Journal of Mathematical Economics 75 (2018) 108–115

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Essential equilibrium in normal-form games with perturbed actions and payoffs[☆]

Oriol Carbonell-Nicolau [*], Nathan Wohl

Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick, NJ 08901, USA

ARTICLE INFO

Article history:
Received 22 February 2017
Received in revised form 10 January 2018
Accepted 22 January 2018
Available online 2 February 2018

Keywords:
Discontinuous normal-form game
Equilibrium refinement
Essential equilibrium
Equilibrium existence
Generalized payoff security

ABSTRACT

A Nash equilibrium of a normal-form game G is essential if it is robust to perturbations of G. A game is essential if all of its Nash equilibria are essential. This paper provides conditions on the primitives of a (possibly) discontinuous game that guarantee the generic existence of essential games. Unlike the extant literature, the present analysis allows for perturbations of the players' action spaces, in addition to the standard payoff perturbations.

1. Introduction

A Nash equilibrium of a normal-form game G is essential if it is robust to perturbations of G. For generic games in the collection of all finite-action games with fixed action spaces, all Nash equilibria are essential (*cf.* Wu and Jiang, 1962). This result has been extended to infinite-action games (*e.g.*, Yu, 1999, Carbonell-Nicolau, 2010, 2015, and Scalzo, 2013). Yu (1999) allows for perturbed action spaces and payoff functions, but requires continuity of payoff functions. Carbonell-Nicolau (2010, 2015) and Scalzo (2013) allow for discontinuous payoffs but require fixed action spaces. In this paper we extend the results in Carbonell-Nicolau (2010) by allowing for perturbed payoffs and actions.

The notion of perturbed game used in this note differs from the definition adopted in Yu (1999). We argue in Section 2 that, in the presence of payoff discontinuities, perturbing actions and payoffs as in Yu (1999) poses problems. In fact, under Yu's approach it is easy to construct games whose perturbations do not include strategies that are of particular strategic significance to the players. Our discussion in Section 2 is framed in terms of a very simple example, which showcases the difficulties of the Yu approach and illustrates the intuitive appeal of the definition of a perturbed game proposed here.

2. Preliminaries

A ***normal-form game*** (or simply a ***game***) $G = (X_i, u_i)_{i=1}^N$ consists of a finite number N of players, a nonempty set of actions X_i for each player i, and a payoff function $u_i : X \to \mathbb{R}$ for each player i defined on the set of action profiles $X := \times_{j=1}^N X_j$.

For each player i, let X_i be a nonempty, compact, convex subset of a metric vector space. Let $X := \times_{i=1}^N X_i$ be endowed with the associated product topology. The sets $X_1, \dots, X_N$ will be fixed throughout the analysis. Let $B(X)$ denote the set of bounded maps $f : X \to \mathbb{R}$. Let $K(X_i)$ denote the hyperspace of nonempty, compact, and convex subsets of X_i. Define

$$\boldsymbol{G}_X := \left(\times_{i=1}^N K(X_i)\right) \times B(X)^N.$$

A typical member of $\boldsymbol{G}_X$ is denoted $(Y, u) = (Y_1, \dots, Y_N, u_1, \dots, u_N)$ and can be viewed as a normal-form game $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$.

In Yu (1999), the space $B(X)^N$ is endowed with the metric $\gamma_X : B(X)^N \times B(X)^N \to \mathbb{R}$ defined by

$$\gamma_X((u_1, \dots, u_N), (v_1, \dots, v_N)) := \sum_{i \in N} \sup_{x \in X} |u_i(x) - v_i(x)|, \tag{1}$$

and, for each i, the space $K(X_i)$ is endowed with the Hausdorff metric topology. The associated product metric space $\boldsymbol{G}_X$, endowed with the corresponding product topology, constitutes the space of games considered in Yu (1999). This topology defines the notion of perturbed game used in Yu (1999), and we wish to argue here that this notion is not appropriate in the presence of payoff discontinuities. To illustrate, consider the one-person game $([0, 1], u)$, where $u(x) := 0$ if $x \in [0, 1)$ and $u(1) := 1$, and the sequence

☆ In memory of Nathan Wohl. Thanks to the anonymous referees and Rich McLean for valuable comments.

* Corresponding author.
E-mail address: carbonell-nicolau@rutgers.edu (O. Carbonell-Nicolau).

https://doi.org/10.1016/j.jmateco.2018.01.002
0304-4068/

$\left([0, 1-\frac{1}{n}], u\right)$, which converges to $([0,1], u)$. Arguably, the strategy $x = 1$, which dominates every other strategy, is particularly important in this game, and it seems hard to justify an approximation that does not include this strategy or another strategy that plays a similar role. In particular, the games $([0,1], u)$ and $\left([0, 1-\frac{1}{n}], u\right)$ appear markedly dissimilar, even for large n, and the sequence $\left([0, 1-\frac{1}{n}], u\right)$ does not seem to well-approximate $([0,1], u)$.[1] By contrast, the sequence $\left([0, 1-\frac{1}{n}], v^n\right)$, where $v^n(x) := 0$ if $x \in [0, 1-\frac{1}{n})$ and $v^n(x) := 1-\frac{1}{n}$ if $x \in [1-\frac{1}{n}, 1]$ seems to better approximate $([0,1], u)$ (for large n). Note that for the above topology, while the sequence $\left([0, 1-\frac{1}{n}], u\right)$ converges to $([0,1], u)$, the sequence $\left([0, 1-\frac{1}{n}], v^n\right)$ does *not* converge to $([0,1], u)$. In the next paragraph, we define a topology that is consistent with the idea that games of the form $\left([0, 1-\frac{1}{n}], v^n\right)$ are close to $([0,1], u)$ (for large n) while games of the form $\left([0, 1-\frac{1}{n}], u\right)$ are not.[2]

Given i and $\{Y_i, Z_i\} \subseteq K(X_i)$, let $\mathcal{H}(Y_i, Z_i)$ be the set of all homeomorphisms h_i from Y_i to Z_i such that $h_i(A) \subseteq Z_i$ is convex if and only if $A \subseteq Y_i$ is convex. Let d_X be a compatible metric for X. Let $\mathfrak{G}_X$ represent the set of normal-form games $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$ such that $(Y, u) \in \boldsymbol{G}_X$. Note that a member of $\boldsymbol{G}_X$ uniquely determines a corresponding element of $\mathfrak{G}_X$, while there is a one-to-many mapping between $\mathfrak{G}_X$ and $\boldsymbol{G}_X$. For the members of $\mathfrak{G}_X$, we write $(Y_i, u_i|_{\times_{j=1}^N Y_j})_{i=1}^N$ and (Y, u) indistinctly, which entails a slight abuse of notation. Define the map $\alpha_X : \mathfrak{G}_X \times \mathfrak{G}_X \to \mathbb{R} \cup \{\infty\}$ by

$$\alpha_X((Y,u),(Z,v)) := \inf \left\{ \epsilon > 0 : \exists h \in \times_{i=1}^N \mathcal{H}(Y_i, Z_i) : \sum_{i=1}^N \sup_{x \in Y} |u_i(x) - v_i(h(x))| \le \epsilon \text{ and } \sup_{x\in Y} d_X(h(x), x) \le \epsilon \right\},$$

if $\times_{i=1}^N \mathcal{H}(Y_i, Z_i) \neq \emptyset$, and $\alpha_X((Y,u),(Z,v)) := \infty$ if $\times_{i=1}^N \mathcal{H}(Y_i, Z_i) = \emptyset$. Now define the metric $\rho_X : \mathfrak{G}_X \times \mathfrak{G}_X \to \mathbb{R}$ by $\rho_X((Y,u),(Z,v)) := \min\{\alpha_X((Y,u),(Z,v)), 1\}$.[3] Throughout the sequel, we endow $\mathfrak{G}_X$ with the metric ρ_X.

Remark 1. As illustrated by the previous example, the metric ρ_X differs from the Yu metric. This discrepancy can even be found within the subdomain of continuous games. Indeed, for $X := [0,1]$ and arbitrary u, the sequence of games $([0, \frac{1}{n}], u)$ in $\boldsymbol{G}_X$ converges to $(\{0\}, u)$ in the sense of Yu, and yet this sequence does not converge with respect to ρ_X in $\mathfrak{G}_X$ because none of its members is homeomorphic to the game $(\{0\}, u)$. Thus, convergence in the sense of Yu need not imply convergence with respect to ρ_X. The converse assertion is also true, as illustrated by the discontinuous game from the previous example.

Definition 1. A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is ***upper hemicontinuous at*** $x \in A$ if the following condition is satisfied: for every neighborhood $V_{\Phi(x)}$ of $\Phi(x)$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \subseteq V_{\Phi(x)}$. Φ is ***upper hemicontinuous*** if it is upper hemicontinuous at every point in A.

Definition 2. A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is ***lower hemicontinuous at*** $x \in A$ if the following condition is satisfied: for every open set $V \subseteq B$ with $V \cap \Phi(x) \neq \emptyset$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \cap V \neq \emptyset$. Φ is ***lower hemicontinuous*** if it is lower hemicontinuous at every point in A.

Definition 3. A strategy profile $x = (x_i, x_{-i})$ in X is a ***Nash equilibrium*** of $G = (X_i, u_i)_{i=1}^N$ if $u_i(y_i, x_{-i}) \le u_i(x)$ for every $y_i \in X_i$ and i.

One can define the Nash equilibrium correspondence as a set-valued map

$$\mathcal{E}_X : \mathfrak{G}_X \rightrightarrows X$$

that assigns to each game (Y, u) in $\mathfrak{G}_X$ the set of Nash equilibria of (Y, u), $\mathcal{E}_X(Y, u)$. Given a family of games $\mathfrak{G} \subseteq \mathfrak{G}_X$, the restriction of $\mathcal{E}_X$ to $\mathfrak{G}$ is denoted by $\mathcal{E}_X|_{\mathfrak{G}}$.

Definition 4. Given a class of games $\mathfrak{G} \subseteq \mathfrak{G}_X$, a Nash equilibrium x of $(Y, u) \in \mathfrak{G}$ is an ***essential equilibrium of*** (Y, u) ***relative to*** $\mathfrak{G}$ if for every neighborhood V_x of x there is a neighborhood $V_{(Y,u)}$ of (Y, u) such that for every $(Z, f) \in V_{(Y,u)} \cap \mathfrak{G}$, $V_x \cap \mathcal{E}_X(Z, f) \neq \emptyset$.

Definition 5. Suppose that $\mathfrak{G} \subseteq \mathfrak{G}_X$. A game (Y, u) in $\mathfrak{G}$ is ***essential relative to*** $\mathfrak{G}$ if every pure-strategy Nash equilibrium of (Y, u) is essential relative to $\mathfrak{G}$. When the domain of reference is clear from the context, we shall simply say that (Y, u) is an ***essential game***.

Remark 2. Suppose that $\mathfrak{G} \subseteq \mathfrak{G}_X$. A game (Y, u) in $\mathfrak{G}$ is essential relative to $\mathfrak{G}$ if and only if $\mathcal{E}_X|_{\mathfrak{G}}$ is lower hemicontinuous at (Y, u).

3. The results

The following definition was introduced in Barelli and Soza (2009).

[1] The idea that "good" approximations to an infinite discontinuous game should include strategies that are of particular strategic significance to the players is already discussed in Simon (1987) and Reny (2011) in the context of finite strategic approximations to infinite games.

[2] This is in fact an example in which a game with a dominant strategy can only be approximated, in the new topology, by games with a dominant strategy. This is obviously false about the Yu topology. We conjecture that this property holds in general, and we thank an anonymous referee for bringing up this point.

[3] It is easily seen that $\rho_X((Y,u),(Z,v)) = 0 \Leftrightarrow (Y,u) = (Z,v)$ for all (Y,u) and (Z,v) in $\mathfrak{G}_X$. Also, it is clearly the case that $\rho_X((Y,u),(Z,v)) = \rho_X((Z,v),(Y,u))$ for all (Y,u) and (Z,v) in $\mathfrak{G}_X$. To verify that the triangle inequality holds for ρ_X, fix (Y,u), (Y',u'), and (Y'',u'') in $\mathfrak{G}_X$ and note that given $h^1 \in \times_{i=1}^N \mathcal{H}(Y_i, Y_i')$ and $h^2 \in \times_{i=1}^N \mathcal{H}(Y_i', Y_i'')$,

$$\begin{aligned}\sum_{i=1}^N \sup_{x\in Y} |u_i(x) - u_i''(h^2(h^1(x)))| &= \sum_{i=1}^N \sup_{x\in Y} |u_i(x) - v_i''(x)| \\ &\le \sum_{i=1}^N \sup_{x\in Y} |u_i(x) - v_i'(x)| + \sum_{i=1}^N \sup_{x\in Y} |v_i'(x) - v_i''(x)| \\ &= \sum_{i=1}^N \sup_{x\in Y} |u_i(x) - u_i'(h^1(x))| + \sum_{i=1}^N \sup_{x\in Y} |u_i'(h^1(x)) - u_i''(h^2(h^1(x)))| \\ &= \sum_{i=1}^N \sup_{x\in Y} |u_i(x) - u_i'(h^1(x))| + \sum_{i=1}^N \sup_{x\in Y'} |u_i'(x) - u_i''(h^2(x))|,\end{aligned}$$

where $v_i' : Y \to Y'$ and $v_i'' : Y \to Y''$ are defined by

$$v_i'(x) := u_i'(h^1(x)) \quad \text{and} \quad v_i''(x) := u_i''(h^2(h^1(x))),$$

and

$$\begin{aligned}\sup_{x\in Y} d_X(h^2(h^1(x)), x) &\le \sup_{x\in Y} d_X(h^1(x), x) + \sup_{x\in Y} d_X(h^2(h^1(x)), h^1(x)) \\ &= \sup_{x\in Y} d_X(h^1(x), x) + \sup_{x\in Y'} d_X(h^2(x), x).\end{aligned}$$

Consequently, $\alpha_X((Y,u),(Y'',u'')) \le \alpha_X((Y,u),(Y',u')) + \alpha_X((Y',u'),(Y'',u''))$ and so $\rho_X((Y,u),(Y'',u'')) \le \rho_X((Y,u),(Y',u')) + \rho_X((Y',u'),(Y'',u''))$. Thus, ρ_X is indeed a metric on $\mathfrak{G}_X$.

Definition 6. A game $(Y, u) \in \mathfrak{G}_X$ is ***generalized payoff secure*** if for each $\epsilon > 0$, $x \in Y$, and i, there exist a neighborhood V_x of x in Y and a nonempty-valued, convex-valued, compact-valued, upper hemicontinuous correspondence $\Phi_i : V_x \rightrightarrows Y_i$ such that $u_i(z_i, y_{-i}) > u_i(x) - \epsilon$ for each $z_i \in \Phi_i(y)$ and $y \in V_x$.

We define $\mathfrak{G}_X^*$ as the collection of games (Y, u) in $\mathfrak{G}_X$ satisfying the following:

1. (Y, u) is quasiconcave, *i.e.*, for each i and $x_{-i} \in Y_{-i}$, the map $x_i \mapsto u_i(x_i, x_{-i})$ defined on Y_i is quasiconcave;
2. (Y, u) is generalized payoff secure; and
3. $\sum_{i=1}^N u_i$ is upper semicontinuous on Y.

Theorem 1. *Every member of a residual subset of $\mathfrak{G}_X^*$ is essential.*

Remark 3. Remark 4 shows that $\mathfrak{G}_X$ fails to be a complete space. Consequently, the residual subset of $\mathfrak{G}_X^*$ given by Theorem 1 cannot be shown to be dense in $\mathfrak{G}_X^*$ as an application of the Baire Category Theorem. Given any complete subset $\mathfrak{G}_X'$ of $\mathfrak{G}_X$, however, the collection $\mathfrak{G}_X' \cap \mathfrak{G}_X^*$ has the property that every member of a residual, dense subset of $\mathfrak{G}_X' \cap \mathfrak{G}_X^*$ is essential (Theorem 2).

To prove Theorem 1, we need three lemmas.

Lemma 1. *Each $(Y, u) \in \mathfrak{G}_X^*$ has a Nash equilibrium.*

Proof. Given $(Y, u) \in \mathfrak{G}_X^*$, the existence of a Nash equilibrium in (Y, u) follows from Proposition 4.18 and Corollary 4.5 in Barelli and Soza (2009). ■

Lemma 2. *The correspondence $\mathcal{E}_X|_{\mathfrak{G}_X^*}$ is compact-valued and upper hemicontinuous.*

Proof. Since X is compact and metric, it suffices to show that $\mathcal{E}_X|_{\mathfrak{G}_X^*}$ has a closed graph (see, *e.g.*, Aliprantis and Border, 2006, Theorem 17.11). Take a sequence (Y^n, u^n) in $\mathfrak{G}_X^*$, and take a sequence (x^n) such that x^n is a Nash equilibrium of (Y^n, u^n) for each n. Suppose that

$$\left(x^n, (Y^n, u^n)\right) \to (x, (Y, u)),$$

for some $(x, (Y, u)) \in X \times \mathfrak{G}_X^*$. We must show that x is a Nash equilibrium of (Y, u).

First note that because $(Y^n, u^n) \to (Y, u)$, $x^n \to x$, and $x^n \in Y^n$ for each n, we have $x \in Y$. To see this, observe first that because (Y^n, u^n) ρ_X-converges to (Y, u), there exist n^* and a sequence $(g^n)_{n \geq n^*}$, where $g^n \in \times_{i=1}^N \mathcal{H}(Y_i, Y_i^n)$ for each $n \geq n^*$, such that $\sup_{y \in Y} d_X(g^n(y), y) \to 0$. Consequently, for $y \in X \setminus Y$, it follows that $Y^n \cap N_\beta(y) = \emptyset$ for infinitely many n and for some $\beta > 0$, and since $x^n \in Y^n$ for each n and $x^n \to x$, we see that $x \in Y$.

In addition, since $(Y^n, u^n) \to (Y, u)$, the following holds: for each n, there exists $h^n \in \times_{i=1}^N \mathcal{H}(Y_i^n, Y_i)$ such that

$$\sum_{i=1}^N \sup_{x \in Y^n} |u_i\left(h^n(x)\right) - u_i^n(x)| \leq \rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}$$

and

$$\sup_{x \in Y^n} d_X\left(h^n(x), x\right) \leq \rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}.$$

Moreover, we may write (passing to a subsequence if necessary) $(x^n, u^n(x^n)) \to (x, \alpha)$ for some $\alpha \in \mathbb{R}^N$. Now suppose that x is not a Nash equilibrium of (Y, u). Then, since $x \in Y$, there exist i and $y_i \in Y_i$ such that $u_i(y_i, x_{-i}) > u_i(x)$. Suppose first that $u_i(x) \geq \alpha_i$. Then, since (Y, u) is generalized payoff secure, there exist a neighborhood $V_{(y_i, x_{-i})}$ of (y_i, x_{-i}) in Y and a nonempty-valued, convex-valued, compact-valued, upper hemicontinuous correspondence $\Phi_i : V_{(y_i, x_{-i})} \rightrightarrows Y_i$ such that $u_i(z_i', z_{-i}) \geq \beta > \alpha_i$ for each $z_i' \in \Phi_i(z)$ and $z \in V_{(y_i, x_{-i})}$ and for some $\beta \in \mathbb{R}$. Consequently, since $h^n(x^n) \to x$, for any large enough n we have $u_i(y_i^n, h_{-i}^n(x_{-i}^n)) \geq \beta > \alpha_i$ for each $y_i^n \in \Phi_i(y_i, h_{-i}^n(x_{-i}^n))$, whence for large enough n, there exists $z_i^n \in Y_i^n$ such that

$$u_i^n\left(h^{n-1}\left(y_i^n, h_{-i}^n(x_{-i}^n)\right)\right) = u_i^n(z_i^n, x_{-i}^n) \geq \gamma > \alpha_i,$$

for some $\gamma \in \mathbb{R}$. Hence, because $u^n(x^n) \to \alpha$, we obtain, for large n, $u_i^n(z_i^n, x_{-i}^n) > u_i^n(x^n)$, thereby contradicting that x^n is a Nash equilibrium of (Y^n, u^n). Next, suppose that $u_i(x) < \alpha_i$. Then, because $u(h^n(x^n)) \to \alpha$, and since $\sum_{j=1}^N u_j$ is upper semicontinuous on Y, there must exist some j such that $u_j(x) > \alpha_j$, and one may proceed as before to derive a contradiction. ■

The next lemma is the classic result of Fort (1951) on generic lower hemicontinuity of nonempty-valued, compact-valued, upper hemicontinuous correspondences.

Lemma 3 (*Fort, 1951, Theorem 2*)**.** *Suppose that X is a metric space and that Y is a topological space. Suppose that $F : Y \rightrightarrows X$ is a nonempty-valued, compact-valued, upper hemicontinuous correspondence. Then there exists a residual subset Q of Y such that F is lower hemicontinuous at every point in Q.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. The correspondence $\mathcal{E}_X|_{\mathfrak{G}_X^*}$ is nonempty-valued (Lemma 1), compact-valued and upper hemicontinuous (Lemma 2). Consequently, Lemma 3 gives a residual subset $\mathfrak{Q}$ of $\mathfrak{G}_X^*$ such that $\mathcal{E}_X|_{\mathfrak{G}_X^*}$ is lower hemicontinuous at every point in $\mathfrak{Q}$, and it follows that for each $(Y, u) \in \mathfrak{Q}$, any Nash equilibrium of (Y, u) is essential relative to $\mathfrak{G}_X^*$ (recall Remark 2). ■

Remark 4. If $\mathfrak{G}_X$ were a complete space and $\mathfrak{G}_X^*$ were closed in $\mathfrak{G}_X$, then $\mathfrak{G}_X^*$ would be a complete, metric space, hence a Baire space (by the Baire Category Theorem).[4] In this case, the set $\mathfrak{Q}$ in the proof of Theorem 1, being a residual subset of a Baire space, would be dense, and so the residual subset given by Theorem 1 would be dense in $\mathfrak{G}_X^*$. Unfortunately, while the set $\mathfrak{G}_X^*$ is closed in $\mathfrak{G}_X$, $\mathfrak{G}_X$ fails to be complete. Indeed, let $X := [0, 1]$ and consider the sequence of games $([0, \frac{1}{n}], u)$ in $\mathfrak{G}_X$. It is easy to see that this sequence is Cauchy.[5] In addition, this sequence has no limit point because none of its members is homeomorphic to the game $(\{0\}, u)$.

Theorem 2. *Given any complete subspace $\mathfrak{G}_X'$ of $\mathfrak{G}_X$, every member of a residual, dense subset of $\mathfrak{G}_X' \cap \mathfrak{G}_X^*$ is essential.*

To prove Theorem 2, we need five lemmas.

The proof of the following lemma is analogous to that of Lemma 2.

Lemma 4. *Suppose that $\mathfrak{G}_X'$ is a complete subspace of $\mathfrak{G}_X$. Then the correspondence $\mathcal{E}_X|_{\mathfrak{G}_X^* \cap \mathfrak{G}_X'}$ is compact-valued and upper hemicontinuous.*

Lemma 5. *Suppose that (Y^n, u^n) is a sequence in $\mathfrak{G}_X$ such that (Y^n, u^n) is quasiconcave for each n. If $(Y^n, u^n) \to (Y, u) \in \mathfrak{G}_X$, then (Y, u) is quasiconcave.*

---

4 A metric space Z is complete if every Cauchy sequence in Z converges to a point in Z. By the Cantor Intersection Theorem, a metric space Z is complete if and only if whenever (E^n) is a decreasing (in the sense of set inclusion) sequence of nonempty closed subsets of Z whose diameter converges to zero, there is a point $z \in Z$ for which $\bigcap_n E^n = \{z\}$.

5 A homeomorphism between $[0, \frac{1}{n}]$ and $[0, \frac{1}{n+k}]$ can be constructed as $x \in [0, \frac{1}{n}] \mapsto \frac{n}{n+k}x \in [0, \frac{1}{n+k}]$.

Proof. Since $(Y^n, u^n) \to (Y, u)$, the following holds: for each n, there exists $h^n \in \times_{i=1}^N \mathcal{H}(Y_i^n, Y_i)$ such that

$$\sum_{i=1}^N \sup_{x \in Y^n} |u_i\left(h^n(x)\right) - u_i^n(x)| \leq \rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}.$$

Fix $\alpha \in \mathbb{R}$. For each i and $y_{-i} \in Y_{-i}$ we have

$$\{x_i \in Y_i : u_i(x_i, y_{-i}) \geq \alpha\} = \bigcap_{n=1}^\infty h_i^n\left(\left\{x_i \in Y_i^n : u_i^n\left(x_i, {h_{-i}^n}^{-1}(y_{-i})\right) \geq \alpha - \rho_X\left((Y^n, u^n), (Y, u)\right) - \frac{1}{n}\right\}\right).$$

To see this, suppose that $x_i \in Y_i$ and $u_i(x_i, y_{-i}) \geq \alpha$. Then, for each n,

$$u_i^n\left(h^{n^{-1}}(x_i, y_{-i})\right) \geq \alpha - \rho_X\left((Y^n, u^n), (Y, u)\right) - \frac{1}{n}.$$

This implies the containment '$\subseteq$'. Conversely, suppose that $y_i \notin \{x_i \in Y_i : u_i(x_i, y_{-i}) \geq \alpha\}$. If $y_i \notin Y_i$, then $h_i^{n^{-1}}(y_i) \notin Y_i^n$ for any n. If $y_i \in Y_i$ and $u_i(y_i, y_{-i}) < \alpha$, then for large enough n,

$$u_i^n\left(h^{n^{-1}}(y_i, y_{-i})\right) < \alpha - \rho_X\left((Y^n, u^n), (Y, u)\right) - \frac{1}{n}.$$

This implies the containment '$\supseteq$'.

Now since (Y^n, u^n) is quasiconcave for each n, the set

$$\left\{x_i \in Y_i^n : u_i^n\left(x_i, {h_{-i}^n}^{-1}(y_{-i})\right) \geq \alpha - \rho_X\left((Y^n, u^n), (Y, u)\right) - \frac{1}{n}\right\}$$

is convex for each n, and so (because $h_i^n \in \mathcal{H}(Y_i^n, Y_i)$) the set

$$h_i^n\left(\left\{x_i \in Y_i^n : u_i^n\left(x_i, {h_{-i}^n}^{-1}(y_{-i})\right) \geq \alpha - \rho_X\left((Y^n, u^n), (Y, u)\right) - \frac{1}{n}\right\}\right)$$

is convex for each n. Thus, $\{x_i \in Y_i : u_i(x_i, y_{-i}) \geq \alpha\}$ is an intersection of convex subsets of Y_i, and hence a convex subset of Y_i itself. Since i was arbitrary, we conclude that (Y, u) is quasiconcave. ■

Lemma 6. *Suppose that (Y^n, u^n) is a sequence in $\mathfrak{G}_X$ such that $\sum_{i=1}^N u_i^n$ is upper semicontinuous on Y^n for each n. If $(Y^n, u^n) \to (Y, u) \in \mathfrak{G}_X$, then $\sum_{i=1}^N u_i$ is upper semicontinuous on Y.*

Proof. For each n, there exists $h^n \in \times_{i=1}^N \mathcal{H}(Y_i^n, Y_i)$ such that

$$\sum_{i=1}^N \sup_{x \in Y^n} |u_i\left(h^n(x)\right) - u_i^n(x)| \leq \rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}.$$

Fix $\alpha \in \mathbb{R}$. We have

$$\left\{x \in Y : \sum_{i=1}^N u_i(x) \geq \alpha\right\} = \bigcap_{n=1}^\infty h^n\left(\left\{x \in Y^n : \sum_{i=1}^N u_i^n(x) \geq \alpha - N\left(\rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}\right)\right\}\right).$$

Because $\sum_{i=1}^N u_i^n$ is upper semicontinuous on Y^n for each n, the set

$$\left\{x \in Y^n : \sum_{i=1}^N u_i^n(x) \geq \alpha - N\left(\rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}\right)\right\}$$

is closed in Y^n for each n, and since each h^n is a homeomorphism, for each n the set

$$h^n\left(\left\{x \in Y^n : \sum_{i=1}^N u_i^n(x) \geq \alpha - N\left(\rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}\right)\right\}\right)$$

is closed in Y. Thus, $\left\{x \in Y : \sum_{i=1}^N u_i(x) \geq \alpha\right\}$ is an intersection of closed sets, and hence a closed set itself, and so $\sum_{i=1}^N u_i$ is upper semicontinuous on Y. ■

Lemma 7. *Suppose that (Y^n, u^n) is a sequence in $\mathfrak{G}_X$ such that (Y^n, u^n) is generalized payoff secure for each n. If $(Y^n, u^n) \to (Y, u) \in \mathfrak{G}_X$, then (Y, u) is generalized payoff secure.*

Proof. Since $(Y^n, u^n) \to (Y, u)$, for each n there exists $h^n \in \times_{i=1}^N \mathcal{H}(Y_i^n, Y_i)$ such that

$$\sum_{i=1}^N \sup_{x \in Y^n} |u_i\left(h^n(x)\right) - u_i^n(x)| \leq \rho_X\left((Y^n, u^n), (Y, u)\right) + \frac{1}{n}.$$

Fix $\epsilon > 0$, $x \in Y$, and i. We must show that there exist a neighborhood V_x of x in Y and a nonempty-valued, convex-valued, compact-valued, upper hemicontinuous correspondence $\Phi_i : V_x \rightrightarrows Y_i$ such that $u_i(z_i, y_{-i}) > u_i(x) - \epsilon$ for each $z_i \in \Phi_i(y)$ and $y \in V_x$. Because (Y^n, u^n) is generalized payoff secure for each n, for each n there is a neighborhood V^n of $h^{n^{-1}}(x)$ in Y^n and a nonempty-valued, convex-valued, compact-valued, upper hemicontinuous correspondence $\Phi_i^n : V^n \rightrightarrows Y_i^n$ such that $u_i^n(z_i, y_{-i}) > u_i^n(h^{n^{-1}}(x)) - \frac{1}{n}$ for each $z_i \in \Phi_i^n(y)$ and $y \in V^n$. Hence, for large n, $h^n(V^n)$ is a neighborhood of x in Y, the correspondence $\Psi_i^n : h^n(V^n) \rightrightarrows Y_i$ defined by $\Psi_i^n(y) := h^n(\Phi_i^n(h^{n^{-1}}(y)))$ is nonempty-valued, convex-valued, compact-valued, and upper hemicontinuous, and we have

$$u_i(z_i, y_{-i}) > u_i^n(h^{n^{-1}}(z_i, y_{-i})) - \frac{\epsilon}{3} > u_i^n(h^{n^{-1}}(x)) - \frac{1}{n} - \frac{\epsilon}{3}$$
$$> u_i(x) - \frac{1}{n} - \frac{2\epsilon}{3} > u_i(x) - \epsilon$$

for each $z_i \in \Psi_i^n(y)$ and $y \in h^n(V^n)$. ■

Lemma 8. *Suppose that $\mathfrak{G}_X'$ is a subspace of $\mathfrak{G}_X$. Then the set $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$ is closed in $\mathfrak{G}_X'$.*

Proof. The assertion is an immediate consequence of Lemmas 5, 6, and 7. ■

We are now ready to prove Theorem 2.

Proof of Theorem 2. Suppose that $\mathfrak{G}_X'$ is a complete subspace of $\mathfrak{G}_X$. The correspondence $\mathcal{E}_X|_{\mathfrak{G}_X^* \cap \mathfrak{G}_X'}$ is nonempty-valued (Lemma 1), and compact-valued and upper hemicontinuous (Lemma 4). Consequently, Lemma 3 gives a residual subset $\mathfrak{Q}$ of $\mathfrak{G}_X^*$ such that $\mathcal{E}_X|_{\mathfrak{G}_X^* \cap \mathfrak{G}_X'}$ is lower hemicontinuous at every point in $\mathfrak{Q}$, and it follows that for each $(Y, u) \in \mathfrak{Q}$, any Nash equilibrium of (Y, u) is essential relative to $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$ (Remark 2). To see that $\mathfrak{Q}$ is dense in $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$, note that because $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$ is a closed subset of $\mathfrak{G}_X'$ (Lemma 8), and since $\mathfrak{G}_X'$ is a complete, metric space, $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$ is itself a complete, metric space. Therefore, $\mathfrak{G}_X^* \cap \mathfrak{G}_X'$ is a Baire space by the Baire Category Theorem. Consequently, $\mathfrak{Q}$, being a residual subset of a Baire space, is dense. ■

In the remainder of the paper, we characterize a family of complete subspaces of $\mathfrak{G}_X$ for which Theorem 2 applies.

Let $\mathfrak{G}_X''$ be a subcollection of $\mathfrak{G}_X$ satisfying the following condition: Suppose that (Y^n, u^n) is a Cauchy sequence in $\mathfrak{G}_X''$, *i.e.*, for each $\epsilon > 0$, there exists M such that for every $m, n \geq M$, $\rho_X((Y^m, u^m), (Y^n, u^n)) < \epsilon$, *i.e.*, for every $m, n \geq M$ there exists $h^{(m,n)} \in \times_{i=1}^N \mathcal{H}(Y_i^m, Y_i^n)$ such that

$$\sum_{i=1}^N \sup_{x \in Y^m} |u_i^m(x) - u_i^n(h^{(m,n)}(x))| < \epsilon \quad \text{and}$$
$$\sup_{x \in Y^m} d_X(h^{(m,n)}(x), x) < \epsilon.$$

Then there exists at least one such sequence $(h^{(m,n)})$ with the following additional property: Suppose that (Y^{n_k}) is a subsequence of (Y^n) such that for each i and $x_i \in Y_i^{n_1}$ the sequence

$$x_i, h_i^{(n_1,n_2)}(x_i), h_i^{(n_2,n_3)}(h_i^{(n_1,n_2)}(x_i)), \ldots$$

converges to a point $y_i(x_i)$ in X_i. Then $y_i(x_i) \neq y_i(x_i')$ for all $x_i \neq x_i'$. Let $\mathcal{G}_X$ be the set of all such subcollections $\mathfrak{G}_X''$ of $\mathfrak{G}_X$.

Example. It is easily seen that the collection

$$\left(\left[0, 1+\frac{1}{n}\right], u^n\right)_{n=1}^{\infty}$$

of one-player games in $\mathfrak{G}_{[0,2]}$, where for each n, $u^n : [0, 2] \to \mathbb{R}$ is defined by

$$u^n(x) := \begin{cases} 1 & \text{if } x = 1+\dfrac{1}{n}, \\ 0 & \text{otherwise,} \end{cases}$$

belongs to $\mathcal{G}_{[0,2]}$.

On the other hand, the collection

$$\left(\left[0, 1+\frac{1}{n}\right], v\right)_{n=1}^{\infty} \tag{2}$$

of one-player games in $\mathfrak{G}_{[0,2]}$, where $v : [0, 2] \to \mathbb{R}$ is defined by

$$v(x) := \begin{cases} 1 & \text{if } x \in [1, 2], \\ 0 & \text{otherwise,} \end{cases}$$

does not belong to $\mathcal{G}_{[0,2]}$. To see this, note first that the sequence in (2) is Cauchy. This can be seen as follows. For m and n, define $h^{(m,n)} : [0, 1+\frac{1}{m}] \to [0, 1+\frac{1}{n}]$ by

$$h^{(m,n)}(x) := \begin{cases} x & \text{if } x \in [0, 1], \\ 1+\dfrac{m}{n}(x-1) & \text{if } x > 1. \end{cases}$$

Given $\epsilon > 0$, and for $m, n \geq M$, where M satisfies $\frac{1}{M} < \epsilon$, one has

$$|v(x) - v(h^{(m,n)}(x))| = \begin{cases} |v(x) - v(x)| = 0 & \text{if } x \in [0, 1], \\ |v(x) - v(1+\dfrac{m}{n}(x-1))| & \\ \quad = |1-1| = 0 & \text{if } x > 1, \end{cases}$$

and

$$|x - h^{(m,n)}(x)| = \begin{cases} |x-x| = 0 & \text{if } x \in [0, 1], \\ \left|x - 1 - \dfrac{m}{n}(x-1)\right| = \left|(x-1)\right. & \\ \quad \left.\times\left(\dfrac{n-m}{n}\right)\right| \leq \left|\dfrac{n-m}{mn}\right| \leq \dfrac{1}{M} < \epsilon & \text{if } x \in \left(1, 1+\dfrac{1}{m}\right]. \end{cases}$$

Hence, since $h^{(m,n)} \in \mathcal{H}([0, 1+\frac{1}{m}], [0, 1+\frac{1}{n}])$, it follows that the sequence in (2) is Cauchy.

Next, suppose that $(g^{(m,n)})$ is a sequence satisfying the following: for each m and n, $g^{(m,n)} \in \mathcal{H}([0, 1+\frac{1}{m}], [0, 1+\frac{1}{n}])$, and for each $\epsilon > 0$, there exists M such that for every $m, n \geq M$,

$$\sup_{x\in[0,1+\frac{1}{m}]} |v(x) - v(g^{(m,n)}(x))| < \epsilon \quad \text{and}$$

$$\sup_{x\in[0,1+\frac{1}{m}]} |g^{(m,n)}(x) - x| < \epsilon.$$

Then, for each m and n, and for every $x \geq 1$, one must have $g^{(m,n)}(x) \geq 1$, and so one may pick two distinct x and y, both in $[1, 2]$, such that the sequences

$$x, g^{(1,2)}(x), g^{(2,3)}(g^{(1,2)}(x)), \ldots \quad \text{and}$$
$$y, g^{(1,2)}(y), g^{(2,3)}(g^{(1,2)}(y)), \ldots$$

converge to 1, implying that the collection in (2) is not a member of $\mathcal{G}_{[0,2]}$.

Lemma 9. *Suppose that $\mathfrak{G}_X'' \in \mathcal{G}_X$. Let $\mathfrak{G}_X'$ be the closure of $\mathfrak{G}_X''$ in $\mathfrak{G}_X$. Then $\mathfrak{G}_X'$ is a complete subspace of $\mathfrak{G}_X$.*

Proof. Suppose that $\mathfrak{G}_X'' \in \mathcal{G}_X$. Let $\mathfrak{G}_X'$ be the closure of $\mathfrak{G}_X''$ in $\mathfrak{G}_X$. Pick a Cauchy sequence (Z^n, v^n) in $\mathfrak{G}_X'$. Then there exists a Cauchy sequence (Y^n, u^n) in $\mathfrak{G}_X''$ such that if (Y^n, u^n) converges to (Y, u) then (Z^n, v^n) converges to (Y, u). Indeed, it suffices to pick a sequence (Y^n, u^n) from $\mathfrak{G}_X''$ such that $\rho_X((Y^n, u^n), (Z^n, v^n)) < \frac{1}{n}$ for each n. Thus, it suffices to show that (Y^n, u^n) converges to a point in $\mathfrak{G}_X'$. But since $\mathfrak{G}_X'$ is closed in $\mathfrak{G}_X$, it suffices to show that the sequence (Y^n, u^n) is convergent.

Because (Y^n, u^n) is a Cauchy sequence, given $\epsilon > 0$, there exists M such that for every $m, n \geq M$, $\rho_X((Y^m, u^m), (Y^n, u^n)) < \epsilon$, *i.e.*, for every $m, n \geq M$ there exists $h^{(m,n)} \in \times_{i=1}^N \mathcal{H}(Y_i^m, Y_i^n)$ such that

$$\sum_{i=1}^N \sup_{x\in Y^m} |u_i^m(x) - u_i^n(h^{(m,n)}(x))| < \epsilon \quad \text{and}$$

$$\sup_{x\in Y^m} d_X(h^{(m,n)}(x), x) < \epsilon.$$

Because (Y^n, u^n) is a Cauchy sequence in $\mathfrak{G}_X''$, there is no loss of generality in assuming that the sequence $(h^{(m,n)})$ has the following additional property: Suppose that (Y^{n_k}) is a subsequence of (Y^n) such that for each i and $x_i \in Y_i^{n_1}$ the sequence

$$x_i, h_i^{(n_1,n_2)}(x_i), h_i^{(n_2,n_3)}(h_i^{(n_1,n_2)}(x_i)), \ldots$$

converges to a point $y_i(x_i)$ in X_i. Then $y_i(x_i) \neq y_i(x_i')$ for all $x_i \neq x_i'$.

Below we show that there exists a subsequence (Y^{n_k}, u^{n_k}) of (Y^n, u^n) satisfying the following: there exists a sequence (g^k) with $g^k \in \times_{i=1}^N \mathcal{H}(Y_i^{n_k}, Y_i^{n_{k+1}})$ for each k such that the sequences

$$\eta^1(x) := x, \eta^2(x) := g^1(x), \eta^3(x) := g^2(g^1(x)), \ldots \tag{3}$$

and

$$u_i^{n_1}(\eta^1(x)), u_i^{n_2}(\eta^2(x)), u_i^{n_3}(\eta^3(x)), \ldots, \quad i \in \{1, \ldots, N\}, \tag{4}$$

for $x \in Y^{n_1}$, satisfy the following: given $\epsilon > 0$, there exists K such that for every $k, l \geq K$,

$$\sum_{i=1}^N \sup_{x\in Y^{n_1}} |u_i^{n_k}(\eta^k(x)) - u_i^{n_l}(\eta^l(x))| < \epsilon \quad \text{and}$$

$$\sup_{x\in Y^{n_1}} d_X(\eta^k(x), \eta^l(x)) < \epsilon. \tag{5}$$

Consequently, for each $x \in Y^{n_1}$, the sequences in (3) and (4) are Cauchy in X and $\mathbb{R}$ respectively, and since these spaces are complete, it follows that the sequence in (3) converges to a point $y(x) = (y_1(x_1), \ldots, y_N(x_N))$ in X and the sequence in (4) converges to a point $\alpha_i(x)$ in $\mathbb{R}$. Note that one has $y_i(x_i) \neq y_i(x_i')$ whenever $x_i \neq x_i'$. Therefore, defining $Y_i := \bigcup_{x_i\in Y_i^{n_1}} \{y_i(x_i)\}$, and given $z_i \in Y_i$, there exists a unique $x_i \in Y_i^{n_1}$ such that $y_i(x_i) = z_i$. Thus, there is a map $f_i : Y_i \to Y_i^{n_1}$ satisfying $y_i(f_i(z_i)) = z_i$ for each $z_i \in Y_i$. Let $Y := \times_{i=1}^N Y_i$ and define, for each i, $u_i : X \to \mathbb{R}$ as follows:

$$u_i(x) := \begin{cases} \alpha_i(f_1(x_1), \ldots, f_N(x_N)) & \text{if } x \in Y, \\ 0 & \text{otherwise.} \end{cases}$$

The proof will be complete if we show that $(Y^n, u^n) \to (Y, u)$ and $(Y, u) \in \mathfrak{G}_X$. First, note that each u_i is bounded. To see this, observe that for each i and $x \in Y^{n_1}$, the sequence in (4) converges to $u_i(y(x))$, and choose K^* such that for every $k, l \geq K^*$,

$$\sum_{i=1}^N \sup_{x\in Y^{n_1}} |u_i^{n_k}(\eta^k(x)) - u_i^{n_l}(\eta^l(x))| < 1.$$

For each i and $x \in Y^{n_1}$,

$$|u_i(y(x))| \leq |u_i^{n_{K^*}}(\eta^{K^*}(x)) - u_i(y(x))| + |u_i^{n_{K^*}}(\eta^{K^*}(x))| \leq 1 + |u_i^{n_{K^*}}(\eta^{K^*}(x))|,$$

implying (since $u_i^{n_{K^*}}$ is bounded) that u_i is bounded.

It remains to show that $(Y^n, u^n) \to (Y, u)$ in $\mathfrak{G}_X$. To this end, first it will be shown that $(Y^{n_k}, u^{n_k}) \to (Y, u)$. For each i and k, define the map $f_i^k : Y_i \to Y_i^{n_k}$ by

$$f_i^k(z_i) := \eta^k(f_i(z_i)),$$

where $f_i : Y_i \to Y_i^{n_1}$ is as defined above. The map f_i is a member of $\mathcal{H}(Y_i, Y_i^{n_1})$. This flows from the following observations. First, it is easily seen that f_i is one-to-one and onto. To see that f_i is continuous, let (z_i^l) be a sequence in Y_i with limit point $z_i \in Y_i$. Then the sequence $(f_i(z_i^l))$ is convergent in $Y_i^{n_1}$ with limit point $f_i(z_i)$. Otherwise, one has (extracting a subsequence if necessary) $f_i(z_i^l) \to x_i$ for some $x_i \in Y_i^{n_1}$ with $x_i \neq f_i(z_i)$, and this implies $y_i(x_i) \neq y_i(f_i(z_i)) = z_i$, and so there exists $\epsilon > 0$ such that $N_\epsilon(y_i(x_i)) \cap N_\epsilon(z_i) = \emptyset$. We now establish the following impossibility: there exist k and l such that $\eta_i^k(f_i(z_i^l)) \in N_\epsilon(z_i) \cap N_\epsilon(y_i(x_i))$. Given $\alpha > 0$, there exists K_α such that for all $k, k' \geq K_\alpha$,

$$\sup_{x \in Y^{n_1}} d_X(\eta^k(x), \eta^{k'}(x)) < \alpha,$$

and so in particular,

$$\sup_{x_i \in \{f_i(z_i^1), f_i(z_i^2), \ldots\}} d_{X_i}(\eta_i^k(x_i), \eta_i^{k'}(x_i)) < \alpha,$$

implying that for each $f_i(z_i^l)$ and $k \geq K_\alpha$,

$$d_{X_i}(\eta_i^k(f_i(z_i^l)), z_i^l) = \lim_{k' \to \infty} d_{X_i}(\eta_i^k(f_i(z_i^l)), \eta_i^{k'}(f_i(z_i^l))) \leq \alpha.$$

Consequently, because $z_i^l \to z_i$, there exist K' and L' such that for all $k \geq K'$ and $l \geq L'$, $\eta_i^k(f_i(z_i^l)) \in N_\epsilon(z_i)$. Now since the sequence $\eta_i^1(x_i) = x_i, \eta_i^2(x_i), \eta_i^3(x_i), \ldots$ converges to $y_i(x_i)$, there exists K'' such that for all $k \geq K''$, $\eta_i^k(x_i) \in N_\epsilon(y_i(x_i))$. In addition, because η^k is continuous for each k and since $f_i(z_i^l) \to x_i$, for each k one has $\eta_i^k(f_i(z_i^l)) \to \eta_i^k(x_i)$ (as $l \to \infty$). Consequently, for $k \geq \max\{K', K''\}$, and for l large enough, one obtains $\eta_i^k(f_i(z_i^l)) \in N_\epsilon(z_i) \cap N_\epsilon(y_i(x_i))$, the sought contradiction. We conclude that f_i is continuous.

To see that f_i^{-1} is continuous, let (x_i^l) be a convergent sequence in $Y_i^{n_1}$ with limit point $x_i \in Y^{n_1}$. Then

$$\begin{aligned} d_{X_i}(f_i^{-1}(x_i^l), f_i^{-1}(x_i)) &= d_{X_i}(y_i(x_i^l), y_i(x_i)) \\ &\leq d_{X_i}(y_i(x_i^l), \eta_i^k(x_i^l)) + d_{X_i}(\eta_i^k(x_i^l), \eta_i^k(x_i)) \\ &\quad + d_{X_i}(\eta_i^k(x_i), y_i(x_i)). \end{aligned} \tag{6}$$

It suffices to show that for each $\varepsilon > 0$ there exists L^* such that for all $l \geq L^*$, $d_{X_i}(y_i(x_i^l), y_i(x_i)) < \varepsilon$. Fix $\varepsilon > 0$. Then there exists K_ε such that for all $k, k' \geq K_\varepsilon$,

$$\sup_{x' \in Y^{n_1}} d_X(\eta^k(x'), \eta^{k'}(x')) < \frac{\varepsilon}{3},$$

and so in particular, for each $k \geq K_\varepsilon$,

$$d_{X_i}(\eta_i^k(x_i), y_i(x_i)) = \lim_{k' \to \infty} d_{X_i}(\eta_i^k(x_i), \eta_i^{k'}(x_i)) \leq \frac{\varepsilon}{3}$$

and

$$d_{X_i}(\eta_i^k(x_i^l), y_i(x_i^l)) = \lim_{k' \to \infty} d_{X_i}(\eta_i^k(x_i^l), \eta_i^{k'}(x_i^l)) \leq \frac{\varepsilon}{3}, \quad l \in \{1, 2, \ldots\}.$$

Now fix $k \geq K_\varepsilon$. Since η_i^k is continuous and $x_i^l \to x_i$, it follows that there exists L^* such that for all $l \geq L^*$, $d_{X_i}(\eta_i^k(x_i^l), \eta_i^k(x_i)) < \frac{\varepsilon}{3}$. Consequently, in light of (6), one obtains, for $l \geq L^*$,

$$d_{X_i}(y_i(x_i^l), y_i(x_i)) \leq d_{X_i}(y_i(x_i^l), \eta_i^k(x_i^l)) + d_{X_i}(\eta_i^k(x_i^l), \eta_i^k(x_i)) + d_{X_i}(\eta_i^k(x_i), y_i(x_i)) < \varepsilon,$$

as desired. We conclude that f_i^{-1} is continuous.

Next, it will be shown that $f_i(A) \subseteq Y_i^{n_1}$ is convex if and only if $A \subseteq Y_i$ is convex. Fix $A \subseteq Y_i$. Suppose that $f_i(A)$ is a convex set. Then A is convex. To see this, fix z_i and z_i' in A and λ in $[0, 1]$. We need to show that $\lambda z_i + (1 - \lambda) z_i' \in A$. Below we show that for each k,

$$\eta^k\left(\left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\}\right) = \left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}, \tag{7}$$

and that for each $\theta \in (0, 1)$, there exists a sequence (θ^m) in $[0, 1]$ such that $\theta^m \to \theta$ and

$$f_i\left(\theta^m z_i + (1 - \theta^m) z_i'\right) \in \left\{\theta' f_i(z_i) + (1 - \theta') f_i(z_i') : \theta' \in [0, 1]\right\}, \text{ for each } m. \tag{8}$$

We now assume that $\lambda z_i + (1 - \lambda) z_i' \notin A$ and derive a contradiction. Note that because

$$\left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\} \subseteq f_i(A)$$

(since $\{f_i(z_i), f_i(z_i')\} \subseteq f_i(A)$ and $f_i(A)$ is, by assumption, a convex set), $\lambda z_i + (1 - \lambda) z_i' \notin A$ implies that $f_i(\lambda z_i + (1 - \lambda) z_i') \notin f_i(A)$ and hence

$$f_i(\lambda z_i + (1 - \lambda) z_i') \notin \left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\}. \tag{9}$$

In addition, there exists a sequence (λ^m) in $[0, 1]$ such that $\lambda^m \to \lambda$ and

$$f_i\left(\lambda^m z_i + (1 - \lambda^m) z_i'\right) \in \left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\}, \text{ for each } m. \tag{10}$$

Since $\lambda^m z_i + (1 - \lambda^m) z_i' \to \lambda z_i + (1 - \lambda) z_i'$ and f_i is continuous, it follows that

$$f_i(\lambda^m z_i + (1 - \lambda^m) z_i') \to f_i(\lambda z_i + (1 - \lambda) z_i'),$$

and so (10) implies that $f_i(\lambda z_i + (1 - \lambda) z_i') = \lambda^* f_i(z_i) + (1 - \lambda^*) f_i(z_i')$ for some $\lambda^* \in [0, 1]$, contradicting (9). We conclude that A is convex.

Next, we show that (7) holds for each k. Fix k. Since $\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\}$ is convex and η^k is a convex preserving map, it follows that $\eta_i^k(\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\})$ is convex, and since $\eta_i^k(f_i(z_i))$ and $\eta_i^k(f_i(z_i'))$ are members of the convex set $\eta_i^k(\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\})$, it follows that

$$\eta_i^k\left(\left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\}\right) \supseteq \left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}.$$

To establish the reverse containment, note that since $\eta_i^k \in \mathcal{H}(Y_i^{n_1}, Y_i^{n_k})$, and since

$$\left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}$$

is convex in $Y_i^{n_k}$, the set

$$\eta_i^{k^{-1}}\left(\left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}\right)$$

is convex in $Y_i^{n_1}$, and because

$$\{f_i(z_i), f_i(z_i')\} \subseteq \eta_i^{k^{-1}}\left(\left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}\right),$$

it follows that

$$\left\{\theta f_i(z_i) + (1 - \theta) f_i(z_i') : \theta \in [0, 1]\right\} \subseteq \eta_i^{k^{-1}}\left(\left\{\theta \eta_i^k(f_i(z_i)) + (1 - \theta)\eta_i^k(f_i(z_i')) : \theta \in [0, 1]\right\}\right),$$

implying that

$$\eta_i^k\left(\{\theta f_i(z_i)+(1-\theta)f_i(z_i'):\theta\in[0,1]\}\right)\subseteq\left\{\theta\eta_i^k(f_i(z_i))+(1-\theta)\eta_i^k(f_i(z_i')):\theta\in[0,1]\right\}.$$

Next, we show that for each $\theta\in(0,1)$, there exists a sequence (θ^m) in $[0,1]$ such that $\theta^m\to\theta$ and (8) holds. Fix $\theta\in(0,1)$. Note that it suffices to show that there exists a subsequence $(\eta_i^{k_\ell})$ of (η_i^k) such that for each $\epsilon>0$, there exist $y_i\in\{\theta' f_i(z_i)+(1-\theta')f_i(z_i'):\theta'\in[0,1]\}$ and L such that $\eta_i^{k_\ell}(y_i)\in N_\epsilon(\theta z_i+(1-\theta)z_i')$ for all $\ell\geq L$. To prove this, note first that since $\eta_i^k(f_i(z_i))\to z_i$ and $\eta_i^k(f_i(z_i'))\to z_i'$, by the continuity of vector addition and scalar multiplication it follows that

$$\theta\eta_i^k(f_i(z_i))+(1-\theta)\eta_i^k(f_i(z_i'))\to\theta z_i+(1-\theta)z_i'. \tag{11}$$

In addition, recall that given $\epsilon>0$, there exists K such that for every $k,l\geq K$, (5) holds. Consequently, there is a subsequence $(\eta_i^{k_\ell})$ of (η_i^k) such that

$$\sup_{x_i\in Y_i^{n_1}} d_{X_i}\left(\eta_i^{k_\ell}(x_i),\eta_i^{k_{\ell+1}}(x_i)\right)<\frac{1}{2^{\ell+1}},\quad\text{for each }\ell.$$

Now fix $\epsilon>0$. Given that (7) holds for each k, and given the convergence in (11), there exist $y_i\in\{\theta' f_i(z_i)+(1-\theta')f_i(z_i'):\theta'\in[0,1]\}$ and L such that $\eta_i^{k_L}(y_i)\in N_{\frac{\epsilon}{2}}(\theta z_i+(1-\theta)z_i')$ and

$$\sum_{\kappa=0}^{\infty} d_{X_i}\left(\eta_i^{k_{L+\kappa}}(y_i),\eta_i^{k_{L+\kappa+1}}(y_i)\right)<\frac{1}{2^{L+1}}+\frac{1}{2^{L+2}}+\cdots=\frac{1}{2^L}<\frac{\epsilon}{2}.$$

In addition, for each $\ell\geq L$ one has

$$d_{X_i}\left(\eta_i^{k_L}(y_i),\eta_i^{k_\ell}(y_i)\right)\leq\sum_{\kappa=0}^{\infty} d_{X_i}\left(\eta_i^{k_{L+\kappa}}(y_i),\eta_i^{k_{L+\kappa+1}}(y_i)\right)<\frac{1}{2^{L+1}}+\frac{1}{2^{L+2}}+\cdots=\frac{1}{2^L}<\frac{\epsilon}{2}.$$

Consequently, since $\eta_i^{k_L}(y_i)\in N_{\frac{\epsilon}{2}}(\theta z_i+(1-\theta)z_i')$, we see that $\eta_i^{k_\ell}(y_i)\in N_\epsilon(\theta z_i+(1-\theta)z_i')$ for all $\ell\geq L$.

It remains to show that $f_i(A)$ is convex if A is convex. Suppose that A is convex and pick x_i and x_i' in $f_i(A)$ and $\lambda\in[0,1]$. Then $x_i=f_i(z_i)$ and $x_i'=f_i(z_i')$ for some z_i and z_i' in A. To see that $\lambda f_i(z_i)+(1-\lambda)f_i(z_i')\in f_i(A)$, it suffices to show that

$$\eta_i^k\left(\lambda f_i(z_i)+(1-\lambda)f_i(z_i')\right)\to\lambda^* z_i+(1-\lambda^*)z_i',\quad\text{for some }\lambda^*\in[0,1] \tag{12}$$

(since this implies that $f_i(\lambda^* z_i+(1-\lambda^*)z_i')=\lambda f_i(z_i)+(1-\lambda)f_i(z_i')$, which, together with the fact that $\lambda^* z_i+(1-\lambda^*)z_i'\in A$ (by convexity of A), yields $\lambda f_i(z_i)+(1-\lambda)f_i(z_i')\in f_i(A)$, as we sought). To see that (12) holds, note that by (7),

$$\eta_i^k\left(\lambda f_i(z_i)+(1-\lambda)f_i(z_i')\right)\in\left\{\theta\eta_i^k(f_i(z_i))+(1-\theta)\eta_i^k(f_i(z_i')):\theta\in[0,1]\right\},\quad\text{for each }k,$$

implying that there exists a sequence (θ^k) in $[0,1]$ such that

$$\eta_i^k\left(\lambda f_i(z_i)+(1-\lambda)f_i(z_i')\right)=\theta^k\eta_i^k(f_i(z_i))+(1-\theta^k)\eta_i^k(f_i(z_i')).$$

Consequently, since the sequence $(\eta_i^k(\lambda f_i(z_i)+(1-\lambda)f_i(z_i')))$ converges, and since $\eta_i^k(f_i(z_i))\to z_i$ and $\eta_i^k(f_i(z_i'))\to z_i'$, we see that

$$\theta^k\eta_i^k(f_i(z_i))+(1-\theta^k)\eta_i^k(f_i(z_i'))\to\lambda^* z_i+(1-\lambda^*)z_i',\quad\text{for some }\lambda^*\in[0,1],$$

as desired.

We conclude that $f_i\in\mathcal{H}(Y_i,Y_i^{n_1})$, and consequently $f_i^k\in\mathcal{H}(Y_i,Y_i^{n_k})$ for each k. To prove that $(Y^{n_k},u^{n_k})\to(Y,u)$, it suffices to show that given $\epsilon>0$ there exists K such that for all $k\geq K$,

$$\sum_{i=1}^N\sup_{z\in Y}|u_i^{n_k}(f^k(z))-u_i(z)|<\epsilon\quad\text{and}\quad\sup_{z\in Y}d_X(z,f^k(z))\leq\epsilon. \tag{13}$$

Fix $\epsilon>0$. Recall that there exists K such that for every $k,l\geq K$,

$$\sum_{i=1}^N\sup_{x\in Y^{n_1}}|u_i^{n_k}(\eta^k(x))-u_i^{n_l}(\eta^l(x))|<\frac{\epsilon}{N}\quad\text{and}\quad\sup_{x\in Y^{n_1}}d_X(\eta^k(x),\eta^l(x))<\epsilon.$$

Given i, and for $k\geq K$ and $z\in Y$,

$$|u_i^{n_k}(f^k(z))-u_i(z)|=|u_i^{n_k}(\eta^k(f(z)))-u_i(z)|=\lim_{l\to\infty}|u_i^{n_k}(\eta^k(f(z)))-u_i^{n_l}(\eta^l(f(z)))|\leq\frac{\epsilon}{N}$$

and

$$d_X(z,f^k(z))=d_X(z,\eta^k(f(z)))=\lim_{l\to\infty}d_X(\eta^l(f(z)),\eta^k(f(z)))\leq\epsilon,$$

implying (13). We conclude that $(Y^{n_k},u^{n_k})\to(Y,u)$.

Next, we show that $(Y^n,u^n)\to(Y,u)$. Fix $\epsilon>0$ and recall that there exists M' such that for every $m,n\geq M'$, there exists $h^{(m,n)}\in\times_{i=1}^N\mathcal{H}(Y_i^m,Y_i^n)$ such that

$$\sum_{i=1}^N\sup_{x\in Y^m}|u_i^m(x)-u_i^n(h^{(m,n)}(x))|<\frac{\epsilon}{2N}\quad\text{and}\quad\sup_{x\in Y^m}d_X(h^{(m,n)}(x),x)<\frac{\epsilon}{2}.$$

In addition, there exists K^* such that for all $k\geq K^*$,

$$\sum_{i=1}^N\sup_{z\in Y}|u_i^{n_k}(f^k(z))-u_i(z)|<\frac{\epsilon}{2N}\quad\text{and}\quad\sup_{z\in Y}d_X(z,f^k(z))\leq\frac{\epsilon}{2}.$$

For $n\geq M'$, and given i and $z\in Y$, and $k\geq K^*$ with $n_k\geq M'$,

$$|u_i^n(h^{(n_k,n)}(f^k(z)))-u_i(z)|\leq|u_i^{n_k}(f^k(z))-u_i(z)|+|u_i^n(h^{(n_k,n)}(f^k(z)))-u_i^{n_k}(f^k(z))|\leq\frac{\epsilon}{N}$$

and

$$d_X(z,h^{(n_k,n)}(f^k(z)))=d_X(z,f^k(z))+d_X(h^{(n_k,n)}(f^k(z)),f^k(z))\leq\epsilon.$$

Consequently, for $n\geq M'$ one has $\rho_X((Y^n,u^n),(Y,u))\leq\epsilon$, and we conclude that $(Y^n,u^n)\to(Y,u)$.

It remains to show that there exists a subsequence (Y^{n_k},u^{n_k}) of (Y^n,u^n) satisfying the following: there exists a sequence (g^k) with $g^k\in\times_{i=1}^N\mathcal{H}(Y_i^{n_k},Y_i^{n_{k+1}})$ for each k such that the sequences (3) and (4) (for $x\in Y^{n_1}$) satisfy the following: given $\epsilon>0$, there exists K such that for every $k,l\geq K$,

$$\sum_{i=1}^N\sup_{x\in Y^{n_1}}|u_i^{n_k}(\eta^k(x))-u_i^{n_l}(\eta^l(x))|<\epsilon\quad\text{and}\quad\sup_{x\in Y^{n_1}}d_X(\eta^k(x),\eta^l(x))<\epsilon. \tag{14}$$

Recall that given $\epsilon > 0$, there exists M such that for every $m, n \geq M$ there exists $h^{(m,n)} \in \times_{i=1}^N \mathcal{H}(Y_i^m, Y_i^n)$ such that

$$\sum_{i=1}^N \sup_{x\in Y^m} |u_i^m(x) - u_i^n(h^{(m,n)}(x))| < \epsilon \quad \text{and}$$

$$\sup_{x\in Y^m} d_X(h^{(m,n)}(x), x) < \epsilon.$$

Therefore, one can select a subsequence (n_k) of (n) satisfying the following: for each k,

$$\sum_{i=1}^N \sup_{x\in Y^{n_k}} |u_i^{n_k}(x) - u_i^{n_{k+1}}(h^{(n_k,n_{k+1})}(x))| < \frac{1}{2^{k+1}N} \quad \text{and}$$

$$\sup_{x\in Y^{n_k}} d_X(h^{(n_k,n_{k+1})}(x), x) < \frac{1}{2^{k+1}}.$$

For each k, set $g^k := h^{(n_k,n_{k+1})} \in \times_{i=1}^N \mathcal{H}(Y_i^{n_k}, Y_i^{n_{k+1}})$. Now fix $\epsilon > 0$ and choose K with $\frac{1}{2^{K-1}} < \epsilon$. For $k, l \geq K$ and for $x \in Y^{n_1}$ we have

$$\begin{aligned}|u_i^{n_k}(\eta^k(x)) - u_i^{n_l}(\eta^l(x))| &\leq |u_i^{n_K}(\eta^K(x)) - u_i^{n_{K+1}}(\eta^{K+1}(x))| \\ &\quad + |u_i^{n_{K+1}}(\eta^{K+1}(x)) - u_i^{n_{K+2}}(\eta^{K+2}(x))| + \cdots \\ &< \sum_{k'=K}^{\infty} \frac{1}{2^{k'}N} = \frac{1}{N}\cdot\frac{1}{2^{K-1}}\left(\frac{1}{2}+\frac{1}{2^2}+\cdots\right) \\ &< \frac{\epsilon}{N}\end{aligned}$$

and

$$\begin{aligned}d_X(\eta^k(x), \eta^l(x)) &\leq d_X(\eta^K(x), \eta^{K+1}(x)) + d_X(\eta^{K+1}(x), \eta^{K+2}(x)) + \cdots \\ &< \frac{1}{2^K} + \frac{1}{2^{K+1}} + \cdots = \frac{1}{2^{K-1}} < \epsilon.\end{aligned}$$

Consequently, for every $k, l \geq K$, (14) holds, as we sought. ■

Theorem 2, together with Lemma 9, immediately gives the following corollary, which generalizes Theorem 2 in Carbonell-Nicolau (2010).[6]

Corollary (*to Theorem 2*)**.** *Suppose that $\mathfrak{G}_X'' \in \mathcal{G}_X$. Let $\mathfrak{G}_X'$ be the closure of $\mathfrak{G}_X''$ in $\mathfrak{G}_X$. Then every member of a residual, dense subset of $\mathfrak{G}_X' \cap \mathfrak{G}_X^*$ is essential.*

We conclude with a comment. A natural question is whether the techniques for fixed action spaces developed in Carbonell-Nicolau (2015) to extend the results in Carbonell-Nicolau (2010) can be adapted to the more general notion of perturbed game considered in this paper. This suggests a natural avenue towards a generalization of Theorem 2 and the Corollary. A detailed discussion of this topic lies outside the scope of this paper and is left for future research.

References

Aliprantis, C.D., Border, K.C., 2006. Infinite Dimensional Analysis. Springer-Verlag, Berlin.
Barelli, P., Soza, I., 2009. On the Existence of Nash Equilibria in Discontinuous and Qualitative Games. Mimeo.
Carbonell-Nicolau, O., 2010. Essential equilibria in normal-form games. J. Econom. Theory 145, 421–431.
Carbonell-Nicolau, O., 2015. Further results on essential Nash equilibria in normal-form games. Econom. Theory 59, 277–300.
Fort, M.K., 1951. Points of continuity of semi-continuous functions. Publ. Math. Debrecen 2, 100–102.
Reny, P.J., 2011. Strategic approximations of discontinuous games. Econom. Theory 48, 17–29.
Scalzo, V., 2013. Essential equilibria of discontinuous games. Econom. Theory 54, 27–44.
Simon, L.K., 1987. Games with discontinuous payoffs. Rev. Econom. Stud. 54, 569–597.
Wu, W.-T., Jiang, J.-H., 1962. Essential equilibrium points of n-person non-cooperative games. Sci. Sin. 11, 1307–1322.
Yu, J., 1999. Essential equilibria of n-person noncooperative games. J. Math. Econom. 31, 361–372.

[6] The Corollary reduces to Theorem 2 in Carbonell-Nicolau (2010) in the special case when $\mathfrak{G}_X'' = \{X_1\} \times \cdots \times \{X_N\} \times B(X)^N$.