Dynamic Lorenz Dominance

Oriol Carbonell-Nicolau*

28 February 2026

Abstract

The Lorenz criterion, rooted in the classic Pigou–Dalton transfer principle, provides the foundational consensus for static inequality measurement. However, evaluating the intensity of inequality changes across distributional transitions requires a broader framework. This paper extends the axiomatic approach to a dynamic setting, proposing a pre-order to unambiguously rank income transitions based on the magnitude of structural inequality reduction. By imposing Symmetry, the framework treats the initial and final distributions anonymously, abstracting away from individual identities. We introduce the Dynamic Pigou–Dalton Transfer Principle, which operationalizes transfer intensity across the entire distribution. We then propose Dynamic Lorenz dominance—driven by absolute rank-based income increments—and establish a fundamental equivalence: a dynamic inequality pre-order satisfies Symmetry and the Dynamic Pigou–Dalton Transfer Principle if and only if it is consistent with Dynamic Lorenz dominance. Finally, we extend this base characterization to unrestricted domains with variable total incomes and population sizes by introducing dynamic analogues of scale invariance, translation invariance, and the population principle. Through these extensions, our axiomatic approach provides a universal, index-free ordinal foundation for measuring the intensity of structural progressivity.

Keywords: static Lorenz dominance, dynamic Lorenz dominance, dynamic inequality.
JEL classifications: D63, D71.

*Department of Economics, Rutgers University, 75 Hamilton St., New Brunswick, NJ 08901. E-mail: carbonell-nicolau@rutgers.edu.

1 Introduction

The Lorenz criterion is widely accepted for unambiguous inequality comparisons across income distributions. It not only draws on appealing intuition but is also the only criterion fully characterized by the classic Pigou–Dalton transfer principle (Pigou, 1912; Dalton, 1920) via the fundamental majorization result of Hardy, Littlewood, and Pólya (cf. Hardy et al., 1952; Atkinson, 1970; Rothschild and Stiglitz, 1973; Dasgupta et al., 1973; Fields and Fei, 1978). It serves as a common ground for virtually all standard inequality measures: while specific indices may sometimes disagree among themselves, they uniformly align with the core Lorenz criterion.

This paper extends the axiomatic framework for static comparisons to a dynamic setting to measure the intensity of changes in inequality. Thus, the primary domain of evaluation becomes the set of transitions from an initial income distribution $\boldsymbol{x}$ to a final distribution $\boldsymbol{y}$.

A *dynamic inequality pre-order* $\succcurlyeq$ defined on the set of transitions is a reflexive and transitive relation, where $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}, \boldsymbol{y})$ represents a weakly higher degree of inequality reduction in the first transition. Such measures naturally cover the static case, represented by the null transition $(\boldsymbol{x}, \boldsymbol{x})$, where evaluating $(\boldsymbol{x}, \boldsymbol{y}) \succcurlyeq (\boldsymbol{x}, \boldsymbol{x})$ collapses precisely to the standard static comparison between $\boldsymbol{x}$ and $\boldsymbol{y}$.

The analysis begins on the restricted domain of transitions with a fixed mean and population size. On this domain, we postulate two foundational axioms: Symmetry (S) and the Dynamic Pigou–Dalton Transfer Principle (DPDTP).

Axiom S is a natural extension of the standard symmetry property, positing that dynamic inequality evaluations are entirely invariant to permutations of individuals' identities within either the initial or final distributions.

The intuition behind DPDTP is built upon the concept of elementary transfers. In a static setting, the classic Pigou–Dalton principle focuses solely on the *direction* of a transfer—whether it is progressive (shifting income from richer to poorer) or regressive. In a dynamic setting, a new dimension emerges: the *intensity* of the transfer. DPDTP operationalizes this via a transfer dominance relation, denoted $\sqsupseteq$, defined over valid elementary transfers (those that preserve the global non-decreasing sortedness of the income distribution). We stipulate that one valid elementary transfer dominates another if it exerts a stronger equalizing effect—meaning it moves a larger absolute amount of income across a weakly wider rank gap if the transfers are progressive, or a smaller amount across a narrower gap if they are regressive (with any progressive transfer naturally dominating a regressive one). DPDTP simply postulates that if the transition $\boldsymbol{x}' \to \boldsymbol{y}'$ is driven by an elementary transfer that unambiguously dominates the transfer driving $\boldsymbol{x} \to \boldsymbol{y}$ via $\sqsupseteq$, then $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}, \boldsymbol{y})$—implying a weakly greater inequality reduction.

Recovering the classical static Pigou–Dalton principle from this dynamic framework involves a nuanced aggregation. Because DPDTP evaluates only single, valid elementary transfers, its direct application merely guarantees that one such progressive step strictly dominates the null transition—since shifting a strictly positive amount of income progressively dominates shifting nothing. However, because our framework requires transfers to preserve global sortedness, a standard classical progressive transfer (which imposes no such rank-preservation constraint) may decompose into a sequence of multiple valid elementary steps. To conclude that the overall classical transfer unambiguously reduces inequality relative to the status quo, we must chain these step-by-step improvements together. This aggregation relies on a transitivity-like closure property, which we formalize as Cumulative Progressivity (CP). A feature of our analysis is that this property need not be postulated independently; as our main characterization reveals, it emerges endogenously from the core dynamic axioms. Thus, classical static fairness is subsumed by the dynamic framework.

The proposed dynamic extension of the classic Pigou–Dalton principle, DPDTP, requires a fundamental modeling choice regarding how transfer intensity is measured. In this paper, our evaluation of transfer intensity is eminently expressed in *absolute* terms (we refer the reader to the concluding remarks for a discussion of relative alternatives).

We introduce *Dynamic Lorenz* (DL) dominance ($\succcurlyeq_{DL}$) as the core dynamic inequality pre-order. While static Lorenz dominance evaluates the cumulative income held by the poorest individuals, DL dominance evaluates the *change* in this cumulative income across periods. This change is entirely captured by a transition's rank-based increment vector, $\boldsymbol{y} - \boldsymbol{x}$, which records the net income change at each rank. Specifically, $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$ requires that, for any rank k, the cumulative sum of these increments for the bottom k individuals is weakly greater in the transition $\boldsymbol{x}' \to \boldsymbol{y}'$ than in $\boldsymbol{x} \to \boldsymbol{y}$. This criterion satisfies both S and DPDTP. A fundamental result of this paper establishes the converse structural logic: we prove that DL dominance holds if and only if the underlying rank-based increments can be decomposed into sequences of elementary transfers satisfying the transfer dominance relation $\sqsupseteq$ (Lemma 5).

Once this equivalence is established, a natural question arises: does adherence to S and DPDTP uniquely characterize consistency with the DL pre-order? We answer this in the affirmative. Our main characterization (Theorem 1) proves that any dynamic inequality pre-order satisfies S and DPDTP if and only if it is DL-consistent. This establishes $\succcurlyeq_{DL}$ as the foundational criterion for dynamic inequality measurement, parallel to the role of the static Lorenz criterion.

We then expand the evaluation to the subdomain of sequences with a fixed population size but variable total income. Here, two notions of dynamic invariance are introduced: Dynamic Scale Invariance (DSI) and Dynamic Translation Invariance (DTI). These axioms extend the standard static notions of scale and translation invariance by positing that dynamic inequality evaluations are unaffected by independent proportional scaling or absolute lump-sum additions in either the initial or final periods. To ensure adherence to these postulates, we adjust the core DL pre-order, introducing relative ($\succcurlyeq_{rDL}$) and absolute ($\succcurlyeq_{aDL}$) dynamic Lorenz dominance criteria. Utilizing normalization techniques, we apply Theorem 1 to prove that these extended pre-orders are completely characterized by their corresponding invariance axioms alongside S and DPDTP.

Finally, we move to the unrestricted domain of transitions with arbitrary means and population sizes. We formulate the Dynamic Population Principle (DPP), an extension of Dalton's (1920) classic static principle, asserting that dynamic inequality evaluations are invariant to population replications in either period. Evaluating the null transition recovers the static concept as a special case. We show that both the relative and absolute versions of DL dominance satisfy DPP. By leveraging the base characterization in Theorem 1, we establish the final extensions, proving that these comprehensive pre-orders are fully characterized on the unrestricted domain.

Having outlined the theoretical foundations of our extended pre-orders, it is instructive to contrast this distributional approach with the existing literature. Our framework distinguishes itself from the vast literature on income mobility (e.g., Shorrocks, 1978; Fields and Ok, 1999). Traditional mobility measures rely on the joint distribution of incomes across periods to track the specific economic trajectories of individuals. Consequently, these indices often conflate true structural reductions in inequality (e.g., the aggregate narrowing of income gaps) with "exchange mobility" (individuals merely swapping positional ranks). By contrast, our framework operates strictly on the marginal distributions. Driven by Axiom S, it evaluates the sorted income vectors within each period, deliberately abstracting away from individual identity tracking. This filters out positional churn, isolating the pure intensity of anonymous structural inequality reduction.

In this regard, our approach shares conceptual ground with the literature on dynamic inequality decompositions and pro-poor growth, yet it departs from these works in both methodology and scope. For instance, Jenkins and Van Kerm (2006) demonstrate that changes in standard, parameterized inequality indices over time can be algebraically decomposed into a "re-ranking" penalty—capturing the positional churn of individuals—and a "progressivity" component. Because our framework relies on Axiom S, it operates on the independent, cross-sectional distributions of each period, neutralizing any re-ranking penalty. Consequently, rather than decomposing a specific functional index, Dynamic Lorenz dominance provides the fundamental, index-free ordinal foundation for this exact concept of progressivity.

A related but distinct departure from individual tracking appears in the pro-poor growth literature (e.g., Ravallion and Chen, 2003; Son, 2004), which also evaluates distributional changes at the aggregate level but does so through a poverty lens. Studies in this vein typically propose empirical tools, such as growth incidence curves, to evaluate the relative income growth rates of the poorest quantiles. Despite sharing a common focus on evaluating changes in distributional outcomes over time—rather than static snapshots—our framework differs from the pro-poor growth literature along three dimensions. First, whereas Ravallion and Chen (2003) and Son (2004) propose specific empirical indices derived from the growth incidence curve and the poverty growth curve respectively, we establish an ordinal, axiomatic foundation of dynamic inequality measurement. Moreover, the poverty measures underlying the pro-poor growth literature are typically consistent with generalized Lorenz dominance, which differs from Lorenz even in the static case. Second, the pro-poor growth literature is, by design, concerned with what happens below the poverty line: transfers occurring above it are simply invisible to these measures. Our framework is global—a progressive transfer between two individuals registers as a structural reduction in inequality, regardless of where it occurs on the income scale. Third, these two papers evaluate distributional changes in relative terms, measuring the proportional income growth accruing to each quantile. We instead measure transfer intensity in absolute terms—what matters is the absolute amount transferred and the absolute rank distance spanned.

Finally, DPDTP bears a conceptual relationship to the transfer sensitivity condition of Shorrocks and Foster (1987), which strengthens the classical Pigou–Dalton principle by requiring that progressive transfers occurring lower in the income distribution reduce inequality more. Both notions concern the intensity of progressive transfers, but along orthogonal dimensions: Shorrocks and Foster vary the income-level position of the transfer while holding size and income gap fixed, whereas the DPDTP varies the transfer size and rank gap while abstracting away from absolute income levels. This distinction also explains why the Gini coefficient, which fails Shorrocks and Foster's transfer sensitivity, is fully consistent with DPDTP: its rank-based structure is blind to the absolute income level of a transfer but responsive to rank distance and magnitude.

The paper is organized as follows. Section 2 develops the analysis for the restricted domain of transitions with fixed mean and population size. We first introduce the basic setup and formalize the core dynamic axioms. We then define the Dynamic Lorenz pre-order and establish its equivalence to transfer dominance, concluding with the main characterization of DL-consistency in terms of the core axioms. Section 3 extends the analysis to variable total incomes via dynamic invariance axioms. Section 4 generalizes the framework to varying population sizes. Section 5 offers concluding remarks and paths for future research.

2 Distributions with the same total income and population size

2.1 Basic setup

We represent an *income distribution* as an n-dimensional vector of nonnegative incomes $\boldsymbol{x} = (x_1, \ldots, x_n)$, always expressed in rank-based ordered form via its order statistics $x_{[1]} \leq \cdots \leq x_{[n]}$, where $x_{[i]}$ denotes the income of the individual occupying rank i. The term *rank* refers throughout to this position in the ordered sequence, not to an individual's fixed label, so all statements are invariant under relabeling of individuals.

To analyze inequality over time, we consider *sequences* of distributions $(\boldsymbol{x}, \boldsymbol{y})$ spanning two periods. Our goal is to define a *dynamic inequality pre-order* $\succcurlyeq$ on such sequences to capture changes in inequality between periods:

$$(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}, \boldsymbol{y})$$

means that the reduction in inequality from $\boldsymbol{x}'$ to $\boldsymbol{y}'$ is weakly larger than the reduction from $\boldsymbol{x}$ to $\boldsymbol{y}$.

Throughout the sequel, $\succcurlyeq$ is assumed reflexive and transitive on the space of sequences.

2.2 Symmetry axiom

Because individuals differ only by income, any ordering of distributions should be invariant under relabeling:

Symmetry (S). If $\boldsymbol{x}'$ and $\boldsymbol{y}'$ result from $\boldsymbol{x}$ and $\boldsymbol{y}$ by a permutation of incomes, then

$$(\boldsymbol{x}', \boldsymbol{y}') \sim (\boldsymbol{x}, \boldsymbol{y}).$$

2.3 Transfers and elementary decompositions

Understanding how a single transfer alters an income distribution is central to the analysis of rank-based inequality measures. In what follows, we show that any transfer between two individuals can be expressed as a collection of rank-based increments that preserve the ordering of incomes. This decomposition is useful because it allows us to describe *changes in inequality* entirely in terms of rank movements and income shifts, without referring to individual identities. In particular, it provides the building block for dynamic inequality comparisons: any observed change in distribution over time can be interpreted as the cumulative effect of such elementary, rank-preserving transfers.

2.3.1 Transfer notation and setup

Definition 1 (Elementary transfer)**.** An *elementary transfer* from rank d to rank r of amount $t \geq 0$ applied to distribution $\boldsymbol{x}$ is a transformation $\boldsymbol{x} \mapsto \boldsymbol{y}$ defined as follows.

Let the individual occupying rank r in $\boldsymbol{x}$ receive t units, and let the individual occupying rank d in $\boldsymbol{x}$ give t units. All other individuals' incomes remain unchanged. The resulting distribution is $\boldsymbol{y}$.

The transfer must satisfy the *rank-preservation condition*: after the transfer, the relative income ranking between the donor and recipient is preserved. That is, if a and b denote the individuals at ranks r and d respectively in $\boldsymbol{x}$, then $y_a \leq y_b$ when $r < d$, and $y_a \geq y_b$ when $r > d$.

The transfer is called *progressive* if $r < d$ (income flows from higher to lower rank), and *regressive* if $r > d$ (income flows from lower to higher rank). By convention, null transfers (where $t = 0$) are classified as both progressive and regressive.

The definition above explicitly requires that the relative income ranking between the donor and recipient is preserved. For progressive transfers, this means the recipient remains poorer than the donor after the transfer, ensuring that $y_a \leq y_b$ when $x_a \leq x_b$.

2.3.2 Rank-based increments

Consider an income distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ with order statistics

$$x_{[1]} \le \cdots \le x_{[n]}.$$

Suppose we transfer an amount $t \ge 0$ from an individual at rank d (the donor) to an individual at rank r (the recipient), resulting in a new distribution $\boldsymbol{y}$ with order statistics

$$y_{[1]} \le \cdots \le y_{[n]}.$$

The transfer affects only a contiguous interval of ranks. Specifically, all ranks outside $\{r, r+1, \ldots, d\}$ remain unchanged:

$$y_{[i]} = x_{[i]} \quad \text{for all } i \notin \{r, \ldots, d\}. \tag{1}$$

To characterize the changes within the affected interval, we define the *rank-based increments*

$$\tau_i = y_{[i]} - x_{[i]}, \quad i \in \{1, \ldots, n\}.$$

From equation (1), we immediately have $\tau_i = 0$ for all $i \notin \{r, \ldots, d\}$.

2.3.3 Structure of the increments

The increments within the affected range exhibit a systematic pattern dictated by the mechanics of reordering after the transfer. In particular, the adjustment is not confined to a single rank: several intermediate ranks between the donor and the recipient may experience positive or negative changes in their ordered income levels.

Lemma 1 (cutoff rank)**.** *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be an income vector. Fix ranks $r < d$ and an amount $t \ge 0$ such that the transfer from the individual at rank d to the individual at rank r is elementary (i.e., the relative income ranking between donor and recipient is preserved). Form the vector $\boldsymbol{y}$ by transferring t from the individual at original rank d to the individual at original rank r (before reordering). Define the rank-based increments*

$$\tau_i = y_{[i]} - x_{[i]}, \quad i \in \{1, \ldots, n\}.$$

Then there exists a cutoff rank $c \in \{r, \ldots, d-1\}$ such that

- *$\tau_i \ge 0$ for all $i \le c$, with at least one $i \le c$ satisfying $\tau_i > 0$ if $t > 0$;*
- *$\tau_i \le 0$ for all $i > c$, with at least one $i > c$ satisfying $\tau_i < 0$ if $t > 0$.*

Proof. If $t = 0$, the lemma is trivial, so suppose that $t > 0$.

Since the transfer is elementary and progressive, we have:

$$x_{[r]} + t \leq x_{[d]} - t.$$

Define

$$m = x_{[r]} + t \quad \text{and} \quad M = x_{[d]} - t.$$

Then $m \leq M$ by the progressive transfer condition.

Pick a rank c for $\boldsymbol{x}$ such that

$$x_{[c]} \leq m \leq x_{[c+1]}. \tag{2}$$

Since c is not uniquely defined, we select the lowest rank in $\boldsymbol{x}$ satisfying these inequalities.

Locating the cutoff rank. First, observe that $m > x_{[r]}$ and (2) imply $c \geq r$. Moreover, since $M < x_{[d]}$ and $m \leq M$, we have $c \leq d - 1$. Thus $c \in \{r, \ldots, d-1\}$.

Positive increments below the cutoff. We claim that $\tau_i \geq 0$ for each $i \leq c$, with at least one strict inequality.

To understand this pattern, note that the transfer replaces the two pre-transfer values $x_{[r]}$ and $x_{[d]}$ with m and M and then reorders the resulting multiset. The ordered vector $\boldsymbol{y}$ is obtained from $(x_{[1]}, \ldots, x_{[n]})$ by removing $x_{[r]}$ and $x_{[d]}$, then inserting m and M in their appropriate ordered positions. Consequently, all ranks with $x_{[i]} < m$ remain unchanged ($\tau_i = 0$), and the ranks between the insertion points of m and M are shifted upward in the order, creating nonnegative increments for those positions.

Formally, if $c = r$, then $y_{[c]} = m$, implying $\tau_c = m - x_{[r]} = t > 0$, and $\tau_i = 0$ for $i < c$, yielding the desired result.

Next, suppose that $c > r$. We must have $x_{[c]} < m$. To see this, assume by contradiction that $x_{[c]} = m$. Since $c > r \geq 1$, the rank $c - 1$ exists. Because $\boldsymbol{x}$ is sorted, $x_{[c-1]} \leq x_{[c]} = m$. This implies $x_{[c-1]} \leq m \leq x_{[c]}$, meaning rank $c - 1$ satisfies the bounds in (2), which contradicts the choice of c as the *lowest* such rank.

Because $x_{[c]} < m$, the insertion of m into the ordered sequence occurs at rank c, shifting the remaining elements downward by one index. That is:

$$y_{[c]} = m, y_{[c-1]} = x_{[c]}, y_{[c-2]} = x_{[c-1]}, \ldots$$

Consequently, the increments are:

$$\tau_c = m - x_{[c]} > 0, \tau_{c-1} = x_{[c]} - x_{[c-1]} \geq 0, \tau_{c-2} = x_{[c-1]} - x_{[c-2]} \geq 0, \ldots$$

We conclude that $\tau_i \geq 0$ for each $i \leq c$, with at least one strict inequality ($\tau_c > 0$).

Negative increments above the cutoff. We claim that $\tau_i \leq 0$ for each $i > c$, with at least one strict inequality.

By symmetry with the previous argument, after inserting the two new values m and M into the ordered sequence, the larger value M is positioned at a rank $j > c$. All ranks with $x_{[i]} > M$ retain their original values ($\tau_i = 0$), while the ranks between the insertion points of m and M shift downward in the ordered list. This downward shift reduces the corresponding ordered incomes, producing nonpositive increments for those positions. Therefore $\tau_i \leq 0$ for all $i > c$, and at least one of these inequalities is strict.

Observe that there exists a rank $j \in [c+1, d]$ for $\boldsymbol{x}$ such that

$$x_{[c]} \leq m \leq M \leq x_{[j]} \quad \text{and} \quad x_{[j-1]} \leq M \leq x_{[j]}.$$

Since j is not uniquely defined, we pick the lowest possible rank j in $\boldsymbol{x}$ satisfying the above inequalities.

Since $M < x_{[d]}$, if $j = d$, then

$$y_{[c+1]} = x_{[c+1]}, \ldots, y_{[d-1]} = x_{[d-1]}, y_{[d]} = M,$$

implying

$$\tau_{c+1} = 0, \ldots, \tau_{d-1} = 0, \tau_d = -t < 0,$$

as desired.

If $j < d$, then

$$y_{[c+1]} = x_{[c+1]}, \ldots, y_{[j-1]} = x_{[j-1]}, y_{[j]} = M, y_{[j+1]} = x_{[j]}, y_{[j+2]} = x_{[j+1]}, \ldots,$$

implying

$$\tau_{c+1} = 0, \ldots, \tau_{j-1} = 0, \tau_j = M - x_{[j]} \leq 0, \tau_{j+1} = x_{[j]} - x_{[j+1]} \leq 0, \ldots.$$

Since $M < x_{[d]}$, it follows that at least one of these inequalities is strict.

We conclude that $\tau_i \leq 0$ for each $i > c$, with at least one strict inequality. ∎

An entirely analogous result holds when the elementary transfer is *regressive*, that is, when income is transferred from a lower-ranked individual to a higher-ranked one. The argument proceeds by perfect symmetry with that of Lemma 1: the direction of the inequalities is reversed, but the structure of the reasoning—locating the cutoff rank, tracking the reordering of incomes, and establishing the sign pattern of increments—remains identical.

Lemma 2. *Let $\boldsymbol{y}$ be obtained from $\boldsymbol{x}$ by an elementary* regressive *transfer between ranks r and d ($r < d$), with transfer size $t \geq 0$. Then there exists a cutoff rank $c \in \{r, \ldots, d-1\}$ such that:*

- *$\tau_i \le 0$ for all $i \le c$, with at least one strict inequality if $t > 0$; and*
- *$\tau_i \ge 0$ for all $i > c$, with at least one strict inequality if $t > 0$.*

2.3.4 Interpretation as elementary transfers

Lemma 1 establishes that any progressive transfer induces a systematic sign pattern in the rank-based increment vector: ranks below a cutoff c gain while those above c lose, with total inflow below c balancing total outflow above. Each increment τ_i thus represents an elementary parcel of income flowing from higher to lower ranks, and the vector $(\tau_1, \dots, \tau_n)$ fully determines the transition via $y_{[i]} = x_{[i]} + \tau_i$, preserving rank order throughout.

The following examples illustrate this decomposition.

Example 1 (Elementary decomposition)**.** Consider the initial distribution

$$\boldsymbol{x} = (1, 3, 10),$$

with order statistics

$$x_{[1]} = 1, \quad x_{[2]} = 3, \quad x_{[3]} = 10.$$

We transfer $t = 4$ units from rank 3 (the donor) to rank 1 (the recipient).

After the transfer, the donor's income becomes $10 - 4 = 6$. Since this falls between the original incomes at ranks 1 and 2, the individuals must be reordered. The resulting distribution is

$$\boldsymbol{y} = (3, 5, 6),$$

with order statistics

$$y_{[1]} = 3, \quad y_{[2]} = 5, \quad y_{[3]} = 6.$$

Computing the rank-based increments:

$$\tau_1 = y_{[1]} - x_{[1]} = 3 - 1 = 2,$$
$$\tau_2 = y_{[2]} - x_{[2]} = 5 - 3 = 2,$$
$$\tau_3 = y_{[3]} - x_{[3]} = 6 - 10 = -4.$$

The realized outflow from rank 3 is $s = 4$, and we verify the balance condition:

$$\tau_1 + \tau_2 = 2 + 2 = 4 = s.$$

Note that the original donor (who had income 10) now occupies rank 3 with income 6, while the person originally at rank 2 has received income through the rank reshuffling mechanism. The increments τ_1 and τ_2 capture how the donor's outflow is distributed across the lower ranks in the final ordered distribution.

In this case, the cutoff rank identified in Lemma 1 is $c = 2$: the first two ranks experience nonnegative increments $(\tau_1, \tau_2) = (2, 2)$, while the top rank experiences a negative increment $\tau_3 = -4$. This confirms the predicted sign pattern, with income flowing from higher to lower ranks.

Example 2 (Additional elementary decomposition)**.** Consider the initial distribution

$$\boldsymbol{x} = (1, 9, 10),$$

with order statistics

$$x_{[1]} = 1, \quad x_{[2]} = 9, \quad x_{[3]} = 10.$$

We transfer $t = 4$ units from rank 3 (the donor) to rank 1 (the recipient).

After the transfer, the donor's income becomes $10 - 4 = 6$. Since 6 lies between 1 and 9, the ordered distribution $\boldsymbol{y}$ becomes

$$\boldsymbol{y} = (5, 6, 9),$$

with order statistics

$$y_{[1]} = 5, \quad y_{[2]} = 6, \quad y_{[3]} = 9.$$

Computing the rank-based increments:

$$\begin{aligned}
\tau_1 &= y_{[1]} - x_{[1]} = 5 - 1 = 4, \\
\tau_2 &= y_{[2]} - x_{[2]} = 6 - 9 = -3, \\
\tau_3 &= y_{[3]} - x_{[3]} = 9 - 10 = -1.
\end{aligned}$$

In this example, the realized outflow $s = x_{[3]} - y_{[3]} = 1$ is less than the transfer amount $t = 4$, illustrating the reshuffling where some transfer effect is absorbed by rank movement. The intermediate rank 2 has a negative increment, showing the complexity of rank-based flows, while the recipient rank 1 receives the full positive increment.

Here, the cutoff rank is $c = 1$: the bottom rank gains ($\tau_1 = 4 > 0$) while all higher ranks lose income ($\tau_2 = -3$, $\tau_3 = -1$). Again, the sign structure matches Lemma 1, showing how the reduction at the top affects not only the donor but also adjacent higher ranks once the distribution is reordered.

2.3.5 Canonical decompositions

An *elementary transfer vector* $\boldsymbol{A} \in \mathbb{R}^n$ is defined as

$$\boldsymbol{A} = s(\boldsymbol{e}_r - \boldsymbol{e}_d),$$

where $s \geq 0$ is the transfer amount, and $\boldsymbol{e}_r, \boldsymbol{e}_d$ are standard basis vectors corresponding to ranks r (recipient) and d (donor), respectively.

In component form,

$$\boldsymbol{A}_i = \begin{cases} +s, & \text{if } i = r, \\ -s, & \text{if } i = d, \\ 0, & \text{otherwise.} \end{cases}$$

By construction, the total income is preserved:

$$\sum_{i=1}^{n} \boldsymbol{A}_i = 0.$$

Definition 2 (Canonical decomposition)**.** Let $\boldsymbol{x}, \boldsymbol{y}$ be two income distributions with rank-based increment vector $\boldsymbol{\tau} = \boldsymbol{y} - \boldsymbol{x}$. A *canonical decomposition of* $\boldsymbol{\tau}$ is a finite sequence of elementary transfer vectors $(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_M)$ such that $\boldsymbol{\tau} = \sum_{\ell=1}^{M} \boldsymbol{A}_\ell$, constructed via the following allocation mechanism:

1. Identify the set of recipient ranks $R = \{i : \tau_i > 0\}$ and order it by *decreasing* rank index (highest to lowest). Identify the set of donor ranks $D = \{j : \tau_j < 0\}$ and order it by *increasing* rank index (lowest to highest).
2. The decomposition is generated by sequentially matching the lowest available donor with the highest available recipient. At each step ℓ:
   - Let d be the lowest rank in D with remaining surplus, and r be the highest rank in R with remaining deficit.
   - Execute an elementary transfer $\boldsymbol{A}_\ell$ from d to r with magnitude equal to the maximum feasible amount: specifically, the minimum of the donor's remaining surplus and the recipient's remaining deficit.
   - If the recipient's deficit is fully satisfied, move to the next highest rank in R. If the donor's surplus is exhausted, move to the next lowest rank in D.
3. The process stops when all surplus and deficit masses in $\boldsymbol{\tau}$ are fully allocated.

Remark 1. This canonical decomposition is globally rank-preserving. Let $\boldsymbol{\tau}^{(\ell)} = \sum_{k=1}^{\ell} \boldsymbol{A}_k$ be the cumulative transfer up to step ℓ. At every intermediate step, the resulting vector $\boldsymbol{z}^{(\ell)} = \boldsymbol{x} + \boldsymbol{\tau}^{(\ell)}$ remains completely sorted from lowest to highest income. Because the global order is preserved, every individual transfer $\boldsymbol{A}_\ell$ trivially satisfies the donor-recipient rank-preservation requirement of an elementary transfer (Definition 1).

To see this, let $\boldsymbol{z}^{(\ell)} = \boldsymbol{x} + \boldsymbol{\tau}^{(\ell)}$ denote the intermediate income vector after step ℓ, with $\boldsymbol{z}^{(0)} = \boldsymbol{x}$. We proceed by induction. Assume $\boldsymbol{z}^{(\ell-1)}$ is globally sorted ($z_k^{(\ell-1)} \leq z_{k+1}^{(\ell-1)}$ for all

k). We must show that applying the transfer $\boldsymbol{A}_\ell$ yields a sorted $\boldsymbol{z}^{(\ell)}$.

First, observe that $i \in R$ implies $z_i^{(\ell)} \leq y_i$; $i \in D$ implies $z_i^{(\ell)} \geq y_i$; and $\tau_i = 0$ implies $z_i^{(\ell)} = y_i$.

At step ℓ, the transfer strictly increases the recipient's income ($z_r^{(\ell)} > z_r^{(\ell-1)}$) and strictly decreases the donor's income ($z_d^{(\ell)} < z_d^{(\ell-1)}$). All other coordinates remain unchanged. To verify global sorting, we only need to check the boundaries of the modified ranks: we must ensure $z_r^{(\ell)} \leq z_{r+1}^{(\ell)}$ and $z_{d-1}^{(\ell)} \leq z_d^{(\ell)}$.

We know $z_r^{(\ell)} \leq y_r$. Consider the adjacent higher rank $r+1$. Because recipients are processed in strictly descending order, if $r+1 \in R$, its deficit has already been fully satisfied in a previous step, meaning $z_{r+1}^{(\ell)} = y_{r+1}$. If $r+1 \notin R$, it is either a donor or has zero increment, meaning it is bounded below by its target: $z_{r+1}^{(\ell)} \geq y_{r+1}$. In all cases, $z_{r+1}^{(\ell)} \geq y_{r+1}$. Since the target distribution $\boldsymbol{y}$ is fully ordered, $y_r \leq y_{r+1}$. Chaining these yields:

$$z_r^{(\ell)} \leq y_r \leq y_{r+1} \leq z_{r+1}^{(\ell)}.$$

We know $z_d^{(\ell)} \geq y_d$. Consider the adjacent lower rank $d-1$. Because donors are processed in strictly ascending order, if $d-1 \in D$, its surplus has already been fully exhausted in a previous step, meaning $z_{d-1}^{(\ell)} = y_{d-1}$. If $d-1 \notin D$, it is either a recipient or has zero increment, meaning it is bounded above by its target: $z_{d-1}^{(\ell)} \leq y_{d-1}$. In all cases, $z_{d-1}^{(\ell)} \leq y_{d-1}$. Since $\boldsymbol{y}$ is ordered, $y_{d-1} \leq y_d$. Chaining these yields:

$$z_{d-1}^{(\ell)} \leq y_{d-1} \leq y_d \leq z_d^{(\ell)}.$$

Because the modified coordinates do not violate the ordering with their adjacent elements, and all other elements retain their previously sorted order, the entire vector $\boldsymbol{z}^{(\ell)}$ remains globally sorted. By induction, every intermediate distribution is completely sorted.

Remark 2. When the transition from $\boldsymbol{x}$ to $\boldsymbol{y}$ is generated by a single elementary transfer (Definition 1), the canonical decomposition yields a particularly simple and structurally consistent representation.

Suppose the underlying elementary transfer is progressive. By Lemma 1, there exists a cutoff rank c separating the rank-based increments: all positive increments are located at or below c ($\tau_i \geq 0$ for $i \leq c$), and all negative increments are located strictly above c ($\tau_i \leq 0$ for $i > c$). Consequently, the set of recipient ranks R is entirely bounded above by c ($R \subseteq \{1, \dots, c\}$), and the set of donor ranks D is strictly bounded below by c ($D \subseteq \{c+1, \dots, n\}$). Because the canonical decomposition pairs ranks from D with ranks from R, every intermediate transfer $\boldsymbol{A}_\ell$ moves mass from a donor $d \in D$ to a recipient $r \in R$. Since $r \leq c < d$, it strictly holds that $r < d$ for every pair. Thus, the canonical decomposition of a progressive elementary transfer consists exclusively of progressive transfers.

By symmetric reasoning using Lemma 2, if the initial transition is generated by a regressive elementary transfer, the cutoff separates donors below from recipients above. Thus, every matching in the decomposition satisfies $r > d$, yielding a sequence composed exclusively of regressive transfers.

Example 3 (Canonical decomposition of a complex transition)**.** Let the initial and final distributions be $\boldsymbol{x} = (1, 4, 7, 10)$ and $\boldsymbol{y} = (2, 3, 8, 9)$. Both are ordered distributions with the same total income (22). The target rank-based increment vector is

$$\boldsymbol{\tau} = \boldsymbol{y} - \boldsymbol{x} = (1, -1, 1, -1).$$

Unlike a single elementary transfer (Lemma 1), this increment vector alternates in sign and cannot be separated by a single cutoff rank. We apply the canonical decomposition (Definition 2) to resolve it:

1. The set of recipient ranks is $R = \{1, 3\}$. Ordered decreasingly: $3, 1$. The set of donor ranks is $D = \{2, 4\}$. Ordered increasingly: $2, 4$.
2. We pair the lowest available donor, $d = 2$ (surplus 1), with the highest available recipient, $r = 3$ (deficit 1). The transfer amount is 1. We execute the elementary transfer $\boldsymbol{A}_1 = (0, -1, 1, 0)$. Because $r > d$, this is a regressive transfer. The intermediate distribution becomes
$$\boldsymbol{z}^{(1)} = \boldsymbol{x} + \boldsymbol{A}_1 = (1, 3, 8, 10).$$
Notice that despite the regressive transfer, $\boldsymbol{z}^{(1)}$ remains completely globally sorted. Recipient rank 3 and donor rank 2 are both fully satisfied.
3. We advance to the next highest recipient, $r = 1$ (deficit 1), and the next lowest donor, $d = 4$ (surplus 1). The transfer amount is 1. We execute the elementary transfer $\boldsymbol{A}_2 = (1, 0, 0, -1)$. Because $r < d$, this is a progressive transfer. The intermediate distribution becomes
$$\boldsymbol{z}^{(2)} = \boldsymbol{z}^{(1)} + \boldsymbol{A}_2 = (2, 3, 8, 9) = \boldsymbol{y}.$$
Both sets are now fully exhausted. The complex transition is successfully decomposed into $\boldsymbol{\tau} = \boldsymbol{A}_1 + \boldsymbol{A}_2$, maintaining global sorting at every step.

2.4 The transfer dominance relation $\sqsupseteq$

The previous subsection showed that any individual transfer can be represented as a structured vector of rank-based increments. We now use this structure to formally compare the redistributive strength of different transfers. The relation $\sqsupseteq$ captures the idea that one transfer is "at least as progressive" as another—meaning it moves more income, or moves it across a broader span of the distribution.

Let r and d denote the rank of the recipient and donor, respectively. For a progressive transfer, income flows downward, so $r < d$. For a regressive transfer, income flows upward, so $r > d$.

Definition 3 (Transfer dominance relation $\sqsupseteq$ and $\sqsupset$)**.** Let $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x}', \boldsymbol{y}')$ be two sequences with equal total income and population size, generated by elementary transfers of amounts t and t', respectively. Assume both transfers preserve rank sortedness in their initial distributions. Let r, d and r', d' denote the ranks of the recipient and donor in the unprimed and primed transfers, respectively.

We write

$$(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y})$$

if one of the following holds:

(i) Progressive case. Both transfers are progressive with

$$r' \leq r < d \leq d' \qquad \text{and} \qquad t' \geq t.$$

(ii) Regressive case. Both transfers are regressive with

$$r \geq r' > d' \geq d \qquad \text{and} \qquad t' \leq t.$$

(iii) Mixed case. The transfer for $(\boldsymbol{x}', \boldsymbol{y}')$ is progressive and the other transfer is regressive.

We write

$$(\boldsymbol{x}', \boldsymbol{y}') \sqsupset (\boldsymbol{x}, \boldsymbol{y})$$

if $(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y})$ and, in addition:

(a) in the progressive case, at least one of the weak inequalities is strict and $t' > 0$;

(b) in the regressive case, at least one of the weak inequalities is strict and $t > 0$;

(c) in the mixed case, at least one of the two transfers is strictly positive.

In short, when $(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y})$, the primed transition redistributes income more intensely (a larger transfer amount t') or more broadly (a wider rank distance between donor and recipient) than the unprimed transition.

2.5 Dynamic Pigou–Dalton Transfer Principle (DPDTP)

The relation $\sqsupseteq$ provides a natural partial order for evaluating single transfers. To extend this logic to general sequences, we require an axiom linking the direction and intensity of redistribution to the ordering of inequality changes over time.

Dynamic Pigou–Dalton Transfer Principle (DPDTP). For any sequences $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x}', \boldsymbol{y}')$ with equal total income and population size,

$$(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}, \boldsymbol{y}) \quad \text{and} \quad (\boldsymbol{x}', \boldsymbol{y}') \sqsupset (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \succ (\boldsymbol{x}, \boldsymbol{y})$$

To understand the intuition behind the DPDTP, it is helpful to view it as a dynamic extension of the classical Pigou-Dalton principle. While the classical principle evaluates static states—declaring a more equal distribution as socially preferred—the DPDTP evaluates the magnitude of change over time, ranking transitions based on how strongly they equalize the distribution.

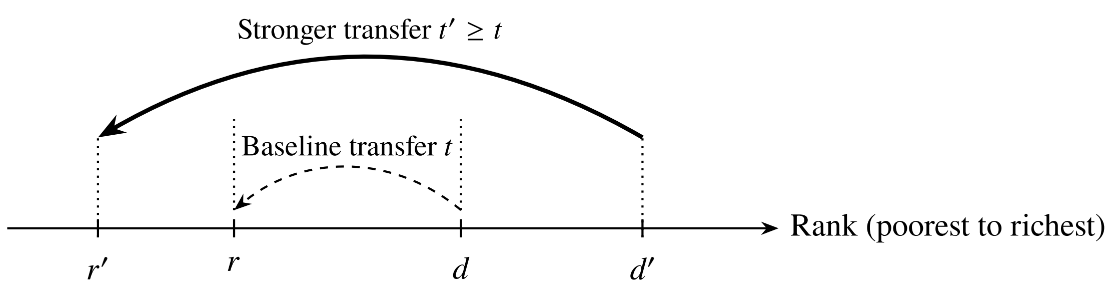

Figure 1: Visualizing the dynamic transfer dominance relation for progressive transfers.

Specifically, this principle establishes a clear normative hierarchy for evaluating dynamic redistribution. Progressive transfers reduce inequality, with a strictly greater reduction when the transfer amount is larger or bridges a wider rank gap (Figure 1). Regressive transfers exacerbate inequality, with a heavier penalty when larger or spanning a greater rank distance. Mixed comparisons are resolved absolutely by direction: any non-trivial progressive transfer strictly dominates any regressive one, regardless of amount or rank distance.

Because the DPDTP evaluates redistributions exclusively through these anonymous, structural metrics (transfer amounts and rank distances), it strips away individual identities. Consequently, the induced ordering depends solely on the shape of the transition across ranks, ensuring it naturally satisfies the Symmetry axiom (S).

Consistency with static fairness. The DPDTP evaluates changes. However, it successfully recovers the classical static Pigou–Dalton Transfer Principle when paired with the following transitivity-like condition:

Cumulative Progressivity (CP). For all income distributions $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$,

$$(\boldsymbol{y}, \boldsymbol{z}) \succcurlyeq (\boldsymbol{x}, \boldsymbol{y}) \succcurlyeq (\boldsymbol{x}, \boldsymbol{x}) \Rightarrow (\boldsymbol{x}, \boldsymbol{z}) \succcurlyeq (\boldsymbol{x}, \boldsymbol{x}),$$

with the relation in the conclusion being strict if at least one relation in the premise is strict.

CP imposes a closure property on inequality-reducing transitions, guaranteeing that if a transition decomposes into a sequence of progressive steps, the aggregate effect remains progressive relative to the status quo.

To see how this restores static fairness, consider a transition $\boldsymbol{x} \to \boldsymbol{y}$ mediated by a progressive elementary transfer $t > 0$ (Definition 1). Let $(\boldsymbol{A}_\ell)$ be the canonical decomposition of the rank-based increment vector $\boldsymbol{\tau}$ (Definition 2). This yields a sequence of intermediate distributions

$$\boldsymbol{x} = \boldsymbol{x}^0 \xrightarrow[\boldsymbol{A}_1]{} \boldsymbol{x}^1 \xrightarrow[\boldsymbol{A}_2]{} \cdots \xrightarrow[\boldsymbol{A}_M]{} \boldsymbol{x}^n = \boldsymbol{y},$$

where each $\boldsymbol{A}_\ell$ corresponds to an elementary progressive transfer preserving ranks in $\boldsymbol{x}$ (Remark 1, Remark 2). By Definition 3 (mixed case), every positive transfer step strictly dominates the null transition $(\boldsymbol{x}, \boldsymbol{x})$:

$$\left(\boldsymbol{x} + \sum_{\ell=1}^{M-1} \boldsymbol{A}_\ell, \boldsymbol{y}\right) \sqsupseteq \cdots \sqsupseteq (\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{A}_1) \sqsupseteq (\boldsymbol{x}, \boldsymbol{x}),$$

with at least one strict relation since $t > 0$. The DPDTP implies the corresponding dominance in the pre-order ($\succcurlyeq$). Iterated application of CP aggregates these improvements, yielding $(\boldsymbol{x}, \boldsymbol{y}) \succ (\boldsymbol{x}, \boldsymbol{x})$, which precisely recovers the classical static Pigou–Dalton Principle.

While CP serves as the logical bridge ensuring composite progressive transitions dominate the status quo, it need not be assumed as an independent axiom. As shown later in our main result (Theorem 1), the coarsest dynamic inequality pre-order satisfying S and DPDTP satisfies CP endogenously. Thus, static fairness emerges naturally from the canonical dynamic framework.

DPDTP and weak transfer sensitivity. The DPDTP is conceptually related to, yet distinct from, the transfer sensitivity condition of Shorrocks and Foster (1987). Both notions concern the *intensity* of progressive transfers, but along different dimensions. Shorrocks and Foster's weak transfer sensitivity holds transfer size and income gap fixed and compares two progressive transfers that differ only in their position in the distribution: the lower-positioned transfer—with both a poorer recipient *and* a poorer donor—is declared more equalizing. This embeds a non-trivial trade-off: a poorer recipient makes the transfer more beneficial, but a poorer donor makes it more costly. Weak transfer sensitivity resolves this trade-off in favor of the recipient, asserting that the net effect of shifting the entire transfer lower down is always favorable. The DPDTP, by contrast, involves no such trade-off: widening the rank gap between donor and recipient unambiguously strengthens the transfer's equalizing effect, since the recipient moves down and the donor moves up simultaneously. The two criteria are therefore orthogonal. This orthogonality also clarifies why the Gini coefficient fails Shorrocks and Foster's transfer sensitivity yet is fully consistent with the DPDTP: being rank-based, the Gini is insensitive to the absolute income level at which a transfer occurs, but does respond to the rank distance and magnitude dimensions that the DPDTP tracks.

We refer the reader to Section 5 for a discussion of how incorporating both dimensions into a unified axiomatic framework might be pursued.

2.6 Dynamic Lorenz dominance

The DPDTP establishes a normative foundation for comparing inequality changes by linking the structure of transfers to the ordering of distributional sequences. To operationalize this principle, we introduce a criterion that aggregates rank-based income changes into a tractable dominance relation.

The key insight is to measure inequality reduction through cumulative gains at the bottom of the distribution. For any transition $(\boldsymbol{x}, \boldsymbol{y})$, we compute the net income gain accruing to the poorest k individuals for each $k \in \{1, \dots, n\}$:

$$d_k(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{k} (y_{[i]} - x_{[i]}).$$

A sequence exhibits greater equalization if it delivers weakly larger cumulative gains to every bottom subset of the population.

Definition 4 (Dynamic Lorenz dominance)**.** We say that sequence $(\boldsymbol{x}', \boldsymbol{y}')$ *dynamically Lorenz dominates* sequence $(\boldsymbol{x}, \boldsymbol{y})$, written $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$, if the primed sequence delivers weakly larger cumulative gains to the bottom k ranks for every k:

$$(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y}) \Leftrightarrow d_k(\boldsymbol{x}', \boldsymbol{y}') \geq d_k(\boldsymbol{x}, \boldsymbol{y}), \quad \forall k \in \{1, \dots, n\}.$$

Recall that we currently assume all distributions share the same total income and population size. The following lemma confirms that $\succcurlyeq_{DL}$ is a well-behaved pre-order that fully implements our core axioms. In particular, more progressive transfers yield greater reductions in inequality, while less regressive transfers impose smaller increases.

Lemma 3. *The dynamic Lorenz dominance relation $\succcurlyeq_{DL}$ is reflexive and transitive and satisfies the axioms S and DPDTP.*

Proof. **Reflexivity and transitivity.** Reflexivity follows immediately since $d_k(\boldsymbol{x}, \boldsymbol{x}) = 0$. Transitivity holds because

$$d_k(\boldsymbol{x}'', \boldsymbol{y}'') \geq d_k(\boldsymbol{x}', \boldsymbol{y}') \geq d_k(\boldsymbol{x}, \boldsymbol{y}), \quad \text{for all } k,$$

implies $d_k(\boldsymbol{x}'', \boldsymbol{y}'') \geq d_k(\boldsymbol{x}, \boldsymbol{y})$.

Symmetry (S). Permuting the incomes in $(\boldsymbol{x}, \boldsymbol{y})$ leaves the order statistics ($x_{[i]}$ and $y_{[i]}$) unchanged. Thus, $d_k(\boldsymbol{x}', \boldsymbol{y}') = d_k(\boldsymbol{x}, \boldsymbol{y})$ for all k, yielding $(\boldsymbol{x}', \boldsymbol{y}') \sim_{DL} (\boldsymbol{x}, \boldsymbol{y})$.

DPDTP. Suppose $(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y})$ or $(\boldsymbol{x}', \boldsymbol{y}') \sqsupset (\boldsymbol{x}, \boldsymbol{y})$, with elementary transfers t and t' and donor/recipient ranks d, d', r, r' as in Definition 3. Let $\tau_i = y_{[i]} - x_{[i]}$ and $\tau'_i = y'_{[i]} - x'_{[i]}$, with partial sums $S_k = d_k(\boldsymbol{x}, \boldsymbol{y})$ and $S'_k = d_k(\boldsymbol{x}', \boldsymbol{y}')$. We must show $S'_k \geq S_k$ for all k, with strict inequality for some k under $\sqsupset$. We evaluate the three cases of Definition 3.

Case 1: Both transfers are progressive. Because both transfers are rank-preserving in their initial distributions, there exist ranks $r < d$ and $r' < d'$ such that $\tau_r = t, \tau_d = -t$, and $\tau'_r = t', \tau'_d = -t'$, with all other increments zero. The partial sums satisfy:

$$S_k = \begin{cases} 0, & k < r, \\ t, & r \leq k < d, \\ 0, & k \geq d, \end{cases} \qquad S'_k = \begin{cases} 0, & k < r', \\ t', & r' \leq k < d', \\ 0, & k \geq d'. \end{cases}$$

Weak dominance ($\sqsupseteq$) implies $r' \leq r < d \leq d'$ and $t' \geq t$. Comparing S'_k and S_k across all possible intervals:

- For $k < r'$ and $k \geq d'$, $S'_k = S_k = 0$.
- For $r' \leq k < r$ and $d \leq k < d'$ (if non-empty), $S'_k = t'$ and $S_k = 0$, so $S'_k \geq S_k$.
- For $r \leq k < d$, $S'_k = t'$ and $S_k = t$, so $S'_k \geq S_k$ since $t' \geq t$.

Thus $S'_k \geq S_k$ for all k, yielding $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$. If the relation is strict ($\sqsupset$), either $r' < r$, $d < d'$, or $t' > t$. In each scenario, $S'_k > S_k$ for some k, yielding strict DL-dominance.

Case 2: Both transfers are regressive. This case is completely analogous. Regressive transfers produce identical step-functions but with negative signs (cumulative sums are $-t$

or $-t'$ on the affected interval). Definition 3 reverses the nesting ($r \geq r' > d' \geq d$) and the transfer magnitude ($t' \leq t$). This sign-reversed geometry ensures $S'_k \geq S_k$ for all k, with strict inequality under $\sqsupset$.

Case 3: Mixed case. Suppose $(\boldsymbol{x}', \boldsymbol{y}')$ is progressive and $(\boldsymbol{x}, \boldsymbol{y})$ is regressive. Then $S'_k \geq 0$ everywhere (strictly positive on $[r', d')$), while $S_k \leq 0$ everywhere (strictly negative on $[d, r)$). Consequently, $S'_k \geq S_k$ everywhere. If at least one transfer is strictly positive, there exists some k where $S'_k > 0$ or $S_k < 0$, guaranteeing strict DL-dominance. ■

Example 4 (Necessity of the rank-preservation condition)**.** Definition 3 explicitly restricts comparisons to elementary transfers that preserve the rank order of their initial distributions. If a transfer is large enough to cause individuals to overtake adjacent ranks, the implication $\sqsupset \Rightarrow \succcurlyeq_{DL}$ may fail.

Consider two sequences for a population $n = 3$. Both involve a transfer of $t = 0.5$ from the richest individual ($d = 3$) to the poorest ($r = 1$).

- Sequence 1: Initial distribution $\boldsymbol{x} = (1, 1.01, 2.01)$. The transfer yields an unsorted vector $(1.5, 1.01, 1.51)$. Sorting produces $\boldsymbol{y} = (1.01, 1.5, 1.51)$, generating rank-based increments $\boldsymbol{\tau} = (0.01, 0.49, -0.5)$. Notice the recipient overtook rank 2.
- Sequence 2: Initial distribution $\boldsymbol{x}' = (1, 1.9, 2)$. The transfer yields an unsorted vector $(1.5, 1.9, 1.5)$. Sorting produces $\boldsymbol{y}' = (1.5, 1.5, 1.9)$, generating increments $\boldsymbol{\tau}' = (0.5, -0.4, -0.1)$. Notice the donor fell below rank 2.

Because both transitions share the same parameters ($r' = r = 1$, $d' = d = 3$, $t' = t = 0.5$), a naive application of Definition 3 ignoring rank-preservation would conclude $(\boldsymbol{x}', \boldsymbol{y}') \sqsupseteq (\boldsymbol{x}, \boldsymbol{y})$. However, evaluating the cumulative gains $d_k = \sum_{i=1}^{k} \tau_i$:

k	$d_k(\boldsymbol{x}, \boldsymbol{y})$	$d_k(\boldsymbol{x}', \boldsymbol{y}')$	Dominance Check ($d'_k \geq d_k$)
1	0.01	0.5	Holds
2	0.50	0.1	Violation ($0.1 \ngeq 0.50$)
3	0	0	Holds

Because rank reshuffling alters the cutoff rank, the bottom two individuals collectively gain less in Sequence 2. This example demonstrates why the DPDTP requires transfers to be structurally rank-preserving to ensure coherent dominance.

While Lemma 3 confirms that dynamic Lorenz dominance respects single transfer dominance, a natural question follows: does $\succcurlyeq_{DL}$ completely characterize the $\sqsupseteq$ ordering?

The next two lemmas establish an exact equivalence: dynamic Lorenz dominance holds if and only if a transition can be decomposed into elementary steps satisfying the $\sqsupseteq$ criterion.

Lemma 4. *Suppose $(\boldsymbol{y}, \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{x}')$, where both sequences share total income and population size. Let $\boldsymbol{\tau}^{y}$ and $\boldsymbol{\tau}^{x}$ be their rank-based increment vectors. There exist sequences of elementary transfer vectors $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_M)$ and $(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_M)$ such that:*

1. $\sum_{\ell=1}^{M} \boldsymbol{Q}_\ell = \boldsymbol{\tau}^{x}$ *and* $\sum_{\ell=1}^{M} \boldsymbol{T}_\ell = \boldsymbol{\tau}^{y}$.
2. *For each $\kappa \in \{1, \ldots, M\}$, we have*
$$\left(\boldsymbol{y} + \sum_{\ell=1}^{\kappa-1} \boldsymbol{T}_\ell, \boldsymbol{y} + \sum_{\ell=1}^{\kappa} \boldsymbol{T}_\ell\right) \sqsupseteq \left(\boldsymbol{x} + \sum_{\ell=1}^{\kappa-1} \boldsymbol{Q}_\ell, \boldsymbol{x} + \sum_{\ell=1}^{\kappa} \boldsymbol{Q}_\ell\right),$$
where $\sum_{\ell=1}^{0} \boldsymbol{T}_\ell = \boldsymbol{0} = \sum_{\ell=1}^{0} \boldsymbol{Q}_\ell$.

If $(\boldsymbol{y}, \boldsymbol{y}') \succ_{DL} (\boldsymbol{x}, \boldsymbol{x}')$, the relation is strict ($\sqsupset$) for at least one κ.

The proof of Lemma 4 is relegated to Appendix A.1. This lemma is instrumental to proving our main path-equivalence result.

Lemma 5. *Let $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x}', \boldsymbol{y}')$ share population size and total income. Then $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$ if and only if there exist finite sequences of distributions*
$$\boldsymbol{x} = \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)} = \boldsymbol{y}, \qquad \boldsymbol{x}' = \boldsymbol{x}'^{(1)}, \ldots, \boldsymbol{x}'^{(m)} = \boldsymbol{y}',$$
such that
$$(\boldsymbol{x}'^{(\ell-1)}, \boldsymbol{x}'^{(\ell)}) \sqsupseteq (\boldsymbol{x}^{(\ell-1)}, \boldsymbol{x}^{(\ell)}) \quad \textit{for all } \ell \in \{2, \ldots, m\}.$$
A strict DL relation implies strict $\sqsupset$ dominance for at least one index step, and vice versa.

Proof. ($\Leftarrow$) Suppose the requisite sequences exist. By Lemma 3, $\succcurlyeq_{DL}$ satisfies DPDTP, meaning each step satisfies $(\boldsymbol{x}'^{(\ell-1)}, \boldsymbol{x}'^{(\ell)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(\ell-1)}, \boldsymbol{x}^{(\ell)})$. Because $\succcurlyeq_{DL}$ is reflexive and transitive, Lemma 1*a of Sen (2017) yields $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$ (with strict dominance if at least one step is strict).

($\Rightarrow$) Assume $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$. Lemma 4 guarantees the existence of elementary transfer vectors $\boldsymbol{Q}_\ell$ and $\boldsymbol{T}_\ell$. Setting $m = M + 1$, we initialize $\boldsymbol{x}^{(1)} = \boldsymbol{x}$ and $\boldsymbol{x}'^{(1)} = \boldsymbol{x}'$, and construct the intermediate distributions iteratively:
$$\boldsymbol{x}^{(\ell)} = \boldsymbol{x}^{(\ell-1)} + \boldsymbol{Q}_{\ell-1} \quad \text{and} \quad \boldsymbol{x}'^{(\ell)} = \boldsymbol{x}'^{(\ell-1)} + \boldsymbol{T}_{\ell-1}.$$
Telescoping the sums recovers the final distributions ($\boldsymbol{x}^{(m)} = \boldsymbol{y}$ and $\boldsymbol{x}'^{(m)} = \boldsymbol{y}'$). Because the intermediate distributions represent the cumulative sums described in Lemma 4, the required nested path dominance holds by construction. ∎

Given this structural equivalence, must any pre-order satisfying symmetry and our dynamic transfer principle agree with dynamic Lorenz dominance? To formalize this, let $\mathcal{S}_{(n,a)}$ denote the space of sequences with fixed population n and fixed total income a.

A pre-order $\succcurlyeq$ on $\mathcal{S}_{(n,a)}$ is *DL-consistent* if

$$(\boldsymbol{x}', \boldsymbol{y}') \succ_{DL} (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \succ (\boldsymbol{x}, \boldsymbol{y}) \quad \text{and} \quad (\boldsymbol{x}', \boldsymbol{y}') \sim_{DL} (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \sim (\boldsymbol{x}, \boldsymbol{y}).$$

The following theorem establishes that dynamic Lorenz dominance is the *unique* canonical ordering satisfying our foundational axioms.

Theorem 1. *A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}_{(n,a)}$ satisfies S and DPDTP if and only if it is DL-consistent.*

Proof. ($\Leftarrow$) Assume DL-consistency. By Lemma 3, $\succcurlyeq_{DL}$ satisfies S and DPDTP. Because $\succcurlyeq$ agrees with $\succcurlyeq_{DL}$ in both its strict and indifference components, $\succcurlyeq$ inherits these properties.

($\Rightarrow$) Assume $\succcurlyeq$ satisfies S and DPDTP. Suppose $(\boldsymbol{x}', \boldsymbol{y}') \succ_{DL} (\boldsymbol{x}, \boldsymbol{y})$. By Lemma 5, there exist parallel sequences of intermediate distributions such that each step satisfies $(z'_{\ell-1}, z'_{\ell}) \sqsupseteq (z_{\ell-1}, z_{\ell})$, with at least one strict step. Because $\succcurlyeq$ satisfies DPDTP, this implies $(z'_{\ell-1}, z'_{\ell}) \succcurlyeq (z_{\ell-1}, z_{\ell})$ at each step, and strictly for at least one step. Transitivity yields $(\boldsymbol{x}', \boldsymbol{y}') \succ (\boldsymbol{x}, \boldsymbol{y})$. Applying identical reasoning in both directions for the indifference case establishes $(\boldsymbol{x}', \boldsymbol{y}') \sim_{DL} (\boldsymbol{x}, \boldsymbol{y}) \Rightarrow (\boldsymbol{x}', \boldsymbol{y}') \sim (\boldsymbol{x}, \boldsymbol{y})$. ■

Theorem 1 underscores the robustness of $\succcurlyeq_{DL}$. To appreciate the implications, it is instructive to contrast dynamic comparisons with their static counterparts. The static Lorenz criterion evaluates states, declaring $\boldsymbol{x}$ more equal than $\boldsymbol{x}'$ regardless of origins. The dynamic framework evaluates changes, assessing whether the transition to $\boldsymbol{y}$ from $\boldsymbol{x}$ represents a more potent reduction (or smaller exacerbation) in inequality than the transition to $\boldsymbol{y}'$ from $\boldsymbol{x}'$.

The perspective is dynamic rather than static. Static Lorenz dominance ranks levels; dynamic Lorenz dominance ranks changes. Example 5 illustrates two points: (i) starting from greater inequality, the same transfer reduces inequality more, since it sidesteps rank-reshuffling; (ii) a transition may deliver a larger inequality reduction even if its final distribution is statically less equal than another.

Example 5 (Equality reduction despite higher absolute equality)**.** Consider two populations of three individuals with a fixed total income of 15. We apply a progressive transfer of $t = 1.5$ from rank 3 to rank 1 in both cases.

- Sequence 1 (highly unequal): $\boldsymbol{x} = (1, 4, 10)$. The transfer yields $\boldsymbol{y} = (2.5, 4, 8.5)$.
- Sequence 2 (highly equal): $\boldsymbol{x}' = (4, 5, 6)$. The transfer yields $\boldsymbol{y}' = (5.5, 5, 4.5)$, which reorders to $\boldsymbol{y}' = (4.5, 5, 5.5)$.

First, compare the static Lorenz curves. The primed sequence is more equal at the start and remains more equal at the end.

Now, evaluate the dynamic equalizing effect using the rank-based increment vectors:

$$\boldsymbol{\tau} = (1.5, 0, -1.5) \quad \text{and} \quad \boldsymbol{\tau}' = (0.5, 0, -0.5)$$

Note that for Sequence 2, although 1.5 units were moved, the "effective" rank-based change is only 0.5 because the donor and recipient swapped positions. The cumulative income changes are:

$$(1.5, 1.5, 0) \quad \text{and} \quad (0.5, 0.5, 0).$$

Because $d_k(x, y) \geq d_k(x', y')$ for all k (with strict inequalities at $k = 1, 2$), we have:

$$(x, y) \succ_{DL} (x', y').$$

This single example demonstrates two points. First, the point of departure matters: the same $3 \to 1$ transfer is more effective in the unequal society because it doesn't cause a "leapfrog" effect. Second, Sequence 1 represents a strictly larger reduction in inequality (greater equalization), even though Sequence 2 is strictly more equal in every period.

Comparison with identical initial distributions. When initial distributions are identical ($\boldsymbol{x} = \boldsymbol{x}'$), the condition $(\boldsymbol{x}, \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{y})$ simplifies to whether $\boldsymbol{y}'$ statically Lorenz dominates $\boldsymbol{y}$. This establishes a vital consistency property: the dynamic ranking of transitions from a common baseline aligns perfectly with the static ranking of their endpoints. This allows us to compare the relative merit of two policies even if neither represents a clear static improvement over the status quo. Specifically, if $\boldsymbol{y}'$ dominates $\boldsymbol{y}$, the transition to $\boldsymbol{y}'$ is definitively a larger reduction in inequality than the transition to $\boldsymbol{y}$, regardless of whether the Lorenz curves of $\boldsymbol{y}$ or $\boldsymbol{y}'$ cross that of the baseline $\boldsymbol{x}$. Because the common initial term cancels out, this comparative ranking depends solely on the final distributions, isolating the effectiveness of the change from the initial conditions.

3 Distributions with fixed population size and varying total income

We now extend our analysis to sequences where the population size n is fixed, but total income may vary. Let $\mathcal{S}_n$ denote the space of all such sequences. We consider dynamic inequality pre-orders defined on $\mathcal{S}_n$.

Definition 5. A *dynamic inequality pre-order* on $\mathcal{S}_n$ is a reflexive and transitive binary relation $\succcurlyeq$ on $\mathcal{S}_n$.

3.1 Invariance and extensions

Dynamic Scale Invariance (DSI). For each $\boldsymbol{x}$, $\boldsymbol{y}$, and $\alpha, \beta > 0$, $(\boldsymbol{x}, \boldsymbol{y}) \sim (\alpha\boldsymbol{x}, \beta\boldsymbol{y})$.

The intuition behind DSI extends the standard static scale invariance axiom—which states that scaling an income distribution by a strictly positive constant does not alter its underlying inequality—to a dynamic setting in two logical steps.

First, consider a transition from an initial distribution $\boldsymbol{x}$ to a final distribution $\boldsymbol{y}$. Because the endpoints $\boldsymbol{y}$ and $\alpha\boldsymbol{y}$ possess identical static inequality, the *change* in inequality experienced when transitioning from $\boldsymbol{x}$ to $\boldsymbol{y}$ must equal the change experienced when transitioning from $\boldsymbol{x}$ to $\alpha\boldsymbol{y}$. This yields an asymmetric invariance condition: $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x}, \alpha\boldsymbol{y})$.

Second, applying this reasoning symmetrically to the initial period, the distributions $\boldsymbol{x}$ and $\beta\boldsymbol{x}$ also share identical static inequality. Therefore, the transition from $\boldsymbol{x}$ to $\alpha\boldsymbol{y}$ must represent the exact same dynamic inequality change as the transition from $\beta\boldsymbol{x}$ to $\alpha\boldsymbol{y}$. Combining these steps yields the formal definition of DSI: $(\boldsymbol{x}, \boldsymbol{y}) \sim (\beta\boldsymbol{x}, \alpha\boldsymbol{y})$. Under DSI, dynamic inequality evaluations are entirely independent of proportional income scaling in either period.

Furthermore, DSI naturally particularizes to static scale invariance: applying it to a null transition $(\boldsymbol{x}, \boldsymbol{x})$—representing zero inequality change—yields $(\boldsymbol{x}, \boldsymbol{x}) \sim (\boldsymbol{x}, \alpha\boldsymbol{x})$. This equivalence confirms that the original and scaled distributions are equally equal.

We formulate a completely analogous extension for translation invariance, a property characterizing static absolute inequality measures wherein uniform additions ($\boldsymbol{y} = \boldsymbol{x} + \alpha\mathbf{1}$ for $\mathbf{1} = (1, \dots, 1)$) leave inequality unaltered. Symmetrically extending this logic to independent lump-sum transfers in both periods yields the following axiom:

Dynamic Translation Invariance (DTI). For each $\boldsymbol{x}$, $\boldsymbol{y}$, and $\alpha, \beta > 0$, $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x} + \alpha\mathbf{1}, \boldsymbol{y} + \beta\mathbf{1})$.

Like its scale-invariant counterpart, DTI implies that dynamic inequality evaluations are strictly independent of absolute, uniform income shifts in either the baseline or target period, recovering static translation invariance via $(\boldsymbol{x}, \boldsymbol{x}) \sim (\boldsymbol{x}, \boldsymbol{x} + \alpha\mathbf{1})$.

3.2 Relative and absolute dynamic Lorenz dominance

The base dynamic Lorenz dominance criterion $\succcurlyeq_{DL}$ adapts to varying total incomes through relative and absolute variants.

Definition 6 (Relative dynamic Lorenz dominance)**.** A sequence $(\boldsymbol{x}', \boldsymbol{y}')$ *relatively DL-dominates* $(\boldsymbol{x}, \boldsymbol{y})$, denoted $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{rDL} (\boldsymbol{x}, \boldsymbol{y})$, if

$$\sum_{i=1}^{k} \left(\frac{y'_{[i]}}{\mu_{\boldsymbol{y}'}} - \frac{x'_{[i]}}{\mu_{\boldsymbol{x}'}} \right) \geq \sum_{i=1}^{k} \left(\frac{y_{[i]}}{\mu_{\boldsymbol{y}}} - \frac{x_{[i]}}{\mu_{\boldsymbol{x}}} \right), \quad \forall k,$$

where $\mu_z = \frac{1}{n} \sum_i z_i$ is the mean of distribution $\boldsymbol{z}$.

Definition 7 (Absolute dynamic Lorenz dominance)**.** A sequence $(\boldsymbol{x}', \boldsymbol{y}')$ *absolutely DL-dominates* $(\boldsymbol{x}, \boldsymbol{y})$, denoted $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{aDL} (\boldsymbol{x}, \boldsymbol{y})$, if

$$\sum_{i=1}^{k} \left[y'_{[i]} - \mu_{\boldsymbol{y}'} - (x'_{[i]} - \mu_{\boldsymbol{x}'}) \right] \geq \sum_{i=1}^{k} \left[y_{[i]} - \mu_{\boldsymbol{y}} - (x_{[i]} - \mu_{\boldsymbol{x}}) \right], \quad \forall k.$$

These criteria extend fixed-income dominance by imposing distinct invariance requirements. The relative criterion ($\succcurlyeq_{rDL}$) tracks the evolution of income shares by mean-normalizing, effectively comparing changes in the standard relative Lorenz curves. This normalization systematically removes aggregate income levels, ensuring $\succcurlyeq_{rDL}$ satisfies DSI.

Conversely, the absolute criterion ($\succcurlyeq_{aDL}$) tracks changes in absolute income deviations from the mean, effectively comparing absolute Lorenz curves. Because deviations $(z_{[i]} - \mu_z)$ are invariant to uniform lump-sum transfers, $\succcurlyeq_{aDL}$ satisfies DTI. Both criteria reduce to the base $\succcurlyeq_{DL}$ when total incomes are perfectly fixed across periods.

Lemma 6. *The pre-order $\succcurlyeq_{rDL}$ defined on $\mathcal{S}_n$ satisfies* S, DPDTP, *and* DSI.

Proof. By definition, $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{rDL} (\boldsymbol{x}, \boldsymbol{y})$ if and only if the mean-normalized sequences satisfy $\succcurlyeq_{DL}$.

S. Permutations of $\boldsymbol{x}$ and $\boldsymbol{y}$ preserve their respective means. Thus, permuting the raw distributions yields an exact permutation of the mean-normalized distributions. Since the base relation $\succcurlyeq_{DL}$ satisfies S (Lemma 3), $\succcurlyeq_{rDL}$ immediately satisfies S.

DPDTP. For sequences restricted to fixed total income, all distributions share a common mean $\mu > 0$, causing the relative criterion to coincide exactly with the base criterion ($\succcurlyeq_{rDL} \Leftrightarrow \succcurlyeq_{DL}$). Since $\succcurlyeq_{DL}$ satisfies DPDTP (Lemma 3), $\succcurlyeq_{rDL}$ inherits this property on the restricted domain.

DSI. For any arbitrary scalars $\alpha, \beta > 0$, the scaled distributions $\alpha\boldsymbol{x}$ and $\beta\boldsymbol{y}$ yield means $\alpha\mu_{\boldsymbol{x}}$ and $\beta\mu_{\boldsymbol{y}}$. Because $\alpha x_{[i]}/(\alpha\mu_{\boldsymbol{x}}) = x_{[i]}/\mu_{\boldsymbol{x}}$ (and similarly for $\boldsymbol{y}$), mean-normalized vectors are invariant to scalar multiplication. Hence, the evaluation remains identical and $(\boldsymbol{x}, \boldsymbol{y}) \sim_{rDL} (\alpha\boldsymbol{x}, \beta\boldsymbol{y})$. ∎

Lemma 7. *The pre-order $\succcurlyeq_{aDL}$ defined on $\mathcal{S}_n$ satisfies S, DPDTP, and DTI.*

The proof employs an identical reduction strategy to Lemma 6 and is omitted.

3.3 Extensions to relative and absolute dynamic Lorenz dominance

Definition 8 (DL-consistency on $\mathcal{S}_n$)**.** A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}_n$ is *rDL-consistent* if the strict ($\succ$) and indifference ($\sim$) relations of $\succcurlyeq$ perfectly mirror those of $\succcurlyeq_{rDL}$.

(An analogous definition holds for aDL-consistency using $\succcurlyeq_{aDL}$).

Theorem 2. *A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}_n$ satisfies S, DPDTP, and DSI if and only if it is rDL-consistent.*

Proof. ($\Leftarrow$) If $\succcurlyeq$ is rDL-consistent, its relations completely mirror $\succcurlyeq_{rDL}$. Since $\succcurlyeq_{rDL}$ satisfies S, DPDTP, and DSI (Lemma 6), $\succcurlyeq$ inherits all three axioms.

($\Rightarrow$) Suppose $\succcurlyeq$ satisfies S, DPDTP, and DSI. Consider any arbitrary sequences $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{S}_n$. For a chosen target mean $\mu^* > 0$, define scaled distributions $\hat{\boldsymbol{z}} = (\mu^*/\mu_{\boldsymbol{z}})\boldsymbol{z}$ for $\boldsymbol{z} \in \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}', \boldsymbol{y}'\}$, thereby aligning all distributions to the exact same total income.

By DSI, the dynamic evaluation is invariant to these proportional scalings:

$$(\boldsymbol{x}, \boldsymbol{y}) \sim (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \quad \text{and} \quad (\boldsymbol{x}', \boldsymbol{y}') \sim (\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}'). \tag{3}$$

Now suppose $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq_{rDL} (\boldsymbol{x}, \boldsymbol{y})$. By definition, the normalized, fixed-income sequences satisfy $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}') \succcurlyeq_{DL} (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. Because $\succcurlyeq$ satisfies S and DPDTP, Theorem 1 dictates it must be DL-consistent on the fixed-income domain, meaning the base dominance directly implies $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}') \succcurlyeq (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$. Finally, transitivity applied to (3) yields $(\boldsymbol{x}', \boldsymbol{y}') \succcurlyeq (\boldsymbol{x}, \boldsymbol{y})$. Identical logic holds for strict dominance ($\succ$) and indifference ($\sim$). Thus, $\succcurlyeq$ is rDL-consistent. ∎

Theorem 3. *A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}_n$ satisfies S, DPDTP, and DTI if and only if it is aDL-consistent.*

The proof for Theorem 3 follows a reduction strategy analogous to Theorem 2, substituting additive translations for multiplicative scalars. Let $\mu^* \geq \max\{\mu_{\boldsymbol{x}}, \mu_{\boldsymbol{y}}, \mu_{\boldsymbol{x}'}, \mu_{\boldsymbol{y}'}\}$ be a sufficiently large target mean, and define non-negative translations $\hat{\boldsymbol{z}} = \boldsymbol{z} + (\mu^* - \mu_{\boldsymbol{z}})\mathbf{1}$ for all sequences to strictly preserve the non-negative domain $\mathbb{R}^n_+$. These translated distributions share the exact same total income. Invoking DTI in place of DSI maps the fixed-income dominance back to the original sequences, establishing equivalence with aDL-consistency.

4 Distributions with varying population size and total income

Let $\mathcal{S}$ be the space of all sequences with varying population size and total income. We now consider dynamic inequality pre-orders defined on $\mathcal{S}$, extending the relative ($\succcurlyeq_{rDL}$) and absolute ($\succcurlyeq_{aDL}$) dynamic Lorenz dominance criteria to this unrestricted domain.

4.1 Population principle and extensions

Given a distribution $\boldsymbol{x} = (x_1, \ldots, x_n)$ and integer $m \geq 1$, the *m-replica* is

$$\boldsymbol{x}^{\{m\}} = (\underbrace{x_1, \ldots, x_1}_{m \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{m \text{ times}}).$$

Dynamic Population Principle (DPP). For each $\boldsymbol{x}$, $\boldsymbol{y}$, l, and m, $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x}^{\{l\}}, \boldsymbol{y}^{\{m\}})$.

The intuition behind DPP extends the standard static population principle (Dalton, 1920)—which states that replicating a population does not alter its inequality—to a dynamic setting in two steps.

First, consider a transition from $\boldsymbol{x}$ to $\boldsymbol{y}$. Because the target distribution $\boldsymbol{y}$ and its m-replica $\boldsymbol{y}^{\{m\}}$ possess identical static inequality, the change in inequality experienced transitioning from $\boldsymbol{x}$ to $\boldsymbol{y}$ must equal the change transitioning from $\boldsymbol{x}$ to $\boldsymbol{y}^{\{m\}}$. This yields the asymmetric invariance: $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x}, \boldsymbol{y}^{\{m\}})$.

Second, applying this logic symmetrically to the baseline period, $\boldsymbol{x}$ and $\boldsymbol{x}^{\{l\}}$ exhibit identical static inequality. Therefore, transitioning from $\boldsymbol{x}$ to $\boldsymbol{y}^{\{m\}}$ must represent the exact same dynamic inequality change as transitioning from $\boldsymbol{x}^{\{l\}}$ to $\boldsymbol{y}^{\{m\}}$. Combining these steps yields DPP: $(\boldsymbol{x}, \boldsymbol{y}) \sim (\boldsymbol{x}^{\{l\}}, \boldsymbol{y}^{\{m\}})$. Under this axiom, dynamic inequality evaluations are independent of population replications in either period.

Furthermore, DPP particularizes to the static population principle: applying it to the null transition $(\boldsymbol{x}, \boldsymbol{x})$ yields $(\boldsymbol{x}, \boldsymbol{x}) \sim (\boldsymbol{x}, \boldsymbol{x}^{\{m\}})$, confirming that the original and replicated distributions exhibit identical static inequality.

We now formally verify that $\succcurlyeq_{rDL}$ satisfies these axioms on the unrestricted domain $\mathcal{S}$.

Lemma 8. *The pre-order $\succcurlyeq_{rDL}$ defined on $\mathcal{S}$ satisfies S, DPDTP, DSI, and DPP.*

Proof. We verify each property on the variable-population domain $\mathcal{S}$.

DPP. Population replication leaves standard Lorenz curves unaltered; for any replication factors l and m, the normalized partial sums at any population proportion $p \in [0, 1]$ remain identical. Thus, $L_{\boldsymbol{x}^{\{l\}}}(p) = L_{\boldsymbol{x}}(p)$ and $L_{\boldsymbol{y}^{\{m\}}}(p) = L_{\boldsymbol{y}}(p)$. Consequently, the relative dynamic Lorenz curves match perfectly for the replicated and original sequences: $L_{\boldsymbol{y}^{\{m\}}}(p) - L_{\boldsymbol{x}^{\{l\}}}(p) = L_{\boldsymbol{y}}(p) - L_{\boldsymbol{x}}(p)$ for all p. Hence, $(\boldsymbol{x}, \boldsymbol{y}) \sim_{rDL} (\boldsymbol{x}^{\{l\}}, \boldsymbol{y}^{\{m\}})$.

S and DPDTP. Analogous to the arguments in the proof of Lemma 6.

DSI. For scalars $\alpha, \beta > 0$, we must show $(\boldsymbol{x}, \boldsymbol{y}) \sim_{rDL} (\alpha\boldsymbol{x}, \beta\boldsymbol{y})$. First, normalize population sizes to a common multiple $N = n_{\boldsymbol{x}} n_{\boldsymbol{y}}$. The expanded sequences $(\boldsymbol{x}^{\{n_y\}}, \boldsymbol{y}^{\{n_x\}})$ and $(\alpha\boldsymbol{x}^{\{n_y\}}, \beta\boldsymbol{y}^{\{n_x\}})$ share the exact same population size N. By fixed-population DSI (Lemma 6), $(\boldsymbol{x}^{\{n_y\}}, \boldsymbol{y}^{\{n_x\}}) \sim_{rDL} (\alpha\boldsymbol{x}^{\{n_y\}}, \beta\boldsymbol{y}^{\{n_x\}})$. By DPP, these expanded sequences are indifferent to their original variable-population counterparts. Transitivity immediately yields $(\boldsymbol{x}, \boldsymbol{y}) \sim_{rDL} (\alpha\boldsymbol{x}, \beta\boldsymbol{y})$. ∎

DL-consistency on $\mathcal{S}$ is defined analogously to Definition 8.

Theorem 4. *A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}$ satisfies S, DPDTP, DSI, and DPP if and only if it is rDL-consistent.*

Proof. ($\Leftarrow$) If $\succcurlyeq$ is rDL-consistent, its relations perfectly mirror $\succcurlyeq_{rDL}$. Since $\succcurlyeq_{rDL}$ satisfies S, DPDTP, DSI, and DPP (Lemma 8), $\succcurlyeq$ inherits these axioms.

($\Rightarrow$) Suppose $\succcurlyeq$ satisfies the four axioms. For any sequences $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{S}$, we use a two-step normalization to map them to the fixed-population, fixed-income domain.

First, let N be a common multiple of the populations $n_{\boldsymbol{x}}, n_{\boldsymbol{y}}, n_{\boldsymbol{x}'}, n_{\boldsymbol{y}'}$. We construct expanded distributions by replicating each to size N (e.g., $\tilde{\boldsymbol{x}} = \boldsymbol{x}^{\{N/n_x\}}$). By DPP, the evaluation is invariant to these replications:

$$(\boldsymbol{x}, \boldsymbol{y}) \sim (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \quad \text{and} \quad (\boldsymbol{x}', \boldsymbol{y}') \sim (\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{y}}').$$

Second, select an arbitrary target mean $\mu^* > 0$. We scale the expanded distributions to this common mean by defining $\hat{z} = (\mu^*/\mu_{\tilde{z}})\tilde{z}$ for each $\tilde{z} \in \{\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}'\}$. These fully normalized distributions now share the exact same population N and total income. By DSI, the evaluation is invariant to these scalings:

$$(\tilde{x}, \tilde{y}) \sim (\hat{x}, \hat{y}) \quad \text{and} \quad (\tilde{x}', \tilde{y}') \sim (\hat{x}', \hat{y}').$$

Now suppose $(x', y') \succcurlyeq_{rDL} (x, y)$. Because $\succcurlyeq_{rDL}$ evaluates normalized income shares and population proportions, it is invariant to both replication and scaling. Thus, the fully normalized sequences satisfy the base criterion: $(\hat{x}', \hat{y}') \succcurlyeq_{DL} (\hat{x}, \hat{y})$. Because $\succcurlyeq$ satisfies S and DPDTP, Theorem 1 dictates it must be DL-consistent on this restricted domain, implying $(\hat{x}', \hat{y}') \succcurlyeq (\hat{x}, \hat{y})$. Transitivity applied through the DPP and DSI indifferences unwinds the normalizations, yielding $(x', y') \succcurlyeq (x, y)$. Identical logic holds for strict dominance ($\succ$) and indifference ($\sim$), confirming rDL-consistency. ■

The analogues of Lemma 8 and Theorem 4 for the absolute case can be established similarly.

Lemma 9. *The pre-order $\succcurlyeq_{aDL}$ defined on $\mathcal{S}$ satisfies S, DPDTP, DTI, and DPP.*

Theorem 5. *A dynamic inequality pre-order $\succcurlyeq$ on $\mathcal{S}$ satisfies S, DPDTP, DTI, and DPP if and only if it is aDL-consistent.*

5 Concluding remarks

This paper introduces a formal framework for inequality measurement that advances the theory from a standard static setting to a dynamic one. For sequences with fixed total income and population size, we characterized a core dynamic pre-order grounded in the DPDTP (Dynamic Pigou-Dalton Transfer Principle), an axiom that serves as the dynamic extension of the classical Pigou-Dalton principle to evaluate the magnitude of inequality changes. We subsequently extended these base results to environments with heterogeneous total incomes via novel dynamic invariance notions—yielding relative and absolute dynamic Lorenz dominance criteria—and generalized the framework to varying population sizes by introducing a dynamic population principle.

The analysis in this paper points to the following paths for further investigation.

First, the new dynamic framework introduces novel dimensions in the ways we evaluate income transformations and economic growth. In a static setting, evaluating progressive transfers relies exclusively on their direction. In our dynamic setting, the *intensity* of the transfer plays an essential role, as greater intensity is directly associated with a greater change in inequality. In this paper, our evaluation of transfer intensity is eminently expressed in *absolute* terms, as evidenced by DPDTP (see also Figure 1). However, alternative *relative* measures based on percentages can also be perceived as natural. While our notion of DPDTP aligns with absolute measures of transfer intensity, such relative alternatives would require a suitable modification of the core axiom, consequently altering the characterization of the dynamic inequality measure. Thus, one can identify at least four distinct dimensions relevant to measuring dynamic inequality changes: relative versus absolute transfer intensity, coupled with the standard invariance notions applied to widespread income growth or degrowth (scale versus translation invariance). While we have considered both dimensions for the latter, we restricted our attention to a fundamentally absolute way of measuring transfer intensity.

Second, future work should systematically explore the definition of dynamic inequality indices consistent with our core dynamic pre-orders, and their relationship to standard families of static inequality indices.

A *dynamic inequality index* is defined as a real-valued map $J(\boldsymbol{x}, \boldsymbol{y})$ that evaluates the change in inequality in the transition $\boldsymbol{x} \to \boldsymbol{y}$. We can anchor this evaluation to the purely static case by comparing the transition $\boldsymbol{x} \to \boldsymbol{y}$ with the null transition $\boldsymbol{x} \to \boldsymbol{x}$: if $J(\boldsymbol{x}, \boldsymbol{y}) > J(\boldsymbol{x}, \boldsymbol{x}) = 0$, then $\boldsymbol{y}$ exhibits more inequality than $\boldsymbol{x}$.

A natural question is whether a $\succcurlyeq_{DL}$-consistent J (defined over the set of transitions with fixed total income and population size) perfectly aligns with a standard static inequality index I satisfying the classic Pigou–Dalton principle, in the following sense:

$$J(\boldsymbol{x}, \boldsymbol{y}) \geq 0 \Leftrightarrow I(\boldsymbol{y}) \geq I(\boldsymbol{x}).$$

It is straightforward to observe that not all such indices will work. Consider a two-person economy and the static variance index, which satisfies the Pigou–Dalton principle. Let $\boldsymbol{x} = (10, 30)$ transition to $\boldsymbol{y} = (20, 20)$, and let $\boldsymbol{x}' = (0, 40)$ transition to $\boldsymbol{y}' = (10, 30)$. Both transitions are purely progressive and feature the exact same absolute rank-based increment vector: $\boldsymbol{y} - \boldsymbol{x} = \boldsymbol{y}' - \boldsymbol{x}' = (10, -10)$. Because $\succcurlyeq_{DL}$ evaluates fixed absolute increments identically regardless of the base distribution, a consistent dynamic index mandates $J(\boldsymbol{x}, \boldsymbol{y}) = J(\boldsymbol{x}', \boldsymbol{y}')$. However, evaluating the transitions via the difference of the static variance yields strictly unequal changes ($I(\boldsymbol{y}) - I(\boldsymbol{x}) = -100 \neq -300 = I(\boldsymbol{y}') - I(\boldsymbol{x}')$), breaking congruence.

There are, however, standard indices that meet the above criteria.

For instance, the static Gini index provides a natural bridge to J when the dynamic index is defined as the difference of Ginis: $J(\boldsymbol{x},\boldsymbol{y}) = G(\boldsymbol{y}) - G(\boldsymbol{x})$. Because this difference (under fixed mean and population) evaluates linearly the absolute rank-based income increments independently of the initial distribution $\boldsymbol{x}$, it satisfies the congruence condition while fully respecting the absolute transfer intensity embedded in $\succcurlyeq_{DL}$.

Third, the relationship between DPDTP and the transfer sensitivity condition of Shorrocks and Foster (1987) suggests a further dimension for future investigation. As discussed in Section 2.5, the two criteria capture orthogonal aspects of transfer intensity: DPDTP rewards larger transfers spanning wider rank gaps, while weak transfer sensitivity rewards transfers occurring lower in the income distribution. Because DPDTP is entirely silent on the income-level position of a transfer, it is fully compatible with weak transfer sensitivity: imposing the latter as an additional axiom would *refine* it, resolving comparisons that DL dominance leaves ambiguous in precisely the same way that Shorrocks and Foster's criterion refines the static Lorenz pre-order by ranking distributions with single-crossing Lorenz curves. Dynamic Lorenz dominance thus remains the foundational core pre-order, admitting a natural strengthening that incorporates positional transfer sensitivity as an additional layer.

A Auxiliary lemmas

A.1 Proof of Lemma 4

Lemma 10. *Suppose that $(\boldsymbol{y},\boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x},\boldsymbol{x}')$. Let $\boldsymbol{\tau}^{\boldsymbol{y}} = \boldsymbol{y}' - \boldsymbol{y}$ and $\boldsymbol{\tau}^{\boldsymbol{x}} = \boldsymbol{x}' - \boldsymbol{x}$ be the rank-based increment vectors. Construct the sequences*

$$\boldsymbol{y} = \boldsymbol{y}^{(1)} \longrightarrow \boldsymbol{y}^{(2)} \longrightarrow \cdots \longrightarrow \boldsymbol{y}^{(n)} = \boldsymbol{y}'$$

and

$$\boldsymbol{x} = \boldsymbol{x}^{(1)} \longrightarrow \boldsymbol{x}^{(2)} \longrightarrow \cdots \longrightarrow \boldsymbol{x}^{(n)} = \boldsymbol{x}',$$

where $\boldsymbol{y}^{(k+1)}$ (respectively, $\boldsymbol{x}^{(k+1)}$) is obtained from $\boldsymbol{y}^{(k)}$ (respectively, $\boldsymbol{x}^{(k)}$) by applying the elementary transfers that clear the net increment of the individual in rank k.

Specifically, the matching protocol for the transfers is defined as follows:

- *For the $\boldsymbol{x}$-sequence: If rank k is a net recipient ($\tau^{\boldsymbol{x}}_k \geq 0$), donors are chosen as close as possible to rank k (i.e., lowest available ranks $> k$) until the donated quantities match the total received. If rank k is a net donor ($\tau^{\boldsymbol{x}}_k < 0$), recipients are chosen as* *far as possible from rank k (i.e., highest available ranks).*

- *For the* $\boldsymbol{y}$*-sequence: If rank k is a net recipient (*$\tau_k^{\boldsymbol{y}} \geq 0$*), donors are chosen as far as possible from rank k (i.e., highest available ranks). If rank k is a net donor (*$\tau_k^{\boldsymbol{y}} < 0$*), recipients are chosen as close as possible to rank k (i.e., lowest available ranks* $> k$*).*

Then

$$(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}), \quad \textit{for all } k \in \{2, \ldots, n\}.$$

Proof. Since $(\boldsymbol{y}, \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{x}')$, we have

$$d_j(\boldsymbol{y}, \boldsymbol{y}') \geq d_j(\boldsymbol{x}, \boldsymbol{x}'), \quad \text{for all } j \in \{1, \ldots, n\},$$

where $d_j(\boldsymbol{y}, \boldsymbol{y}') = \sum_{i=1}^{j} \tau_i^{\boldsymbol{y}}$ and $d_j(\boldsymbol{x}, \boldsymbol{x}') = \sum_{i=1}^{j} \tau_i^{\boldsymbol{x}}$.

At step k, the distributions $\boldsymbol{y}^{(k)}$ and $\boldsymbol{x}^{(k)}$ represent the states after the net increments for ranks $1, \ldots, k-1$ have been fully matched. Let

$$P_k^{\boldsymbol{y}} = \sum_{i=1}^{k-1} \tau_i^{\boldsymbol{y}} = d_{k-1}(\boldsymbol{y}, \boldsymbol{y}') \quad \text{and} \quad P_k^{\boldsymbol{x}} = \sum_{i=1}^{k-1} \tau_i^{\boldsymbol{x}} = d_{k-1}(\boldsymbol{x}, \boldsymbol{x}')$$

denote the accumulated net surplus (or deficit) up to rank $k-1$. Global dominance guarantees $P_k^{\boldsymbol{y}} \geq P_k^{\boldsymbol{x}}$.

For any rank $j < k$, the increments are resolved to their final values, so

$$d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) = d_j(\boldsymbol{y}, \boldsymbol{y}') \geq d_j(\boldsymbol{x}, \boldsymbol{x}') = d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}).$$

Thus, we only need to evaluate $j \geq k$. Based on the extremal matching rules, we derive the exact closed-form expressions for the intermediate cumulative curves at rank $j \geq k$:

Bounding $\boldsymbol{y}^{(k)}$. There are two cases to consider.

$P_k^{\boldsymbol{y}} \geq 0$. In this case, $\boldsymbol{y}$ resolves the surplus using donors from the *highest* available ranks $(n, n-1, \ldots)$. The cumulative gain $d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)})$ is the total initial surplus $P_k^{\boldsymbol{y}}$ minus the mass drawn from donor ranks in $\{k, \ldots, j\}$. Since the protocol selects donors from the highest available ranks, it exhausts the capacity of ranks $\{j+1, \ldots, n\}$ before drawing from $\{k, \ldots, j\}$. The capacity of $\{j+1, \ldots, n\}$ is $-\sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{y}}\}$. The mass drawn from ranks strictly greater than j is exactly

$$\min\left\{P_k^{\boldsymbol{y}}, -\sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{y}}\}\right\}.$$

Because the total mass drawn across all ranks must equal $P^{\boldsymbol{y}}_k$, the mass drawn from $\{k, \dots, j\}$ is

$$P^{\boldsymbol{y}}_k - \min\left\{P^{\boldsymbol{y}}_k, -\sum_{i=j+1}^{n} \min\{0, \tau^{\boldsymbol{y}}_i\}\right\}.$$

Therefore, the cumulative gain up to j is:

$$\begin{aligned} & d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \\ & \quad = P^{\boldsymbol{y}}_k - \left(P^{\boldsymbol{y}}_k - \min\left\{P^{\boldsymbol{y}}_k, -\sum_{i=j+1}^{n} \min\{0, \tau^{\boldsymbol{y}}_i\}\right\}\right) = \min\left\{P^{\boldsymbol{y}}_k, -\sum_{i=j+1}^{n} \min\{0, \tau^{\boldsymbol{y}}_i\}\right\}. \end{aligned} \tag{4}$$

$P^{\boldsymbol{y}}_k < 0$. In this case, $\boldsymbol{y}$ resolves the deficit using recipients from the *lowest* available ranks $(k, k+1, \dots)$. The cumulative deficit at j is simply the initial deficit offset by the capacities filled up to j:

$$d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) = \min\left\{0, P^{\boldsymbol{y}}_k + \sum_{i=k}^{j} \max\{0, \tau^{\boldsymbol{y}}_i\}\right\}.$$

Bounding $\boldsymbol{x}^{(k)}$. We consider two cases.

$P^{\boldsymbol{x}}_k \geq 0$. In this case, $\boldsymbol{x}$ resolves the surplus using donors from the *lowest* available ranks $(k, k+1, \dots)$. The unabsorbed surplus remaining at j is:

$$d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}) = \max\left\{0, P^{\boldsymbol{x}}_k + \sum_{i=k}^{j} \min\{0, \tau^{\boldsymbol{x}}_i\}\right\}.$$

$P^{\boldsymbol{x}}_k < 0$. In this case, $\boldsymbol{x}$ resolves the deficit using recipients from the *highest* available ranks $(n, n-1, \dots)$. Symmetrically to $\boldsymbol{y}$'s surplus rule, the deficit at j is the initial deficit minus the capacities filled from ranks strictly greater than j:

$$d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}) = \max\left\{P^{\boldsymbol{x}}_k, -\sum_{i=j+1}^{n} \max\{0, \tau^{\boldsymbol{x}}_i\}\right\}. \tag{5}$$

We now prove that $d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \geq d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)})$ for all $j \geq k$ by evaluating the sign combinations of $P^{\boldsymbol{y}}_k$ and $P^{\boldsymbol{x}}_k$:

Case A. $P^{\boldsymbol{y}}_k \geq 0$ **and** $P^{\boldsymbol{x}}_k \geq 0$**.** We must show

$$\min\left\{P^{\boldsymbol{y}}_k, -\sum_{i=j+1}^{n}\min\{0,\tau^{\boldsymbol{y}}_i\}\right\} \geq \max\left\{0, P^{\boldsymbol{x}}_k + \sum_{i=k}^{j}\min\{0,\tau^{\boldsymbol{x}}_i\}\right\}.$$

It suffices to show that both arguments of the min on the left-hand side (LHS) are greater than or equal to both arguments of the max on the RHS. First, because $\min\{0,\tau^{\boldsymbol{x}}_i\} \leq 0$, the RHS is bounded above by $P^{\boldsymbol{x}}_k$. Thus, $P^{\boldsymbol{y}}_k \geq P^{\boldsymbol{x}}_k \geq \text{RHS}$. Second, we must show

$$-\sum_{i=j+1}^{n}\min\{0,\tau^{\boldsymbol{y}}_i\} \geq P^{\boldsymbol{x}}_k + \sum_{i=k}^{j}\min\{0,\tau^{\boldsymbol{x}}_i\}. \tag{6}$$

Using the identity $\tau_i = \max\{0,\tau_i\} + \min\{0,\tau_i\}$, we can rewrite the RHS of (6) as:

$$P^{\boldsymbol{x}}_k + \sum_{i=k}^{j}\min\{0,\tau^{\boldsymbol{x}}_i\} = P^{\boldsymbol{x}}_k + \sum_{i=k}^{j}\tau^{\boldsymbol{x}}_i - \sum_{i=k}^{j}\max\{0,\tau^{\boldsymbol{x}}_i\} = d_j(\boldsymbol{x},\boldsymbol{x}') - \sum_{i=k}^{j}\max\{0,\tau^{\boldsymbol{x}}_i\}.$$

Since $\sum_{i=1}^{n}\tau^{\boldsymbol{y}}_i = 0$, we have $d_j(\boldsymbol{y},\boldsymbol{y}') = -\sum_{i=j+1}^{n}\tau^{\boldsymbol{y}}_i$. Thus, the LHS of (6) rewrites as:

$$-\sum_{i=j+1}^{n}\min\{0,\tau^{\boldsymbol{y}}_i\} = -\sum_{i=j+1}^{n}\tau^{\boldsymbol{y}}_i + \sum_{i=j+1}^{n}\max\{0,\tau^{\boldsymbol{y}}_i\} = d_j(\boldsymbol{y},\boldsymbol{y}') + \sum_{i=j+1}^{n}\max\{0,\tau^{\boldsymbol{y}}_i\}.$$

Substituting these into (6) yields:

$$d_j(\boldsymbol{y},\boldsymbol{y}') + \sum_{i=j+1}^{n}\max\{0,\tau^{\boldsymbol{y}}_i\} \geq d_j(\boldsymbol{x},\boldsymbol{x}') - \sum_{i=k}^{j}\max\{0,\tau^{\boldsymbol{x}}_i\}.$$

Because $d_j(\boldsymbol{y},\boldsymbol{y}') \geq d_j(\boldsymbol{x},\boldsymbol{x}')$ by global dominance, and the sums of the non-negative max terms independently shift the LHS strictly up and the RHS strictly down, the inequality holds.

Case B. $P^{\boldsymbol{y}}_k < 0$ **and** $P^{\boldsymbol{x}}_k < 0$**.** We must show

$$\min\left\{0, P^{\boldsymbol{y}}_k + \sum_{i=k}^{j}\max\{0,\tau^{\boldsymbol{y}}_i\}\right\} \geq \max\left\{P^{\boldsymbol{x}}_k, -\sum_{i=j+1}^{n}\max\{0,\tau^{\boldsymbol{x}}_i\}\right\}.$$

It suffices to show that both arguments of the min on the LHS are greater than or equal to both arguments of the max on the RHS. First, 0 is trivially greater than or equal to both arguments on the RHS because $P^{\boldsymbol{x}}_k < 0$ and $-\sum\max\{0,\tau^{\boldsymbol{x}}_i\} \leq 0$. Second,

$$P^{\boldsymbol{y}}_k + \sum_{i=k}^{j}\max\{0,\tau^{\boldsymbol{y}}_i\} \geq P^{\boldsymbol{y}}_k \geq P^{\boldsymbol{x}}_k.$$

Third, we must show

$$P_k^{\boldsymbol{y}} + \sum_{i=k}^{j} \max\{0, \tau_i^{\boldsymbol{y}}\} \geq -\sum_{i=j+1}^{n} \max\{0, \tau_i^{\boldsymbol{x}}\}.$$

Rewriting the terms using $\tau_i = \max\{0, \tau_i\} + \min\{0, \tau_i\}$ yields:

$$d_j(\boldsymbol{y}, \boldsymbol{y}') - \sum_{i=k}^{j} \min\{0, \tau_i^{\boldsymbol{y}}\} \geq d_j(\boldsymbol{x}, \boldsymbol{x}') + \sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{x}}\}.$$

Since $d_j(\boldsymbol{y}, \boldsymbol{y}') \geq d_j(\boldsymbol{x}, \boldsymbol{x}')$, the inequality holds.

Case C. $P_k^{\boldsymbol{y}} \geq 0$ and $P_k^{\boldsymbol{x}} < 0$. Here, $d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \geq 0$ (see (4)), and $d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}) \leq 0$ (see (5)). The inequality $d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \geq d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)})$ trivially holds.

Case D. $P_k^{\boldsymbol{y}} < 0$ and $P_k^{\boldsymbol{x}} \geq 0$. This case is impossible because it violates the dominance assumption $P_k^{\boldsymbol{y}} \geq P_k^{\boldsymbol{x}}$.

Therefore, across all valid surplus tracking configurations, the prescribed extremal matching protocol prevents crossing, guaranteeing that $d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \geq d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)})$ for all $j \in \{1, \ldots, n\}$. This concludes the proof:

$$(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(k)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(k)}). \qquad \blacksquare$$

We are now ready to prove Lemma 4.

Lemma 4. *Suppose $(\boldsymbol{y}, \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{x}')$, where both sequences share total income and population size. Let $\boldsymbol{\tau}^{\boldsymbol{y}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$ be their rank-based increment vectors. There exist sequences of elementary transfer vectors $(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_M)$ and $(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_M)$ such that:*

1. *$\sum_{\ell=1}^{M} \boldsymbol{Q}_\ell = \boldsymbol{\tau}^{\boldsymbol{x}}$ and $\sum_{\ell=1}^{M} \boldsymbol{T}_\ell = \boldsymbol{\tau}^{\boldsymbol{y}}$.*
2. *For each $\kappa \in \{1, \ldots, M\}$, we have*

$$\left(\boldsymbol{y} + \sum_{\ell=1}^{\kappa-1} \boldsymbol{T}_\ell, \boldsymbol{y} + \sum_{\ell=1}^{\kappa} \boldsymbol{T}_\ell\right) \sqsupseteq \left(\boldsymbol{x} + \sum_{\ell=1}^{\kappa-1} \boldsymbol{Q}_\ell, \boldsymbol{x} + \sum_{\ell=1}^{\kappa} \boldsymbol{Q}_\ell\right),$$

where $\sum_{\ell=1}^{0} \boldsymbol{T}_\ell = \boldsymbol{0} = \sum_{\ell=1}^{0} \boldsymbol{Q}_\ell$.

If $(\boldsymbol{y}, \boldsymbol{y}') \succ_{DL} (\boldsymbol{x}, \boldsymbol{x}')$, the relation is strict ($\sqsupset$) for at least one κ.

Proof. Assume $(\boldsymbol{y}, \boldsymbol{y}') \succcurlyeq_{DL} (\boldsymbol{x}, \boldsymbol{x}')$. By Lemma 10, we construct sequences

$$\boldsymbol{y} = \boldsymbol{y}^{(1)} \to \boldsymbol{y}^{(2)} \to \cdots \to \boldsymbol{y}^{(n)} = \boldsymbol{y}'$$

and

$$\boldsymbol{x} = \boldsymbol{x}^{(1)} \to \boldsymbol{x}^{(2)} \to \cdots \to \boldsymbol{x}^{(n)} = \boldsymbol{x}',$$

where each transition $\boldsymbol{y}^{(m)} \to \boldsymbol{y}^{(m+1)}$ (respectively $\boldsymbol{x}^{(m)} \to \boldsymbol{x}^{(m+1)}$) clears the net increment of the individual in rank m. Lemma 10 guarantees that for all $m \in \{2, \dots, n\}$,

$$(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(m)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m)}).$$

To clarify the exposition and build intuition, we proceed incrementally. We first detail the sequence constructions for the first and second steps ($m = 1$ and $m = 2$), illustrating the core mechanics of decomposing rank increments into matching and difference components. With these foundational cases established, we then generalize the argument to an arbitrary m-th step.

First step (transition from $\boldsymbol{y}^{(1)}$ to $\boldsymbol{y}^{(2)}$). In particular, for the first step where $m = 1$, the cumulative gain evaluated at the first rank gives

$$d_1(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \geq d_1(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}). \tag{7}$$

Let $\boldsymbol{\tau}^{\boldsymbol{y}} = \boldsymbol{y}^{(2)} - \boldsymbol{y}^{(1)}$ and $\boldsymbol{\tau}^{\boldsymbol{x}} = \boldsymbol{x}^{(2)} - \boldsymbol{x}^{(1)}$ denote the rank-based increment vectors for the first transition. By (7), $\tau_1^{\boldsymbol{y}} \geq \tau_1^{\boldsymbol{x}}$.

We show that there is a path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ via $\sqsupseteq$.

Case 1. $\tau_1^{\boldsymbol{y}} \geq 0 \geq \tau_1^{\boldsymbol{x}}$. The $\boldsymbol{y}$-transition is progressive (rank 1 receives) or null, while the $\boldsymbol{x}$-transition is regressive (rank 1 gives) or null. Let $(\boldsymbol{T}_\ell^{\boldsymbol{y}})$ and $(\boldsymbol{T}_\ell^{\boldsymbol{x}})$ be the canonical decompositions of $\boldsymbol{\tau}^{\boldsymbol{y}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$. By the mixed case in Definition 3, any progressive transfer dominates any regressive transfer. Padding the shorter sequence with null transfers ensures:

$$\boldsymbol{y}^{(1)} + \sum_{i=1}^{\ell} \boldsymbol{T}_i^{\boldsymbol{y}} \sqsupseteq \boldsymbol{x}^{(1)} + \sum_{i=1}^{\ell} \boldsymbol{T}_i^{\boldsymbol{x}} \quad \text{for all } \ell.$$

Case 2. $\tau_1^{\boldsymbol{y}} \geq \tau_1^{\boldsymbol{x}} > 0$. Both transitions are progressive. We decompose the $\boldsymbol{y}$ increment into two components: $\boldsymbol{\tau}^{\boldsymbol{y}} = \boldsymbol{\tau}^{\text{match}} + \boldsymbol{\tau}^{\text{diff}}$, where $\tau_1^{\text{match}} = \tau_1^{\boldsymbol{x}}$, and $\tau_1^{\text{diff}} = \tau_1^{\boldsymbol{y}} - \tau_1^{\boldsymbol{x}} \geq 0$.

Let $\boldsymbol{\tau}^{\text{match}}$ represent the transfer of $\tau_1^{\boldsymbol{x}}$ to rank 1 using the $\boldsymbol{y}$-sequence protocol (furthest available donors).

We claim that $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$. By Lemma 10, for $j > 1$ we have

$$d_j(\boldsymbol{\tau}^{\boldsymbol{x}}) = \max\left\{0, \tau_1^{\boldsymbol{x}} + \sum_{i=2}^{j} \min\{0, \tau_i^{\boldsymbol{x}}\}\right\} \quad \text{and} \quad d_j(\boldsymbol{\tau}^{\text{match}}) = \min\left\{\tau_1^{\boldsymbol{x}}, -\sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{y}}\}\right\}.$$

We must show $d_j(\boldsymbol{\tau}^{\text{match}}) \geq d_j(\boldsymbol{\tau}^{\boldsymbol{x}})$. It suffices to show that both arguments of the minimum in $d_j(\boldsymbol{\tau}^{\text{match}})$ are greater than or equal to both arguments of the maximum in $d_j(\boldsymbol{\tau}^{\boldsymbol{x}})$.

First, clearly $d_j(\boldsymbol{\tau}^{\text{match}}) \geq 0$ because $\tau_1^{\boldsymbol{x}} > 0$ and the donor capacities $-\min\{0, \tau_i^{\boldsymbol{y}}\}$ are non-negative. Second,

$$\tau_1^{\boldsymbol{x}} \geq \tau_1^{\boldsymbol{x}} + \sum_{i=2}^{j} \min\{0, \tau_i^{\boldsymbol{x}}\},$$

since the sum of donor terms is non-positive. Third, we must show

$$-\sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{y}}\} \geq \tau_1^{\boldsymbol{x}} + \sum_{i=2}^{j} \min\{0, \tau_i^{\boldsymbol{x}}\}. \tag{8}$$

Using the identity $\min\{0, \tau_i\} = \tau_i - \max\{0, \tau_i\}$, the right-hand side rewrites as

$$\tau_1^{\boldsymbol{x}} + \sum_{i=2}^{j} \min\{0, \tau_i^{\boldsymbol{x}}\} = d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) - \sum_{i=2}^{j} \max\{0, \tau_i^{\boldsymbol{x}}\}.$$

Because $\sum_{i=1}^{n} \tau_i^{\boldsymbol{y}} = 0$, the sum of the tail is the negative of the cumulative sum at j: $-\sum_{i=j+1}^{n} \tau_i^{\boldsymbol{y}} = d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$. Thus, the left-hand side of (8) rewrites as

$$-\sum_{i=j+1}^{n} \min\{0, \tau_i^{\boldsymbol{y}}\} = d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) + \sum_{i=j+1}^{n} \max\{0, \tau_i^{\boldsymbol{y}}\}.$$

Substituting these identities into (8) yields

$$d_j(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) + \sum_{i=j+1}^{n} \max\{0, \tau_i^{\boldsymbol{y}}\} \geq d_j(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) - \sum_{i=2}^{j} \max\{0, \tau_i^{\boldsymbol{x}}\}. \tag{9}$$

Because $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ and the $\max\{0, \cdot\}$ terms independently shift the left-hand side strictly up and the right-hand side strictly down, (9) holds for all j. Therefore, $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$.

Let

$$C_j^{\boldsymbol{x}} = \sum_{i=2}^{j} |\min\{0, \tau_i^{\boldsymbol{x}}\}| \quad \text{and} \quad C_j^{\boldsymbol{y}} = \sum_{i=2}^{j} |\min\{0, \tau_i^{\text{match}}\}|$$

denote the cumulative donor mass drawn from ranks 2 through j. Because the total matched transfer is $\tau_1^{\boldsymbol{x}}$, the cumulative gain curves at rank j can be expressed as $d_j(\boldsymbol{\tau}^{\boldsymbol{x}}) = \tau_1^{\boldsymbol{x}} - C_j^{\boldsymbol{x}}$ and $d_j(\boldsymbol{\tau}^{\text{match}}) = \tau_1^{\boldsymbol{x}} - C_j^{\boldsymbol{y}}$.

The established DL dominance $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$ directly implies $C_j^{\boldsymbol{x}} \geq C_j^{\boldsymbol{y}}$ for all j. This inequality means that the ℓ-th quantile of donor mass in $\boldsymbol{x}$ must come from a rank $d_\ell^{\boldsymbol{x}}$ that is less than or equal to the rank $d_\ell^{\boldsymbol{y}}$ of the corresponding ℓ-th quantile in $\boldsymbol{y}$. Thus, $d_\ell^{\boldsymbol{y}} \geq d_\ell^{\boldsymbol{x}}$ for all ℓ. Hence, by the progressive case of Definition 3, there is a path connecting

$(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ via $\sqsupseteq$.

The difference component $\boldsymbol{\tau}^{\text{diff}}$ consists of additional progressive transfers for $\boldsymbol{y}$, which we pair with null transfers for $\boldsymbol{x}$. By the mixed case of Definition 3, progressive dominates null, there is a path connecting $(\boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}}, \boldsymbol{y}^{(2)})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ via $\sqsupseteq$.

Therefore, concatenating the sequences yields a path from $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})$ via $\sqsupseteq$.

Case 3. $0 > \tau_1^y \geq \tau_1^x$**.** Both transitions are regressive. This case can be handled symmetrically to Case 2.

Second step (transition from $\boldsymbol{y}^{(1)}$ to $\boldsymbol{y}^{(3)}$). We evaluate the cumulative transition over the first two ranks. We consider the aggregate transitions $\boldsymbol{y}^{(1)} \to \boldsymbol{y}^{(3)}$ and $\boldsymbol{x}^{(1)} \to \boldsymbol{x}^{(3)}$, which resolve the net increments for both rank 1 and rank 2. Let $\boldsymbol{\tau}^{\boldsymbol{y}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$ denote their respective increment vectors. This is a slight abuse of notation, since these symbols were used in the previous step for the transitions $\boldsymbol{y}^{(1)} \to \boldsymbol{y}^{(2)}$ and $\boldsymbol{x}^{(1)} \to \boldsymbol{x}^{(2)}$.

Let $S_2^{\boldsymbol{y}} = \tau_1^{\boldsymbol{y}} + \tau_2^{\boldsymbol{y}} = d_2(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(3)})$ and $S_2^{\boldsymbol{x}} = \tau_1^{\boldsymbol{x}} + \tau_2^{\boldsymbol{x}} = d_2(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ denote the cumulative sum of increments for the first two ranks. By the assumption of global DL dominance, we have $S_2^{\boldsymbol{y}} \geq S_2^{\boldsymbol{x}}$. We also know $\tau_1^{\boldsymbol{y}} \geq \tau_1^{\boldsymbol{x}}$.

Case 1. $S_2^{\boldsymbol{y}} \geq 0 \geq S_2^{\boldsymbol{x}}$**.** The net flow for $\boldsymbol{y}$ into $\{1, 2\}$ is progressive (or null), while the net flow for $\boldsymbol{x}$ is regressive (or null). We use the canonical decompositions for the $\boldsymbol{y}^{(1)} \to \boldsymbol{y}^{(3)}$ and $\boldsymbol{x}^{(1)} \to \boldsymbol{x}^{(3)}$, padding the shorter sequence with null transfer vectors, which yields, by the mixed case of Definition 3, a path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(3)})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ via $\sqsupseteq$.

Case 2. $S_2^{\boldsymbol{y}} \geq S_2^{\boldsymbol{x}} > 0$**.** Both sequences exhibit a net progressive flow of income into ranks $\{1, 2\}$ from external donors at ranks 3 and above. We decompose the $\boldsymbol{y}$ transition into a matching component $\boldsymbol{\tau}^{\text{match}}$ and a difference component $\boldsymbol{\tau}^{\text{diff}}$.

- Matching component: $\boldsymbol{\tau}^{\text{match}}$ transfers a total mass of $S_2^{\boldsymbol{x}}$ from ranks ≥ 3 into ranks $\{1, 2\}$ using $\boldsymbol{y}$'s donor protocol (furthest available donors). We explicitly define the increments for the recipients to perfectly replicate $\boldsymbol{x}$'s receiver profile on the first two ranks:

$$\tau_1^{\text{match}} = \tau_1^{\boldsymbol{x}} \quad \text{and} \quad \tau_2^{\text{match}} = \tau_2^{\boldsymbol{x}}.$$

Since $\tau_1^{\boldsymbol{x}} + \tau_2^{\boldsymbol{x}} = S_2^{\boldsymbol{x}}$, this exactly exhausts the transferred mass. This construction ensures that the cumulative gains on the first two ranks match trivially:

$$d_1(\boldsymbol{\tau}^{\text{match}}) = d_1(\boldsymbol{\tau}^{\boldsymbol{x}}) \quad \text{and} \quad d_2(\boldsymbol{\tau}^{\text{match}}) = d_2(\boldsymbol{\tau}^{\boldsymbol{x}}). \tag{10}$$

- Difference component: $\boldsymbol{\tau}^{\text{diff}} = \boldsymbol{\tau}^{\boldsymbol{y}} - \boldsymbol{\tau}^{\text{match}}$ accounts for the surplus mass $S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}} \geq 0$.

We have $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$ by (10) and the fact that dominance for $j \geq 3$ follows exactly from the donor tail inequalities established in the first step, shifted to start at rank 3.

To establish a path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ via $\sqsupseteq$, we compare the decompositions of $\boldsymbol{\tau}^{\text{match}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$.

Let $C_j^{\boldsymbol{y}}$ and $C_j^{\boldsymbol{x}}$ denote the cumulative share of the external donor mass $S_2^{\boldsymbol{x}}$ drawn from ranks 3 through j in $\boldsymbol{\tau}^{\text{match}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$, respectively. Since $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$, we have $C_j^{\boldsymbol{y}} \leq C_j^{\boldsymbol{x}}$ for all $j \geq 3$. This implies that the ranks of the external donors can be chosen at least as large in $\boldsymbol{\tau}^{\text{match}}$ as those in $\boldsymbol{\tau}^{\boldsymbol{x}}$.

By construction, $\boldsymbol{\tau}^{\text{match}}$ replicates exactly the net increments $\tau_1^{\boldsymbol{x}}$ and $\tau_2^{\boldsymbol{x}}$. We can therefore align their decompositions by matching their internal and external flows:

If $\tau_1^{\boldsymbol{x}} < 0$, rank 1 is a net donor to rank 2. In our alignment, the individual in rank 1 transfers the exact same amount to rank 2 in both $\boldsymbol{\tau}^{\text{match}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$. Because this internal transfer is identical across both transitions, it trivially preserves the $\sqsupseteq$-nested relation. Since $S_2^{\boldsymbol{x}} = \tau_1^{\boldsymbol{x}} + \tau_2^{\boldsymbol{x}} > 0$ and $\tau_1^{\boldsymbol{x}} < 0$, it must be that $\tau_2^{\boldsymbol{x}} > 0$. Thus, rank 2 absorbs the internal transfer from rank 1 as well as the entirety of the external mass $S_2^{\boldsymbol{x}}$ from ranks 3 and above. Because $C_j^{\boldsymbol{y}} \leq C_j^{\boldsymbol{x}}$ for all $j \geq 3$, we can perfectly align the external progressive transfers such that they share the same receiver (rank 2) but have weakly higher-ranked donors in $\boldsymbol{\tau}^{\text{match}}$.

If $\tau_1^{\boldsymbol{x}} \geq 0$, the argument is symmetric. Any potential internal transfers from rank 2 to rank 1 (which occur if $\tau_2^{\boldsymbol{x}} < 0$) are identical across both transitions, thus trivially preserving the path. The remaining positive increments are filled by the external mass $S_2^{\boldsymbol{x}}$ from ranks 3 and above. These external transfers can be paired exactly as before: they share identical receivers (in ranks 1 or 2) but are drawn from weakly higher-ranked donors in $\boldsymbol{\tau}^{\text{match}}$.

Applying these paired transfers sequentially—identical internal transfers followed by weakly dominating external progressive transfers—we obtain a valid path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ via $\sqsupseteq$.

We now show that the difference component $\boldsymbol{\tau}^{\text{diff}} = \boldsymbol{\tau}^{\boldsymbol{y}} - \boldsymbol{\tau}^{\text{match}}$ consists exclusively of progressive transfers. By definition, the net increments for the first two ranks in $\boldsymbol{\tau}^{\text{diff}}$ are given by:

$$\tau_1^{\text{diff}} = \tau_1^{\boldsymbol{y}} - \tau_1^{\boldsymbol{x}} \quad \text{and} \quad \tau_2^{\text{diff}} = \tau_2^{\boldsymbol{y}} - \tau_2^{\boldsymbol{x}}.$$

Since $\tau_1^{\boldsymbol{y}} \geq \tau_1^{\boldsymbol{x}}$, we have $\tau_1^{\text{diff}} \geq 0$.

The total mass entering ranks $\{1,2\}$ from external donors in $\boldsymbol{\tau}^{\text{diff}}$ is $(\tau_1^{\boldsymbol{y}}+\tau_2^{\boldsymbol{y}})-(\tau_1^{\boldsymbol{x}}+\tau_2^{\boldsymbol{x}}) = S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}} \geq 0$. Because $\boldsymbol{\tau}^{\text{match}}$ simply exhausts a subset $S_2^{\boldsymbol{x}}$ of $\boldsymbol{y}$'s total external donor mass $S_2^{\boldsymbol{y}}$, the remaining external mass $S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}}$ in $\boldsymbol{\tau}^{\text{diff}}$ is drawn strictly from the leftover donors at ranks $j \geq 3$.

To determine the direction of the transfers in the decomposition of $\boldsymbol{\tau}^{\text{diff}}$, we consider the sign of τ_2^{diff}:

- If $\tau_2^{\text{diff}} \geq 0$, then both rank 1 and rank 2 are net receivers in $\boldsymbol{\tau}^{\text{diff}}$. The entire external mass $S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}}$ flows from donor ranks $j \geq 3$ to receiver ranks $r \in \{1,2\}$. Since all donors are ranked strictly higher than all receivers, the decomposition consists entirely of progressive transfers.
- If $\tau_2^{\text{diff}} < 0$, rank 2 acts as a net donor in $\boldsymbol{\tau}^{\text{diff}}$. Because the total net mass $\tau_1^{\text{diff}} + \tau_2^{\text{diff}} = S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}} \geq 0$, rank 1 is the sole receiver, absorbing both the internal transfer $|\tau_2^{\text{diff}}|$ from rank 2 and the external mass $S_2^{\boldsymbol{y}} - S_2^{\boldsymbol{x}}$ from ranks $j \geq 3$. Since all flows are directed into rank 1 from higher ranks (either rank 2 or ranks ≥ 3), the decomposition again consists entirely of progressive transfers.

Because $\boldsymbol{\tau}^{\text{diff}}$ can be decomposed exclusively into elementary progressive transfers, we can pair these transfers with null transfers for $\boldsymbol{x}$. By the mixed case of Definition 3, progressive transfers dominate null transfers. Therefore, sequentially applying the elementary components of $\boldsymbol{\tau}^{\text{diff}}$ yields a valid path connecting $(\boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}}, \boldsymbol{y}^{(3)})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ via $\sqsupseteq$.

Concatenating this path with the previously established path from $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(3)})$ completes the proof for Case 2.

Case 3. $0 > S_2^{\boldsymbol{y}} \geq S_2^{\boldsymbol{x}}$**.** Symmetric to Case 2.

m-th step (transition from $\boldsymbol{y}^{(1)}$ to $\boldsymbol{y}^{(m+1)}$). We evaluate the aggregate transitions $\boldsymbol{y}^{(1)} \to \boldsymbol{y}^{(m+1)}$ and $\boldsymbol{x}^{(1)} \to \boldsymbol{x}^{(m+1)}$. Slightly abusing notation, let $\boldsymbol{\tau}^{\boldsymbol{y}} = \boldsymbol{y}^{(m+1)} - \boldsymbol{y}^{(1)}$ and $\boldsymbol{\tau}^{\boldsymbol{x}} = \boldsymbol{x}^{(m+1)} - \boldsymbol{x}^{(1)}$ denote their respective increment vectors. These transitions resolve the net increments for all ranks 1 through m.

Let $S_m^{\boldsymbol{y}} = \sum_{i=1}^m \tau_i^{\boldsymbol{y}} = d_m(\boldsymbol{\tau}^{\boldsymbol{y}})$ and $S_m^{\boldsymbol{x}} = \sum_{i=1}^m \tau_i^{\boldsymbol{x}} = d_m(\boldsymbol{\tau}^{\boldsymbol{x}})$ denote the cumulative sum of increments for the first m ranks. By Lemma 10, we have $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(m+1)}) \succcurlyeq_{DL} (\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$, which directly implies $d_i(\boldsymbol{\tau}^{\boldsymbol{y}}) \geq d_i(\boldsymbol{\tau}^{\boldsymbol{x}})$ for all ranks i. In particular, evaluated at m, we have $S_m^{\boldsymbol{y}} \geq S_m^{\boldsymbol{x}}$.

Case 1. $S_m^{\boldsymbol{y}} \geq 0 \geq S_m^{\boldsymbol{x}}$**.** The net flow for $\boldsymbol{y}$ into $\{1, \ldots, m\}$ is progressive (or null), while the net flow for $\boldsymbol{x}$ is regressive (or null). We use the canonical decompositions for $\boldsymbol{\tau}^{\boldsymbol{y}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$, padding the shorter sequence with null transfer vectors. By the mixed case of Definition 3, progressive dominates regressive, yielding a path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(m+1)})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$ via $\sqsupseteq$.

Case 2. $S_m^{\boldsymbol{y}} \geq S_m^{\boldsymbol{x}} > 0$**.** Both sequences exhibit a net progressive flow of income into ranks $\{1, \ldots, m\}$ from external donors at ranks $m+1$ and above. We decompose the $\boldsymbol{y}$ transition into a matching component $\boldsymbol{\tau}^{\text{match}}$ and a difference component $\boldsymbol{\tau}^{\text{diff}}$.

- Matching component: $\boldsymbol{\tau}^{\text{match}}$ transfers a total mass of $S_m^{\boldsymbol{x}}$ from ranks $\geq m+1$ into ranks $\{1, \ldots, m\}$ using $\boldsymbol{y}$'s donor protocol (furthest available donors). We explicitly define the increments for the recipients to perfectly replicate $\boldsymbol{x}$'s receiver profile on the first m ranks:
$$\tau_i^{\text{match}} = \tau_i^{\boldsymbol{x}} \quad \text{for all } i \in \{1, \ldots, m\}.$$
Since $\sum_{i=1}^{m} \tau_i^{\boldsymbol{x}} = S_m^{\boldsymbol{x}}$, this exactly exhausts the transferred mass. This construction ensures that the cumulative gains on the first m ranks match trivially:
$$d_i(\boldsymbol{\tau}^{\text{match}}) = d_i(\boldsymbol{\tau}^{\boldsymbol{x}}) \quad \text{for all } i \in \{1, \ldots, m\}. \tag{11}$$
- Difference component: $\boldsymbol{\tau}^{\text{diff}} = \boldsymbol{\tau}^{\boldsymbol{y}} - \boldsymbol{\tau}^{\text{match}}$ accounts for the surplus external mass $S_m^{\boldsymbol{y}} - S_m^{\boldsymbol{x}} \geq 0$.

We have $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$ by (11) and the fact that dominance for $j \geq m+1$ follows exactly from the donor tail inequalities, shifted to start at rank $m+1$.

To establish a path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$ via $\sqsupseteq$, we compare the decompositions of $\boldsymbol{\tau}^{\text{match}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$.

Let $C_j^{\boldsymbol{y}}$ and $C_j^{\boldsymbol{x}}$ denote the cumulative share of the external donor mass $S_m^{\boldsymbol{x}}$ drawn from ranks $m+1$ through j in $\boldsymbol{\tau}^{\text{match}}$ and $\boldsymbol{\tau}^{\boldsymbol{x}}$, respectively. Since $\boldsymbol{\tau}^{\text{match}} \succcurlyeq_{DL} \boldsymbol{\tau}^{\boldsymbol{x}}$, we have $C_j^{\boldsymbol{y}} \leq C_j^{\boldsymbol{x}}$ for all $j \geq m+1$. This implies that the ranks of the external donors can be chosen at least as large in $\boldsymbol{\tau}^{\text{match}}$ as those in $\boldsymbol{\tau}^{\boldsymbol{x}}$.

By construction, $\boldsymbol{\tau}^{\text{match}}$ replicates exactly the net increments $\tau_i^{\boldsymbol{x}}$ for $i \in \{1, \ldots, m\}$. We can therefore align their decompositions by matching their internal and external flows. Any internal transfers strictly within ranks $\{1, \ldots, m\}$ are identical across both transitions, trivially preserving the $\sqsupseteq$-nested relation. The remaining positive increments in $\{1, \ldots, m\}$ are filled by the external mass $S_m^{\boldsymbol{x}}$ from ranks $m+1$ and above. These external transfers can be paired perfectly: they share identical receivers (within $\{1, \ldots, m\}$) but are drawn from weakly higher-ranked donors in $\boldsymbol{\tau}^{\text{match}}$ because $C_j^{\boldsymbol{y}} \leq C_j^{\boldsymbol{x}}$.

Applying these paired transfers sequentially—identical internal transfers followed by weakly dominating external progressive transfers—we obtain a valid path connecting $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ and $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$ via $\sqsupseteq$.

We now show that the difference component $\boldsymbol{\tau}^{\text{diff}} = \boldsymbol{\tau}^{\boldsymbol{y}} - \boldsymbol{\tau}^{\text{match}}$ consists exclusively of progressive transfers. By definition, the net increments for the first m ranks in $\boldsymbol{\tau}^{\text{diff}}$ are $\tau_i^{\text{diff}} = \tau_i^{\boldsymbol{y}} - \tau_i^{\boldsymbol{x}}$. The cumulative sum of these increments up to any rank $h \in \{1, \ldots, m\}$ is:

$$d_h(\boldsymbol{\tau}^{\text{diff}}) = d_h(\boldsymbol{\tau}^{\boldsymbol{y}}) - d_h(\boldsymbol{\tau}^{\boldsymbol{x}}).$$

Because Lemma 10 guarantees $d_h(\boldsymbol{\tau}^{\boldsymbol{y}}) \geq d_h(\boldsymbol{\tau}^{\boldsymbol{x}})$ for all h, we have $d_h(\boldsymbol{\tau}^{\text{diff}}) \geq 0$ for all $h \in \{1, \ldots, m\}$.

The total mass entering ranks $\{1, \ldots, m\}$ from external donors in $\boldsymbol{\tau}^{\text{diff}}$ is $S_m^{\boldsymbol{y}} - S_m^{\boldsymbol{x}} \geq 0$, which is drawn strictly from the leftover donors at ranks $j \geq m + 1$.

Because the cumulative gains $d_h(\boldsymbol{\tau}^{\text{diff}})$ are non-negative for every rank h, there is no net flow of mass to higher ranks. All internal flows within $\{1, \ldots, m\}$ are directed from higher ranks to lower ranks to maintain these non-negative cumulative buffers, and all external mass flows from ranks $\geq m + 1$ into $\{1, \ldots, m\}$. Consequently, the decomposition of $\boldsymbol{\tau}^{\text{diff}}$ consists entirely of progressive transfers.

Because $\boldsymbol{\tau}^{\text{diff}}$ is purely progressive, we can pair its elementary transfers with null transfers for $\boldsymbol{x}$. By the mixed case of Definition 3, sequentially applying the elementary components of $\boldsymbol{\tau}^{\text{diff}}$ yields a valid path connecting $(\boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}}, \boldsymbol{y}^{(m+1)})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$ via $\sqsupseteq$.

Concatenating this path with the previously established path from $(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(1)} + \boldsymbol{\tau}^{\text{match}})$ to $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(m+1)})$ completes the proof for Case 2.

Case 3. $0 > S_m^{\boldsymbol{y}} \geq S_m^{\boldsymbol{x}}$**.** Symmetric to Case 2.

Strict dominance. Finally, suppose $(\boldsymbol{y}, \boldsymbol{y}') \succ_{DL} (\boldsymbol{x}, \boldsymbol{x}')$. By definition, weak dominance holds and there exists at least one rank j^* such that $d_{j^*}(\boldsymbol{y}, \boldsymbol{y}') > d_{j^*}(\boldsymbol{x}, \boldsymbol{x}')$. We have established that the sequences of elementary transfer vectors satisfy the weak relation $\sqsupseteq$ at every step κ, and that $\sum_{\ell=1}^{M} \boldsymbol{T}_\ell = \boldsymbol{\tau}^{\boldsymbol{y}}$ and $\sum_{\ell=1}^{M} \boldsymbol{Q}_\ell = \boldsymbol{\tau}^{\boldsymbol{x}}$. If the weak relation $\sqsupseteq$ held with exact indifference at every step, the cumulative effect of the summed transfers would be identical, implying $d_j(\boldsymbol{y}, \boldsymbol{y}') = d_j(\boldsymbol{x}, \boldsymbol{x}')$ for all ranks j. This directly contradicts the strict global dominance at j^*. Therefore, the relation must be strict ($\sqsupset$) for at least one step κ. ■

References

Atkinson, A. B. (1970) “On the measurement of inequality,” *Journal of Economic Theory*, 2 (3), 244–263, 10.1016/0022-0531(70)90039-6.

Dalton, Hugh (1920) “The Measurement of the Inequality of Incomes,” *The Economic Journal*, 30 (119), 348–361, 10.2307/2223525.

Dasgupta, Partha, Amartya K. Sen, and David Starrett (1973) “Notes on the Measurement of Inequality,” *Journal of Economic Theory*, 6 (2), 180–187, 10.1016/0022-0531(73)90033-1.

Fields, Gary S. and John C. H. Fei (1978) “On Inequality Comparisons,” *Econometrica*, 46 (2), 303–316, 10.2307/1913902.

Fields, Gary S and Efe A Ok (1999) “The measurement of income mobility: An introduction to the literature,” in *Handbook of income inequality measurement*, 557–598: Springer.

Hardy, G. H., J. E. Littlewood, and G. Pólya (1952) *Inequalities*, Cambridge: Cambridge University Press, 2nd edition.

Jenkins, Stephen P. and Philippe Van Kerm (2006) “Trends in income inequality, pro-poor income growth, and income mobility,” *Oxford Economic Papers*, 58 (3), 531–548, 10.1093/oep/gpl014.

Pigou, Arthur Cecil (1912) *Wealth and Welfare*, London: Macmillan and Co. `https://archive.org/details/cu31924032613386`.

Ravallion, Martin and Shaohua Chen (2003) “Measuring pro-poor growth,” *Economics Letters*, 78 (1), 93–99, 10.1016/S0165-1765(02)00205-7.

Rothschild, Michael and Joseph E. Stiglitz (1973) “Some further results on the measurement of inequality,” *Journal of Economic Theory*, 6 (2), 188–204, 10.1016/0022-0531(73)90034-3.

Sen, Amartya (2017) *Collective Choice and Social Welfare*, Cambridge, MA: Harvard University Press.

Shorrocks, Anthony F. (1978) “The measurement of mobility,” *Econometrica*, 46 (5), 1013–1024, 10.2307/1911433.

Shorrocks, Anthony F. and James E. Foster (1987) “Transfer Sensitive Inequality Measures,” *The Review of Economic Studies*, 54 (3), 485–497, 10.2307/2297571.

Son, Hyun H. (2004) “A note on pro-poor growth,” *Economics Letters*, 82 (3), 307–314, 10.1016/j.econlet.2003.08.003.