geqslant \frac{1}{2} + \varepsilon$. But, since (t, τ) is in the support of $\mu_1 \times \mu_2$, we have

$$\left(\mu_1 \times \mu_2\right)\left(O \cap (\mathcal{N}^{prog} \times \mathcal{T}^{reg})\right) > 0 \quad \text{and} \quad \inf w(O \cap (\mathcal{N}^{prog} \times \mathcal{T}^{reg})) \geqslant \tfrac{1}{2} + \varepsilon,$$

so the assertion follows readily from Fubini's Theorem. □

Proof of Theorem 3. Fix any $(F, r) \in \mathcal{E}^{*+}$. (To simplify the notation, we write $\mathcal{T}$, $\mathcal{N}^{prog}$ and $\mathcal{T}^{reg}$ for $\mathcal{T}_{(F,r)}$, $\mathcal{N}^{prog}_{(F,r)}$ and $\mathcal{T}^{reg}_{(F,r)}$, respectively.) The game $\mathfrak{v}_{(F,r)}$ is compact (Lemma 12). Further, its mixed extension is payoff secure (Lemma 14) and $u_1 + u_2$, being constant on $\mathcal{T}^2$, is obviously continuous on $\mathcal{T}^2$. It follows from Corollary 5.2 and Proposition 5.1 of [28] that $\mathfrak{v}_{(F,r)}$ possesses a mixed strategy Nash equilibrium. Lemma 16 then ensures that $\mathfrak{h}_{(F,r)}$ also possesses a mixed strategy Nash equilibrium.

Next, we wish to show that

$$\mu_1(\mathcal{N}^{prog}) = 1 = \mu_2(\mathcal{N}^{prog})$$

for any equilibrium (μ_1, μ_2) of $\mathfrak{h}_{(F,r)}$.

Take a profile $\mu = (\mu_1, \mu_2) \in \mathbb{P}(\mathcal{C}_{(F,r)})^2$ (not necessarily an equilibrium). Pick any $i \in \{1, 2\}$, say, $i = 2$. Suppose $\mu_2(\mathcal{T}^{reg}) > 0$, and note that we will be done if we can show that μ is not a mixed strategy Nash equilibrium for $\mathfrak{h}_{(F,r)}$. If $\int_0^1 t\,dF > r$ for some $t \in \text{supp}\{\mu_1\} \cup \text{supp}\{\mu_2\}$, it is easy to see that μ cannot be an equilibrium. To focus on the nontrivial case, then, we assume $\int_0^1 t\,dF = r$ for each $t \in \text{supp}\{\mu_1\} \cup \text{supp}\{\mu_2\}$.

Suppose $U_1(\mu) > 0$. Then $U_2(\mu_1, \mu_1) = 0 > U_2(\mu)$, so μ is not a Nash equilibrium. On the other hand, if $U_1(\mu) \leqslant 0$ and $\mu_2(\mathcal{N}^{prog}) = 0$, then, for an arbitrarily chosen $t \in \mathcal{N}^{prog}$, Lemma 17 yields $U_1(t, \mu_2) > 0$, so, again, μ cannot be a Nash equilibrium.

Finally, we turn to the case $U_1(\mu) \leqslant 0$ and $\mu_2(\mathcal{N}^{prog}) > 0$. Define the Borel probability measure σ_1 on $\mathcal{C}_{(F,r)}$ by

$$\sigma_1(\mathcal{B}) := \tfrac{\mu_2(\mathcal{B} \cap \mathcal{N}^{prog})}{\mu_2(\mathcal{N}^{prog})}.$$

Observe that

$$\begin{aligned}
U_1(\sigma_1, \mu_2) &= \int_{\mathcal{T}^{reg}} \int_{\mathcal{N}^{prog}} u_1\,d\sigma_1\,d\mu_2 + \int_{\mathcal{N}^{prog}} \int_{\mathcal{N}^{prog}} u_1\,d\sigma_1\,d\mu_2 \\
&> \int_{\mathcal{N}^{prog}} \int_{\mathcal{N}^{prog}} u_1\,d\sigma_1\,d\mu_2 \\
&= \tfrac{1}{\mu_2(\mathcal{N}^{prog})} \int_{\mathcal{N}^{prog}} \int_{\mathcal{N}^{prog}} u_1\,d\sigma_1\,d\mu_2 \\
&= 0 \\
&\geqslant U_1(\mu),
\end{aligned}$$

where the first inequality holds by Lemma 17. Thus, in this case too, we find that μ is not a Nash equilibrium for $\mathfrak{h}_{(F,r)}$. □

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