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Short communication

On essential, (strictly) perfect equilibria☆

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ABSTRACT

It is known that generic games within certain collections of infinite-action normal-form games have only essential equilibria. We point to a difficulty in showing that essential equilibria in generic games are (strictly) perfect, and we identify collections of games whose generic members have only essential and (strictly) perfect equilibria.

1. Introduction

Given a collection $\mathfrak{g}$ of normal-form games, and given a game G in $\mathfrak{g}$, a Nash equilibrium μ of G is *essential relative to* $\mathfrak{g}$ if neighboring games *within* $\mathfrak{g}$ have Nash equilibria close to μ. It is well-known that for generic games in the collection of all finite-action games, all Nash equilibria are essential and strictly perfect (*cf.* Wu and Jiang (1962)). Generic members of certain collections of infinite-action games have only essential equilibria (*e.g.*, Yu (1999) and Carbonell-Nicolau (2010)). However, it has not been shown that essential equilibria in generic games are (strictly) perfect.

In this paper, we first point out that the collections of games considered in Yu (1999) and Carbonell-Nicolau (2010) are not closed under Selten perturbations, implying that (strict) perfection of essential equilibria in generic games does not follow from known results. We then identify, in Theorem 4, a collection of games whose members have only essential, perfect mixed-strategy equilibria. This collection is closed under some but not all Selten perturbations (Example 1), and this again points to a difficulty in showing that essential equilibria are strictly perfect. The analysis in Carbonell-Nicolau (2011a) implies that there is a subcollection of games whose members have only essential, strictly perfect mixed-strategy equilibria. The formal statement is given in Theorem 5.

2. Preliminaries

A *normal-form game* (or simply a *game*) is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $u_i : X \to \mathbb{R}$ represents player i's payoff function, where $X := \times_{i=1}^N X_i$. By a slight abuse of notation, N will represent both the number of players and the set of players.

If u_i is bounded and X_i is a nonempty subset of a metric space for each i, G is said to be a *metric game*. If in addition X_i is compact for each i, then G is called a *compact, metric game*. If X_i is a nonempty subset of a metric space and u_i is bounded and Borel measurable for each i, then G is said to be a *metric, Borel game*.

For each i, let $X_{-i} := \times_{j \neq i} X_j$. Given i and a strategy profile $x = (x_1, \dots, x_N)$ in X, the subprofile

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$$

in X_{-i} is denoted by x_{-i}, and we sometimes represent x by (x_i, x_{-i}), which is a slight abuse of notation.

Definition 1. A strategy profile $x = (x_i, x_{-i})$ in X is a *Nash equilibrium* of $G = (X_i, u_i)_{i \in N}$ if $u_i(y_i, x_{-i}) \le u_i(x)$ for every $y_i \in X_i$ and each i.

Given a compact, metric game $G = (X_i, u_i)_{i \in N}$, the *mixed extension* of G is the game

$$\overline{G} = (\Delta(X_i), u_i)_{i \in N}, \tag{1}$$

where each $\Delta(X_i)$ represents the set of regular Borel probability measures on X_i, endowed with the weak* topology, and, abusing

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notation, we let $u_i : \times_{j=1}^N \Delta(X_j) \to \mathbb{R}$ be defined by

$$u_i(\mu) := \int_X u_i d\mu.$$

With a slight abuse of notation, we define $\Delta(X) := \times_{j\in N} \Delta(X_j)$. This Cartesian product is endowed with the product topology.

A *mixed-strategy Nash equilibrium* of $G = (X_i, u_i)_{i\in N}$ is a Nash equilibrium of the mixed extension $\overline{G}$ as defined in (1).

The next definition is taken from Carbonell-Nicolau and McLean (2013).

Definition 2. A metric game $G = (X_i, u_i)_{i\in N}$ satisfies *sequential better-reply security* if the following condition is satisfied: if $(x^n, u(x^n)) \in X \times \mathbb{R}^N$ is a convergent sequence with limit $(x, \gamma) \in X \times \mathbb{R}^N$, and if x is not a Nash equilibrium of G, then there exist an i, an $\eta > \gamma_i$, a subsequence (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i(y_i^k, x_{-i}^k) \geq \eta$.

The following condition appears in Monteiro and Page (2007).

Definition 3. A metric game $G = (X_i, u_i)_{i\in N}$ is *uniformly payoff secure* if for each i, $\varepsilon > 0$, and $x_i \in X_i$, there exists $y_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(y_i, z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for every $z_{-i} \in V_{y_{-i}}$.

For each player i, let X_i be a nonempty, compact, metric space, and let $X := \times_{i\in N} X_i$. Let $B(X)$ denote the set of bounded, Borel measurable maps $f : X \to \mathbb{R}$. We view $(B(X)^N, d_X)$ as a metric space, where $d_X : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$d_X((f_1, \ldots, f_N), (g_1, \ldots, g_N)) := \sum_{i\in N} \sup_{x\in X} |f_i(x) - g_i(x)|. \tag{2}$$

It is clear that a metric Borel game of the form $(X_i, u_i)_{i\in N}$ can be viewed as member of $(B(X)^N, d_X)$, and we can define the mixed-strategy Nash equilibrium correspondence over $B(X)^N$ as a set-valued map

$$\mathfrak{E}_X : B(X)^N \rightrightarrows \Delta(X)$$

that assigns to each game G in $B(X)^N$ the set $\mathfrak{E}_X(G)$ of mixed-strategy Nash equilibria of G, i.e., the set of Nash equilibria of the mixed extension $\overline{G}$. Given a family of games $\mathfrak{g} \subseteq B(X)^N$, the restriction of $\mathfrak{E}_X$ to $\mathfrak{g}$ is denoted by $\mathfrak{E}_X|_{\mathfrak{g}}$.

Definition 4. Given a class of games $\mathfrak{g} \subseteq B(X)^N$, a mixed-strategy Nash equilibrium μ of $G \in \mathfrak{g}$ is an *essential equilibrium of G relative to* $\mathfrak{g}$ if for every neighborhood V_μ of μ, there is a neighborhood V_G of G such that for every $g \in V_G \cap \mathfrak{g}$, $V_\mu \cap \mathfrak{E}_X(g) \neq \emptyset$.

The notion of essentiality was introduced for finite games by Wu and Jiang (1962).

A probability measure $\mu_i \in \Delta(X_i)$ is said to be *strictly positive* if $\mu_i(O) > 0$ for every nonempty open set O in X_i.

For each i, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive members of $\Delta(X_i)$. The set of regular Borel measures on X_i is denoted by $M(X_i)$. Let $\widehat{M}(X_i)$ be the set of p_i in $M(X_i)$ such that $p_i(O) > 0$ for every nonempty open set O in X_i. Define

$$\widehat{\Delta}(X) := \times_{i\in N} \widehat{\Delta}(X_i) \quad \text{and} \quad \widehat{M}(X) := \times_{i\in N} \widehat{M}(X_i).$$

For $p = (p_1, \ldots, p_N) \in \widehat{M}(X)$, let

$$\Delta(X_i, p_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq p_i\}$$

and define

$$\overline{G}_p := (\Delta(X_i, p_i), u_i)_{i\in N}.$$

The game $\overline{G}_p$ is called a *Selten perturbation* of G. For $\nu = (\nu_1, \ldots, \nu_N) \in \widehat{\Delta}(X)$ and $\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N$, define the Selten perturbation $\overline{G}_{\delta*\nu}$ as

$$\overline{G}_{\delta*\nu} = (\Delta(X_i, \delta_i\nu_i), u_i)_{i\in N}.$$

Definition 5. A strategy profile $\mu \in \Delta(X)$ is *perfect* in $G = (X_i, u_i)_{i\in N}$ if there are sequences (δ^n), (ν^n), and (μ^n) such that $\delta^n \in (0, 1)^N$ and $\nu^n \in \widehat{\Delta}(X)$ for each n, $\delta^n \to 0$, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of $\overline{G}_{\delta^n*\nu^n}$.

Definition 6. A strategy profile $\mu \in \Delta(X)$ is *strictly perfect* in $G = (X_i, u_i)_{i\in N}$ if for all sequences (δ^n) and (ν^n) such that $\delta^n \in (0, 1)^N$ and $\nu^n \in \widehat{\Delta}(X)$ for each n, and $\delta^n \to 0$, there is a sequence (μ^n) such that $\mu^n \to \mu$ and each μ^n is a Nash equilibrium of $\overline{G}_{\delta^n*\nu^n}$.

The notions of perfection and strict perfection were introduced for finite-action games by Selten (1975) and Okada (1984), respectively.[1]

Given a compact, metric game $G = (X_i, u_i)_{i=1}^N$, we will endow $\Delta(X)$ with the product topology induced by the Prokhorov metric on $\Delta(X_i)$.[2] If ϱ_i denotes the Prokhorov metric on $\Delta(X_i)$, then given $\{\mu, \nu\} \subseteq \Delta(X_i)$,

$$\varrho_i(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon, \text{ for all } B\},$$

where

$$B^\varepsilon := \{x \in X_i : d_i(x, y) < \varepsilon \text{ for some } y \in B\},$$

and d_i denotes the metric associated with X_i. The product metric induced by $(\varrho_1, \ldots, \varrho_N)$ on $\Delta(X)$ is denoted by ϱ.

For $\varepsilon > 0$ and $\emptyset \neq E \subseteq \Delta(X)$, a profile $\mu \in \Delta(X)$ is said to be *ε-close to E* if

$$\varrho(\mu, E) := \inf\{\varrho(\mu, \nu) : \nu \in E\} < \varepsilon.$$

Here and below, $N_\varepsilon(\mu)$ denotes the ε-neighborhood of μ.

Let $\mathcal{S}_G$ be the family of all nonempty closed sets E of Nash equilibria of $\overline{G}$ satisfying the following: for each $\varepsilon > 0$, there exists $\alpha \in (0, 1]$ such that for each $\delta \in (0, \alpha)^N$ and every $\nu \in \widehat{\Delta}(X)$ the perturbed game $\overline{G}_{\delta*\nu}$ has a Nash equilibrium ε-close to E.

Given $x_i \in X_i$, let θ_{x_i} represent the Dirac measure on X_i with support $\{x_i\}$. Similarly, for $x \in X$, θ_x denotes the Dirac measure on X with support $\{x\}$. The map $x_i \mapsto \theta_{x_i}$ (resp. $x \mapsto \theta_x$) is an embedding, so X_i (resp. X) can be topologically identified with a subspace of $\Delta(X_i)$ (resp. $\Delta(X)$). We sometimes abuse notation and refer to $\theta_{x_i} \in \Delta(X_i)$ (resp. $\theta_x \in \Delta(X)$) simply as x_i (resp. x).

Definition 7. A set of mixed strategy profiles in $\Delta(X)$ is a *stable set* of G if it is a minimal element of the set $\mathcal{S}_G$ ordered by set inclusion.

The notion of stability was introduced for finite-action games by Kohlberg and Mertens (1986).

Remark 1. A profile μ is a strictly perfect equilibrium if, and only if, the set $\{\mu\}$ is stable.

Given $(\delta, \mu) \in [0, 1)^N \times \widehat{\Delta}(X)$ and $G = (X_i, u_i)_{i\in N}$, let $G_{(\delta,\mu)}$ be a game defined as

$$G_{(\delta,\mu)} := (X_i, u_i^{(\delta,\mu)})_{i\in N},$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, \ldots, (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Here, $(1-\delta_i)x_i + \delta_i\mu_i$ represents the measure σ_i in $\Delta(X_i)$ such that

$$\sigma_i(B) = (1-\delta_i)\theta_{x_i}(B) + \delta_i\mu_i(B).$$

[1] Infinite-game generalizations of these notions were introduced in Simon and Stinchcombe (1995) and studied in the context of discontinuous games in Carbonell-Nicolau (2011b,c,d).

[2] For compact metric games, this product topology coincides with the product topology induced by the weak* topology on $\Delta(X_i)$.

3. Essential equilibria

For each $i \in N$, let X_i be an action space, and let $X := \times_{i\in N} X_i$. Define the set $\mathfrak{g}_X^u$ of games $(X_i, u_i)_{i=1}^N$ that are compact, metric, Borel, and uniformly payoff secure, with $\sum_{i\in N} u_i$ upper semicontinuous.

We view $\mathfrak{g}_X^u$ as a subspace of the metric space $(B(X)^N, d_X)$ with its relative topology.

We first recapture a result from Carbonell-Nicolau (2010).

Theorem 1. *For any G in a dense, residual subset of $\mathfrak{g}_X^u$, any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}_X^u$.*

We do not know whether generic games in $\mathfrak{g}_X^u$ can be guaranteed to have only essential, (strictly) perfect equilibria. We remark that the statement that generic games in $\mathfrak{g}_X^u$ have only essential, (strictly) perfect equilibria is not a corollary of the above result. In fact, Example 3 in Carbonell-Nicolau (2011c) shows that there is at least one member G of $\mathfrak{g}_X^u$ whose Selten perturbations do not belong to $\mathfrak{g}_X^u$. While G may well be non-generic, it has not been proven that *generically* the collection of games $\mathfrak{g}_X^u$ is closed under Selten perturbations.

In the remainder of the paper, we adapt ideas from Carbonell-Nicolau (2011a) to show that there are subcollections of $\mathfrak{g}_X^u$ that are closed under some (resp. all) Selten perturbations. This observation, together with the above result, implies that generic games in these subcollections are not only essential but also perfect (resp. strictly perfect).

4. Essential and perfect equilibria

The following condition is taken from Carbonell-Nicolau (2011b).[3]

Condition (A). There exists $(\mu_1, \ldots, \mu_N) \in \widehat{\Delta}(X)$ such that for each i and every $\varepsilon > 0$ there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:

(a) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) > u_i(x) - \varepsilon$ for all $y_{-i} \in N_{x_{-i}}$.
(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following condition: for each $x_i \in Y_i$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in V_{x_{-i}}$.

Define the set $\mathfrak{g}_X^A$ of compact, metric, Borel games $G = (X_i, u_i)_{i\in N}$ with $\sum_{i\in N} u_i$ upper semicontinuous such that Condition (A) is satisfied.

Theorem 2 (*Carbonell-Nicolau (2011c, Theorem 2)*)**.** *All members G of $\mathfrak{g}_X^A$ have a perfect equilibrium, and all perfect profiles of G are mixed-strategy Nash equilibria of G.*

Lemma 1. *Suppose that (g^n) is a sequence in $B(X)$ with limit $g \in B(X)$. If g^n is upper semicontinuous for each n, then g is upper semicontinuous.*

Proof. Suppose that (g^n) is a sequence of upper semicontinuous functions in $B(X)$ with limit $g \in B(X)$. Fix $\alpha \in \mathbb{R}$. Then the set $\{x : g(x) \geq \alpha\}$ can be written as

$$\bigcap_n \left\{x : g^n(x) \geq \alpha - \sup_{x\in X} |g^n(x) - g(x)|\right\},$$

a countable intersection of closed sets. It follows that $\{x : g(x) \geq \alpha\}$ is closed or, equivalently, that g is upper semicontinuous. ■

[3] The condition is called (A′) in footnote 8 of Carbonell-Nicolau (2011b).

Lemma 2. *The set $\mathfrak{g}_X^A$ is closed in $B(X)^N$.*

Proof. Take a sequence (u^n) in $B(X)^N$ such that the sequence $\left((X_i, u_i^n)_{i\in N}\right)$ belongs to $\mathfrak{g}_X^A$. Suppose that $u^n \to u$ for some $u \in B(X)^N$. We show that $(X_i, u_i)_{i\in N}$ belongs to $\mathfrak{g}_X^A$.

To lighten notation, let

$$G := (X_i, u_i)_{i\in N} \quad \text{and} \quad G^n := (X_i, u_i^n)_{i\in N}.$$

Because $G^n \in \mathfrak{g}_X^A$ for each n, $\sum_{i\in N} u_i^n$ is upper semicontinuous for each n. Consequently, since $\sum_{i\in N} u_i^n \to \sum_{i\in N} u_i$, $\sum_{i\in N} u_i$ is upper semicontinuous as a consequence of Lemma 1.

It remains to show that G satisfies Condition (A). Since $G^n \in \mathfrak{g}_X^A$ for each n, for each n there exists $(\mu_1^n, \ldots, \mu_N^n) \in \widehat{\Delta}(X)$ such that for each i and every $\varepsilon > 0$, there is a sequence $(f_k^n)_{k=1}^\infty$ of Borel measurable maps $f_k^n : X_i \to X_i$ such that the following is satisfied:

(a) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i^n(f_k^n(x_i), y_{-i}) > u_i^n(x) - \frac{\varepsilon}{2}$ for all $y_{-i} \in N_{x_{-i}}$.
(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i^n of X_i with $\mu_i^n(Y_i^n) = 1$ satisfying the following condition: for each $x_i \in Y_i^n$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i^n(f_k^n(x_i), y_{-i}) < u_i^n(x_i, y_{-i}) + \frac{\varepsilon}{2}$ for all $y_{-i} \in V_{x_{-i}}$.

Since $u^n \to u$, for any large enough n we have

$$u_i^n(z) + \tfrac{\varepsilon}{4} > u_i(z) > u_i^n(z) - \tfrac{\varepsilon}{4}, \quad \text{for all } z \in X.$$

It follows that for any large enough n the following is satisfied:

(a) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that

$$u_i(f_k^n(x_i), y_{-i}) > u_i^n(f_k^n(x_i), y_{-i}) - \tfrac{\varepsilon}{4} > u_i^n(x) - \frac{3\varepsilon}{4} > u_i(x) - \varepsilon,$$

for all $y_{-i} \in N_{x_{-i}}$.
(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i^n of X_i with $\mu_i^n(Y_i^n) = 1$ satisfying the following condition: for each $x_i \in Y_i^n$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(f_k^n(x_i), y_{-i}) < u_i^n(f_k^n(x_i), y_{-i}) + \tfrac{\varepsilon}{4} < u_i^n(x_i, y_{-i}) + \tfrac{3\varepsilon}{4} < u_i(x_i, y_{-i}) + \varepsilon,$$

for all $y_{-i} \in V_{x_{-i}}$.

We conclude that given i and $\varepsilon > 0$, and for large n, the sequence $(f_k^n)_{k=1}^\infty$ of satisfies the following:

(a) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i(f_k^n(x_i), y_{-i}) > u_i(x) - \varepsilon$ for all $y_{-i} \in N_{x_{-i}}$.
(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i^n of X_i with $\mu_i^n(Y_i^n) = 1$ satisfying the following condition: for each $x_i \in Y_i^n$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i(f_k^n(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in V_{x_{-i}}$.

Therefore G satisfies Condition (A). ■

The next lemma follows immediately from the following facts: (i) sequential better-reply security is weaker than Reny's (1999) better-reply security; and (ii) the mixed extension of a game is better-reply secure if the game has an upper semicontinuous sum of payoffs and satisfies Condition (A).

Lemma 3. *Suppose that $G \in \mathfrak{g}_X^A$. Then the mixed extension $\overline{G}$ of G satisfies sequential better-reply security.*

Lemma 4. *Suppose that X is compact and metric. For $\mathfrak{g} \subseteq B(X)^N$, if the mixed extension $\overline{G}$ of G satisfies sequential better-reply security for every $G \in \mathfrak{g}$, then $\mathfrak{E}_X|_{\mathfrak{g}}$ is compact-valued and upper hemicontinuous.*

Proof. Since X is compact and metric, $\Delta(X)$ is compact. Therefore, it suffices to show that $\mathfrak{E}_X|_{\mathfrak{g}}$ has a closed graph (e.g., Aliprantis and Border (2006, Theorem 17.11)). Take a sequence (u^n) in $B(X)^N$ such that the sequence $\left((X_i, u_i^n)_{i\in N}\right)$ belongs to $\mathfrak{g}$, and take a sequence (μ^n) such that μ^n is a mixed-strategy Nash equilibrium of $(X_i, u_i^n)_{i\in N}$ for each n. Suppose that

$$(\mu^n, u^n) \to (\mu, u),$$

for some $(\mu, u) \in \Delta(X) \times B(X)^N$ such that $(X_i, u_i)_{i\in N}$ is a member of $\mathfrak{g}$. We must show that μ is a mixed-strategy Nash equilibrium of $(X_i, u_i)_{i\in N}$.

Suppose that μ is not a mixed-strategy Nash equilibrium of $(X_i, u_i)_{i\in N}$. Because $\mu^n \to \mu$ and u_i is bounded for each i, we may write (passing to a subsequence if necessary)

$$(\mu^n, u(\mu^n)) \to (\mu, \gamma), \tag{3}$$

for some $\gamma \in \mathbb{R}^N$. Therefore, because μ is not a mixed-strategy Nash equilibrium of $(X_i, u_i)_{i\in N}$, and since the mixed extension of $(X_i, u_i)_{i\in N}$ is sequentially better-reply secure (Lemma 3), there exist an i, an $\eta > \gamma_i$, a subsequence (μ^k) of (μ^n), and a sequence (ν_i^k) such that for each k, $\nu_i^k \in \Delta(X_i)$ and $u_i(\nu_i^k, \mu_{-i}^k) \geq \eta$. This, together with (3), gives, for some $\alpha \in \mathbb{R}$ and some $\beta \in \mathbb{R}$, and for any large enough k,

$$u_i(\nu_i^k, \mu_{-i}^k) > \alpha > \beta > u_i(\mu^k).$$

Consequently, since $u_i^n \to u_i$, there exists k such that

$$u_i^k(\nu_i^k, \mu_{-i}^k) > u_i^k(\mu^k),$$

contradicting that μ^k is a mixed-strategy Nash equilibrium of $(X_j, u_j^k)_{j\in N}$. ■

Lemmas 3 and 4 immediately yield the following lemma.

Lemma 5. $\mathfrak{E}_X|_{\mathfrak{g}_X^A}$ *is compact-valued and upper hemicontinuous.*

The proof of the following lemma is relegated to Section 6.

Lemma 6. *Suppose that G is a compact, metric, Borel game satisfying Condition (A). Then there exists $\mu \in \widehat{\Delta}(X)$ such that for every $\delta \in (0,1)^N$, $G_{(\delta,\mu)}$ is a compact, metric, Borel game satisfying Condition (A).*

Lemma 7. *Suppose that $G \in \mathfrak{g}_X^A$. Then there exists $\mu \in \widehat{\Delta}(X)$ such that for every $\delta \in (0,1)^N$, $G_{(\delta,\mu)} \in \mathfrak{g}_X^A$.*

Proof. Suppose that $G = (X_i, u_i)_{i\in N} \in \mathfrak{g}_X^A$. By Lemma 6, there exists $\mu \in \widehat{\Delta}(X)$ such that for every $\delta \in (0,1)^N$, $G_{(\delta,\mu)}$ is a compact, metric, Borel game satisfying Condition (A). In addition, because $\sum_{i\in N} u_i$ is upper semicontinuous, the map $\nu \mapsto \sum_{i\in N} u_i(\nu)$ defined on $\Delta(X)$ is upper semicontinuous (e.g., Aliprantis and Border (2006, Theorem 15.5)). It follows that $\sum_{i\in N} u_i^{(\delta,\mu)}$ is upper semicontinuous. ■

Lemma 8. *If $G \in \mathfrak{g}_X^A$ and μ is an essential equilibrium of G relative to $\mathfrak{g}_X^A$, then μ is perfect.*

Proof. Let $G = (X_i, u_i)_{i\in N}$ be a member of $\mathfrak{g}_X^A$. By Lemma 7, there exists $\mu \in \widehat{\Delta}(X)$ such that for every $\delta \in (0,1)^N$, $G_{(\delta,\mu)} \in \mathfrak{g}_X^A$. Suppose that ν is an essential equilibrium of G relative to $\mathfrak{g}_X^A$. Then, for every neighborhood V_ν of ν, there is a neighborhood V_G of G such that for every $g \in V_G \cap \mathfrak{g}_X^A$, $V_\nu \cap \mathfrak{E}_X(g) \neq \emptyset$. Consequently, since for every $\beta > 0$ one can choose a small enough $\delta \in (0,1)^N$ such that $d_X(u, u^{(\delta,\mu)}) < \beta$, and because $G_{(\delta,\mu)} \in \mathfrak{g}_X^A$ for every $\delta \in (0,1)^N$, we see that there are sequences (δ^n) and (ν^n) such that $\delta^n \in (0,1)^N$ for each n, $\delta^n \to 0$, ν^n is a mixed-strategy Nash equilibrium of $G_{(\delta^n,\mu)}$ for each n, and $\nu^n \to \nu$. It is now easy to see that for each n the strategy profile

$$\left((1-\delta_1^n)\nu_1^n + \delta_1^n\mu_1, \ldots, (1-\delta_N^n)\nu_N^n + \delta_N^n\mu_N\right)$$

is a Nash equilibrium of the Selten perturbation $\overline{G}_{\delta^n * \mu}$. We conclude that ν is a perfect profile. ■

Theorem 3 (*Fort (1951, Theorem 2)*)**.** *Suppose that X is a metric space and that Y is a topological space. Suppose that $F : Y \rightrightarrows X$ is a nonempty-valued, compact-valued, upper hemicontinuous correspondence. Then there exists a residual subset Q of Y such that F is lower hemicontinuous at every point in Q.*

Theorem 4. *All members G of $\mathfrak{g}_X^A$ have a perfect equilibrium, and all perfect profiles of G are mixed-strategy Nash equilibria of G. In addition, for any G in a dense, residual subset of $\mathfrak{g}_X^A$, any mixed-strategy Nash equilibrium of G is perfect and essential relative to $\mathfrak{g}_X^A$.*

Proof. The first statement follows from Theorem 2. The correspondence $\mathfrak{E}_X|_{\mathfrak{g}_X^A}$ is nonempty-valued (Theorem 2), compact-valued and upper hemicontinuous (Lemma 5). Consequently, Theorem 3 gives a residual subset $\mathfrak{q}$ of $\mathfrak{g}_X^A$ such that $\mathfrak{E}_X|_{\mathfrak{g}_X^A}$ is lower hemicontinuous at every point in $\mathfrak{q}$. Since $\mathfrak{E}_X|_{\mathfrak{g}_X^A}$ is upper hemicontinuous and lower hemicontinuous at every point in $\mathfrak{q}$, for each $G \in \mathfrak{q}$ any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}_X^A$. Consequently, by Lemma 8, for each $G \in \mathfrak{q}$ any mixed-strategy Nash equilibrium of G is perfect and essential relative to $\mathfrak{g}_X^A$. To see that $\mathfrak{q}$ is dense in $\mathfrak{g}_X^A$, note that because $\mathfrak{g}_X^A$ is a closed subset of $B(X)^N$ (Lemma 2), and since $B(X)^N$ is a complete, metric space, $\mathfrak{g}_X^A$ is a complete, metric space. Therefore, $\mathfrak{g}_X^A$ is a Baire space by the Baire category theorem. Consequently, $\mathfrak{q}$, being a residual subset of a Baire space, is dense. ■

5. Essential and strictly perfect equilibria

Unfortunately, as the following example illustrates, the collection $\mathfrak{g}_X^A$ is not closed under *all* Selten perturbations, so it is not immediately apparent that one can replace "perfect" by "strictly perfect" in the last statement of Theorem 4.

Example 1. Consider the two-player game $G = ([0,1],[0,1],u_1,u_2)$, where

$$u_1(x_1,x_2) := \begin{cases} 1 & \text{if } x_1 = 1 \text{ or } (x_1,x_2) = \left(\frac{1}{2},\frac{1}{2}\right), \\ 0 & \text{elsewhere,} \end{cases}$$

and u_2 is identically zero. The game G is a member of $\mathfrak{g}_X^A$.

Next, we show that there exists $\mu \in \widehat{\Delta}([0,1]^2)$ such that for any $\delta \in (0,1)^2$, $G_{(\delta,\mu)}$ does not belong to $\mathfrak{g}_X^A$. This means that even if G has an essential equilibrium ν, it does not follow from Theorem 4 that the perturbations $G_{(\delta,\mu)}$ will have a mixed-strategy Nash equilibrium close to ν. Since Nash equilibria of the Selten perturbation $\overline{G}_{\delta*\mu}$ are mixed-strategy Nash equilibria of $G_{(\delta,\mu)}$, it follows that Theorem 4 does not imply that there are sequences (δ^n) and (ν^n) with $\delta^n \in (0,1)^2$ for each n and $\delta^n \to 0$ such that $\nu^n \to \nu$ and ν^n is a Nash equilibrium of $\overline{G}_{\delta^n*\mu}$ for each n. Thus, one cannot conclude that the essential equilibrium ν is strictly perfect.

To see that there exists $\mu \in \widehat{\Delta}([0,1]^2)$ such that for any $\delta \in (0,1)^2$, $G_{(\delta,\mu)}$ does not belong to $\mathfrak{g}_X^A$, it suffices to show that given $(\delta,\mu) \in (0,1)^2 \times \widehat{\Delta}([0,1]^2)$ with

$$\mu_1 = \tfrac{1}{2}\theta_{\frac{1}{2}} + \tfrac{1}{2}\lambda \quad \text{and} \quad \mu_2 = \lambda,$$

where λ denotes Lebesgue measure over $[0, 1]$, and given any $(p_1, p_2) \in \widehat{\Delta}([0, 1]^2)$ and any map $f : [0, 1] \to [0, 1]$, the following two conditions cannot hold simultaneously for

$$\varepsilon \in \left(0, \min\left\{\delta_1(1-\delta_2)\tfrac{1}{2}, 1-\delta_1\right\}\right).$$

(a) For each $(x_1, x_2) \in [0, 1]^2$, there is a neighborhood N_{x_2} of x_2 such that

$$u_1((1-\delta_1)f(x_1) + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) > u_1((1-\delta_1)x_1 + \delta_1\mu_1, (1-\delta_2)x_2 + \delta_2\mu_2) - \varepsilon$$

for all $y_2 \in N_{x_2}$.

(b) For each $x_2 \in [0, 1]$, there is a subset I of $[0, 1]$ with $p_1(I) = 1$ satisfying the following condition: for each $x_1 \in I$, there is a neighborhood V_{x_2} of x_2 such that

$$u_1((1-\delta_1)f(x_1) + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) < u_1((1-\delta_1)x_1 + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) + \varepsilon$$

for all $y_2 \in V_{x_2}$.

Suppose that $x_2 = \frac{1}{2}$. Then, given $x_1 \in [0, 1]$, (a) implies $f(x_1) = 1$. To see this, note that if $f(x_1) \neq 1$ and $y_2 \neq \frac{1}{2}$ we have

$$\begin{aligned} u_1((1-\delta_1)f(x_1) + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) &= 0 \\ &< \delta_1(1-\delta_2)\tfrac{1}{2} - \varepsilon \\ &\leq u_1((1-\delta_1)x_1 + \delta_1\mu_1, (1-\delta_2)x_2 + \delta_2\mu_2) - \varepsilon. \end{aligned}$$

But if $f(x_1) = 1$ for each $x_1 \in [0, 1]$ then (b) cannot hold. Indeed, if $f(x_1) = 1$ for each $x_1 \in [0, 1]$, then for each $x_1 \in [0, 1) \setminus \{\frac{1}{2}\}$ and every $y_2 \in [0, 1] \setminus \{\frac{1}{2}\}$,

$$\begin{aligned} u_1((1-\delta_1)f(x_1) + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) &\geq 1-\delta_1 \\ &> \varepsilon \\ &= u_1((1-\delta_1)x_1 + \delta_1\mu_1, (1-\delta_2)y_2 + \delta_2\mu_2) + \varepsilon, \end{aligned}$$

contradicting condition (b).

The following condition is taken from Carbonell-Nicolau (2011a).

Condition (B). For each i and every $\varepsilon > 0$, there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:

(a) For each $x \in X$ and each k, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) > u_i(x) - \varepsilon$ for all $y_{-i} \in N_{x_{-i}}$.

(b) For each $x \in X$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in V_{x_{-i}}$.

Define the set $\mathfrak{g}_X^B$ of compact, metric, Borel games $G = (X_i, u_i)_{i\in N}$ with $\sum_{i\in N} u_i$ upper semicontinuous such that Condition (B) is satisfied.

Remark 2. It is easy to see that $\mathfrak{g}_X^B \subseteq \mathfrak{g}_X^A$.

Example 2. The following is an example of a game in $\mathfrak{g}_X^A \setminus \mathfrak{g}_X^B$. Let $G = ([0, 1], [0, 1], u_1, u_2)$ be a two-player game with

$$u_1(x_1, x_2) := \begin{cases} 1 - x_2 & \text{if } x_1 \text{ is rational,} \\ 1 & \text{if } x_1 \text{ is irrational and } x_2 = 0, \\ 0 & \text{if } x_1 \text{ is irrational and } x_2 > 0, \end{cases}$$

and suppose that u_2 is identically zero. Clearly, $u_1 + u_2$ is upper semicontinuous. Since u_2 is continuous, Condition (A) is clearly satisfied for $i = 2$. To see that Condition (A) holds for $i = 1$, fix any $\mu_2 \in \widehat{\Delta}([0, 1])$ and choose a $\mu_1 \in \widehat{\Delta}([0, 1])$ supported on the set of rational numbers in $[0, 1]$. Fix $\varepsilon > 0$ and define a sequence (f_k) of maps $f_k : [0, 1] \to [0, 1]$ by $f_k := f$ for each k, where $f : [0, 1] \to [0, 1]$ is defined by

$$f(a) := \begin{cases} a & \text{if } a \text{ is rational,} \\ 1 & \text{if } a \text{ is irrational.} \end{cases}$$

We verify items (a) and (b) in Condition (A).

(a) Fix $(x_1, x_2) \in [0, 1]^2$. If x_1 is irrational and $x_2 > 0$, then for all $y_2 \in [0, 1]$,

$$\begin{aligned} u_1(f(x_1), y_2) &= u_1(1, y_2) = 1 - y_2 \geq 0 \\ &= u_1(x_1, x_2) > u_1(x_1, x_2) - \varepsilon. \end{aligned}$$

If x_1 is irrational and $x_2 = 0$, then for all $y_2 \in [0, \frac{\varepsilon}{2})$,

$$\begin{aligned} u_1(f(x_1), y_2) &= u_1(1, y_2) = 1 - y_2 > 1 - \varepsilon \\ &= u_1(x_1, x_2) - \varepsilon. \end{aligned}$$

If x_1 is rational, then for all $y_2 \in (x_2 - \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}) \cap [0, 1]$,

$$\begin{aligned} u_1(f(x_1), y_2) &= u_1(x_1, y_2) = 1 - y_2 > 1 - x_2 - \varepsilon \\ &= u_1(x_1, x_2) - \varepsilon. \end{aligned}$$

(b) For each $x_2 \in [0, 1]$, let Y_1 be the set of rational numbers and note that $\mu_1(Y_1) = 1$. Then for each $x_1 \in Y_1$ we have $f(x_1) = x_1$ and therefore $u_1(f(x_1), y_2) < u_1(x_1, y_2) + \varepsilon$ for all $y_2 \in [0, 1]$.

To see that G fails Condition (B), let $\varepsilon := \frac{1}{2}$ and let (f_k) be a sequence of Borel measurable maps $f_k : [0, 1] \to [0, 1]$. Observe that for $(x_1, x_2) \in [0, 1]^2$ with x_1 irrational and $x_2 = 0$, and given any k, if $f_k(x_1)$ is irrational, then for any neighborhood N_{x_2} of x_2 and for $y_2 \in N_{x_2} \setminus \{x_2\}$ we have

$$u_1(f_k(x_1), y_2) = 0 < \tfrac{1}{2} = 1 - \varepsilon = u_1(x_1, x_2).$$

Hence, item (a) in Condition (B) can only be fulfilled if $f_k(x_1)$ is rational for each k. But if $f_k(x_1)$ is rational for each k, item (b) in Condition (B) must be violated. Indeed, for any neighborhood V_{x_2} of x_2, and for $y_2 \in V_{x_2} \setminus \{x_2\}$ close enough to $x_2 = 0$, we have

$$u_1(f_k(x_1), y_2) = 1 - y_2 > \tfrac{1}{2} = u_1(x_1, y_2) + \varepsilon.$$

The next and the last result follows from the analysis in Carbonell-Nicolau (2011a). We omit the proof.

Theorem 5. *All members G of $\mathfrak{g}_X^B$ have a stable set, and all stable sets of G contain only perfect equilibria. In addition, for any G in a dense, residual subset of $\mathfrak{g}_X^B$, any mixed-strategy Nash equilibrium of G is strictly perfect and essential relative to $\mathfrak{g}_X^B$.*

Remark 3. Theorem 4 (resp. Theorem 5) states that generic games within $\mathfrak{g}_X^A$ (resp. $\mathfrak{g}_X^B$) have only perfect (resp. strictly perfect) and essential equilibria. These assertions have been proven for a particular metric on the space of games $B(X)^N$, namely the sup metric defined in (2). Whether the above statements hold intact when the space of games is endowed with an alternative metric remains an open question. Other natural metrics are those that measure, in some precise way, the distance between the graphs of the members of $B(X)^N$. Such metrics induce topologies weaker than the sup metric and therefore strengthen the definition of essential equilibrium. Note however that when the space $B(X)^N$ is endowed with a weaker topology, it follows from Theorem 4 (resp. Theorem 5) that for any G in a dense subset of $\mathfrak{g}_X^A$ (resp. $\mathfrak{g}_X^B$), any mixed-strategy Nash equilibrium of G is perfect (resp. strictly perfect).

6. Proof of Lemma 6

Prior to proving Lemma 6, we need a preliminary result.

The following lemma is a variation of Lemma 7 in Carbonell-Nicolau (2011b). The proof of item (ii) is similar to that of item (ii) in Lemma 7 of Carbonell-Nicolau (2011b). The proof of item (i) proceeds in the same manner as the proof of Lemma 1 in Carbonell-Nicolau (2011b). We omit the details.

Lemma 9. *Suppose that $G = (X_i, u_i)_{i\in N}$ is a compact, metric, Borel game satisfying Condition (A). Then there exists $(\mu_1, \ldots, \mu_N) \in \widehat{\Delta}(X)$ such that for each i and every $\varepsilon > 0$ there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:*

(i) *For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i^{(\delta,\mu)}(f_k(x_i), y_{-i}) > u_i^{(\delta,\mu)}(x) - \varepsilon$ for all $y_{-i} \in N_{y_{-i}}$.*

(ii) *For each $\sigma_{-i} \in \Delta(X_{-i})$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following condition: for every $x_i \in Y_i$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $u_i(f_k(x_i), p_{-i}) < u_i(x_i, p_{-i}) + \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$.*

We are now ready to prove Lemma 6.

Lemma 6. *Suppose that G is a compact, metric, Borel game satisfying Condition (A). Then, there exists $\mu \in \widehat{\Delta}(X)$ such that for every $\delta \in (0, 1)^N$, $G_{(\delta,\mu)}$ is a compact, metric, Borel game satisfying Condition (A).*

Proof. Suppose that $G = (X_i, u_i)_{i\in N}$ is a compact, metric, Borel game satisfying Condition (A). Let μ be the measure given by Lemma 9, and fix $\delta \in (0, 1)^N$, i, and $\varepsilon > 0$. We must show that there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:

(a) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i^{(\delta,\mu)}(f_k(x_i), y_{-i}) > u_i^{(\delta,\mu)}(x) - \varepsilon$ for all $y_{-i} \in N_{x_{-i}}$.

(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following condition: for each $x_i \in Y_i$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that $u_i^{(\delta,\mu)}(f_k(x_i), y_{-i}) < u_i^{(\delta,\mu)}(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in V_{x_{-i}}$.

Lemma 9 gives a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ satisfying the following:

(i) For each k and $x \in X$, there is a neighborhood $N_{x_{-i}}$ of x_{-i} such that $u_i^{(\delta,\mu)}(f_k(x_i), y_{-i}) > u_i^{(\delta,\mu)}(x) - \varepsilon$ for all $y_{-i} \in N_{x_{-i}}$.

(ii) For each $\sigma_{-i} \in \Delta(X_{-i})$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following condition: for every $x_i \in Y_i$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $u_i(f_k(x_i), p_{-i}) < u_i(x_i, p_{-i}) + \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$.

To prove (b), fix $x_{-i} \in X_{-i}$. Define

$$\sigma_{-i} := ((1-\delta_1)x_1 + \delta_1\mu_1, \ldots, (1-\delta_{i-1})x_{i-1} + \delta_{i-1}\mu_{i-1},$$
$$(1-\delta_{i+1})x_{i+1} + \delta_{i+1}\mu_{i+1}, \ldots, (1-\delta_N)x_N + \delta_N\mu_N).$$

By (ii), there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following condition: for every $x_i \in Y_i$, there exists K such that for each $k \geq K$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $u_i(f_k(x_i), p_{-i}) < u_i(x_i, p_{-i}) + \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$. Consequently, for $k \geq K$, and for every $p_{-i} \in V_{\sigma_{-i}}$, we have

$$u_i((1-\delta_i)f_k(x_i) + \delta_i\mu_i, p_{-i}) - u_i((1-\delta_i)x_i + \delta_i\mu_i, p_{-i})$$
$$= (1-\delta_i)[u_i(f_k(x_i), p_{-i}) - u_i(x_i, p_{-i})] < \varepsilon.$$

This establishes (b). ■

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