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RESEARCH ARTICLE

Further results on essential Nash equilibria in normal-form games

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Abstract A Nash equilibrium x of a normal-form game G is essential if any perturbation of G has an equilibrium close to x. Using payoff perturbations, we identify a new collection of games containing a dense, residual subset of games whose Nash equilibria are all essential. This collection covers economic examples that cannot be handled by extant results and subsumes the sets of games considered in the literature.

Keywords Infinite normal-form game · Equilibrium refinement · Essential equilibrium · Equilibrium existence · Continuous security

JEL Classification C72

1 Introduction

Given a collection $\mathfrak{G}$ of normal-form games and given a game G in $\mathfrak{G}$, a Nash equilibrium x of G is *essential* relative to $\mathfrak{G}$ if neighboring games within $\mathfrak{G}$ have Nash equilibria close to x. For generic games in the collection of all finite-action games with fixed action spaces, all Nash equilibria are essential (cf. Wu and Jiang 1962). This result has been extended to infinite-action games (e.g., Yu 1999; Carbonell-Nicolau 2010; Scalzo 2013). Using the existence results in Barelli and Meneghel (2013), we identify a new collection of games whose generic members have only essential Nash equilibria. This collection covers economic examples that cannot be handled by extant results

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and subsumes the sets of games considered in Yu (1999), Carbonell-Nicolau (2010), Scalzo (2013).

The standard approach to proving generic essentiality of Nash equilibrium points consists in identifying a collection $\mathfrak{G}$ of compact, metric games with the following properties: (i) $\mathfrak{G}$ is a complete space; (ii) the members of $\mathfrak{G}$ possess a Nash equilibrium; and (iii) the Nash equilibrium correspondence, defined on $\mathfrak{G}$, is compact-valued and upper hemicontinuous. Items (i)–(iii), together with the main theorem in Fort (1951), yield the essentiality of Nash equilibrium points in generic games. Fort's (1951) theorem states that a correspondence F from a topological space to a metric space is lower hemicontinuous at a residual subset of its domain if F is nonempty-valued, compact-valued, and upper hemicontinuous.

Call a game an essential game if its Nash equilibria are all essential. Using the standard approach, we first identify a new collection of games $\mathfrak{G}$ whose generic members are essential. We then take advantage of the topological structure of $\mathfrak{G}$ to establish the existence of generic essential games within an extended domain, the topological closure of $\mathfrak{G}$. Methodologically, the proof of this extension is nonstandard in that it does not rely on Fort's (1951) theorem. The extended domain has interesting properties. First, it contains economically meaningful discontinuous games that cannot be handled by extant results. Second, it subsumes the set of continuous games.

While we cannot guarantee that the topological closure of $\mathfrak{G}$ contains the collections of games considered in Carbonell-Nicolau (2010) and Scalzo (2013), we do prove the existence of generic essential games within a strict superset of the sets of games considered in the literature, namely the union of (the closure of) $\mathfrak{G}$ and the collections of games considered in Carbonell-Nicolau (2010) and Scalzo (2013).

2 Preliminaries

A *normal-form game* (or simply a *game*) is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $u_i : X \to \mathbb{R}$ represents player i's payoff function, defined on the set of action profiles $X := \times_{i=1}^N X_i$. By a slight abuse of notation, N will represent both the number of players and the set of players.

If u_i is bounded and X_i is a nonempty subset of a metric space for each i, G is said to be a *metric game*. If in addition X_i is compact for each i, then G is called a *compact, metric game*. If X_i is a nonempty, convex subset of a locally convex vector space for each i, then G is called a *convex game*. If G is a metric, convex game and in addition the map $x_i \mapsto u_i(x_i, x_{-i})$ defined on X_i is quasiconcave for each i and every $x_{-i} \in X_{-i}$, then G is said to be a *metric*, *quasiconcave game*.

For each i, let $X_{-i} := \times_{j \neq i} X_j$. Given i and a strategy profile $x = (x_1, \ldots, x_N)$ in X, the subprofile

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$$

in X_{-i} is denoted by x_{-i}, and we sometimes write, with a slight abuse of notation, (x_i, x_{-i}) for x.

Definition 1 A strategy profile $x = (x_i, x_{-i})$ in X is a *Nash equilibrium* of $G = (X_i, u_i)_{i \in N}$ if $u_i(y_i, x_{-i}) \leq u_i(x)$ for every $y_i \in X_i$ and each i.

A Nash equilibrium of $G = (X_i, u_i)_{i \in N}$ is sometimes referred to as a *pure-strategy Nash equilibrium* of G.

Definition 2 A subset of a topological space is *nowhere dense* if its closure has an empty interior. A subset of a topological space is *meager* if it is a countable union of nowhere dense sets. A subset of a topological space is *residual* if it is the complement of a meager set. A topological space A is said to be a *Baire space* if every residual set in A is dense in A.

Definition 3 A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is *closed* if its graph, $\{(x, y) \in A \times B : y \in \Phi(x)\}$, is closed in $A \times B$.

For each player i, let X_i be a nonempty, compact metric space, and let $X := \times_{i \in N} X_i$. Let $B(X)$ denote the set of bounded maps $f : X \to \mathbb{R}$. We view $(B(X)^N, \rho_X)$ as a metric space, where $\rho_X : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$\rho_X\left((f_1, \ldots, f_N), (g_1, \ldots, g_N)\right) := \sum_{i \in N} \sup_{x \in X} |f_i(x) - g_i(x)|.$$

It is clear that a game of the form $(X_i, u_i)_{i \in N}$ can be viewed as member u of $(B(X)^N, \rho_X)$, and we can define the Nash equilibrium correspondence as a set-valued map

$$\mathcal{E}_X : B(X)^N \rightrightarrows X$$

that assigns to each game u in $B(X)^N$ the set of Nash equilibria of u, $\mathcal{E}_X(u)$. Given a family of games $\mathfrak{G} \subseteq B(X)^N$, the restriction of $\mathcal{E}_X$ to $\mathfrak{G}$ is denoted by $\mathcal{E}_X|_{\mathfrak{G}}$.

By a slight abuse of notation, we often represent a game of the form $(X_i, u_i)_{i \in N}$ simply as u.

Definition 4 Given a class of games $\mathfrak{G} \subseteq B(X)^N$, a Nash equilibrium x of $u \in \mathfrak{G}$ is an *essential equilibrium of u relative to* $\mathfrak{G}$ if for every neighborhood V_x of x, there is a neighborhood V_u of u such that for every $f \in V_u \cap \mathfrak{G}$, $V_x \cap \mathcal{E}_X(f) \neq \emptyset$.

Definition 5 Suppose that $\mathfrak{G} \subseteq B(X)^N$. A game u in $\mathfrak{G}$ is *essential relative to* $\mathfrak{G}$ if every pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}$. When the domain of reference is clear from the context, we shall simply say that u is an *essential game*.

3 The main results

Given a game $(X_i, u_i)_{i \in N}$ and given $Y \subseteq X$, a correspondence $\Phi : Y \rightrightarrows X$ is a *product correspondence* if there exist correspondences

$$\Phi_1 : Y \rightrightarrows X_1, \ldots, \Phi_N : Y \rightrightarrows X_N$$

such that

$$\Phi(x) = \times_{i \in N} \Phi_i(x), \quad \text{for every } x \in Y.$$

The following definition is taken from Barelli and Meneghel (2013).[1,2]

Definition 6 A metric, convex game $G = (X_i, u_i)_{i \in N}$ is *continuously secure* if for every $x \in X$ that is not a Nash equilibrium of G, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $u_i(z_i, y_{-i}) \geq \alpha_i$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i\}$.

We introduce the following strengthening of Definition 6.

Definition 7 Given $\varepsilon > 0$, a metric, convex game $G = (X_i, u_i)_{i \in N}$ is ε-*continuously secure* if for every $x \in X$ that is not a Nash equilibrium of G, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $u_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i\}$.

For each $i \in N$, let X_i be an action space, and let $X := \times_{i \in N} X_i$. Consider the collection $\mathfrak{G}_X^{(c,\varepsilon)}$ of convex, compact, metric games $(X_i, u_i)_{i \in N}$ satisfying ε-continuous security. Henceforth, for $\mathfrak{G} \subseteq B(X)^N$, $\text{cl}(\mathfrak{G})$ will denote the closure of $\mathfrak{G}$ within $B(X)^N$.

Define

$$\mathfrak{G}_X^{\infty} := \bigcup_{n=1}^{\infty} \mathfrak{G}_X^{(c,\frac{1}{n})} \quad \text{and} \quad \mathfrak{G}_X^{*} := \text{cl}(\mathfrak{G}_X^{\infty}).$$

Theorem 1 *Every member of a dense, residual subset of* $\mathfrak{G}_X^{*}$ *is essential.*[3]

3.1 Relating Theorem 1 to extant results

We begin this subsection by introducing a series of definitions. The following definition is taken from Barelli and Soza (2009).

[1] Barelli and Meneghel (2013) use the notion of continuous security to establish the existence of a pure-strategy Nash equilibrium in discontinuous games. For related existence results, see, *inter alia*, McLennan et al. (2011), Reny (1999, 2009), Bagh and Jofre (2006), Carmona (2009, 2011), Bich (2009), Castro (2011), and Prokopovych (2011), Prokopovych (2013).

[2] The careful reader will observe that Definition 6 does not exactly coincide with Definition 2.1 in Barelli and Meneghel (2013). The definition presented here is needed for Theorem 2.2 in Barelli and Meneghel (2013) to hold (see footnote 4 in Carmona and Podczeck 2014).

[3] Theorem 1 holds intact if $\mathfrak{G}_X^{*}$ is replaced by $\mathfrak{G}_X^{\infty}$. See Remark 7 in Subsection 5.3.

Definition 8 A metric game $G = (X_i, u_i)_{i \in N}$ is *generalized payoff secure* if for each $\varepsilon > 0$, $x \in X$, and i, there is a neighborhood V_x of x and a correspondence $\Phi_i : V_x \rightrightarrows X_i$ such that

$$u_i(z_i, y_{-i}) > u_i(x) - \varepsilon, \quad \text{for each } z_i \in \Phi_i(y) \text{ and every } y \in V_x,$$

and Φ_i is nonempty-valued, convex-valued, and closed.

The following weakening of generalized payoff security can be found in Dasgupta and Maskin (1986) and in Carmona (2009).

Definition 9 A metric game $G = (X_i, u_i)_{i \in N}$ is *weakly payoff secure* if for each $\varepsilon > 0$, $x \in X$, and i, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that $y_{-i} \in V_{x_{-i}}$ implies $u_i(y_i, y_{-i}) > u_i(x) - \varepsilon$ for some $y_i \in X_i$.

The following two definitions are taken from Scalzo (2013).

Definition 10 A metric game $G = (X_i, u_i)_{i \in N}$ is *generalized positively quasitransfer continuous* if for every $\alpha > 0$,

$$\sum_{i \in N} \left[u_i(x_i, y_{-i}) - u_i(y)\right] > \alpha, \quad \text{for some } (x, y) \in X \times X$$

implies that there exists a neighborhood V_y of y and a correspondence $\Phi : V_y \rightrightarrows X$ such that

$$\sum_{i \in N} \left[u_i(a_i, z_{-i}) - u_i(z)\right] > \alpha, \quad \text{for each } a \in \Phi(z) \text{ and } z \in V_y,$$

and Φ is nonempty, convex-valued, compact-valued, and upper hemicontinuous.[4]

Definition 11 A metric, convex game $G = (X_i, u_i)_{i \in N}$ is 0-*diagonally quasiconcave* if for each finite subset $\{x^1, \ldots, x^k\}$ of X and each member x of the convex hull of $\{x^1, \ldots, x^k\}$, we have

$$\min_{l \in \{1,\ldots,k\}} \left\{ \sum_{i \in N} \left[u_i(x_i^l, x_{-i}) - u_i(x)\right] \right\} \leq 0.$$

For each $i \in N$, let X_i be an action space, and let $X := \times_{i \in N} X_i$. Consider the following collections of games:

- The set $\mathfrak{G}_X$ of compact, metric games $(X_i, u_i)_{i \in N}$ for which a pure-strategy Nash equilibrium exists.
- The set $\mathfrak{G}_X^p$ of compact, metric, quasiconcave games $(X_i, u_i)_{i \in N}$ satisfying generalized payoff security, with $\sum_{i \in N} u_i$ upper semicontinuous.

[4] The formal definition of upper hemicontinuity of a correspondence is given in Definition 15, Subsection 5.1, below.

- The set $\mathfrak{G}_X^w$ of compact, metric, quasiconcave games $(X_i, u_i)_{i\in N}$ satisfying weak payoff security, with u_i upper semicontinuous for each i.
- The set $\mathfrak{G}_X^g$ of compact, metric games $(X_i, u_i)_{i\in N}$ that possess a Nash equilibrium and satisfy generalized positive quasitransfer continuity.
- The set $\mathfrak{G}_X^q$ of compact, metric games $(X_i, u_i)_{i\in N}$ satisfying generalized positive quasitransfer continuity and 0-diagonal quasiconcavity.

Remark 1 We have $\mathfrak{G}_X^w \subseteq \mathfrak{G}_X$ [Carmona (2009, Corollary 2 and Proposition 3)], $\mathfrak{G}_X^q \subseteq \mathfrak{G}_X$ [Scalzo (2013, Proposition 2)], and $\mathfrak{G}_X^q \subseteq \mathfrak{G}_X^g \supseteq \mathfrak{G}_X^p$ (Scalzo 2013).

Next, we recapture two results from Carbonell-Nicolau (2010) and Scalzo (2013).

Theorem 2 (Carbonell-Nicolau 2010) *Every member of a dense, residual subset of $\mathfrak{G}_X^p$ (resp. $\mathfrak{G}_X^w$) is essential.*

Theorem 3 (Scalzo 2013) *Every member of a dense, residual subset of $\mathfrak{G}_X^g$ (resp. $\mathfrak{G}_X^q$) is essential.*

Recall that

$$\mathfrak{G}_X^\infty := \bigcup_{n=1}^{\infty} \mathfrak{G}_X^{(c,\frac{1}{n})} \quad \text{and} \quad \mathfrak{G}_X^* := \text{cl}(\mathfrak{G}_X^\infty).$$

The following example illustrates that $\mathfrak{G}_X^* \setminus [\mathfrak{G}_X^g \cup \mathfrak{G}_X^w] \neq \emptyset$, implying that Theorem 1 covers games that cannot be handled by Theorem 2 and Theorem 3.

Example 1 Consider the game $([0, 1], u)$, where

$$u(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 - x & \text{if } x \in (0, 1), \\ 1 & \text{if } x = 1. \end{cases}$$

Since u is not upper semicontinuous, we have $u \notin \mathfrak{G}_{[0,1]}^w$. Next, we show that u violates generalized positive quasitransfer continuity so that $u \notin \mathfrak{G}_{[0,1]}^g$. To this end, we show that there exist $\alpha > 0$ and $(x, y) \in [0, 1]^2$ such that $u(x) - u(y) > \alpha$ and for each neighborhood V_y of y, there exists $z \in V_y$ such that $u(a) - u(z) \leq \alpha$ for all $a \in [0, 1]$. It suffices to take $\alpha = \frac{1}{2}$ and $(x, y) = (1, 0)$, for we have $u(x) - u(y) = u(1) - u(0) = 1 > \alpha = \frac{1}{2}$ and each neighborhood of 0 contains a number z sufficiently close to 0 (so that $u(z)$ is sufficiently close to 1) for which, given any $a \in [0, 1]$, $u(a) - u(z) \leq u(1) - u(z) = 1 - u(z) < \alpha = \frac{1}{2}$.

To see that $u \in \mathfrak{G}_{[0,1]}^*$, define a sequence (u^n) in $B([0, 1])$ as follows: for each n,

$$u^n(x) := \begin{cases} u(x) + \frac{1}{n} & \text{if } x = 1, \\ u(x) & \text{otherwise.} \end{cases}$$

We have $u^n \to u$, and it is easy to verify that for each n, $u^n \in \mathfrak{G}^{(c,\varepsilon)}_{[0,1]}$ for some ε. Indeed, fix n and set $\varepsilon := \frac{1}{2n}$. Suppose that x is not a Nash equilibrium of u^n. Then $x \neq 1$. Let $\alpha := 1$, and choose a neighborhood V_x of x with $V_x \cap \{1\} = \emptyset$. Define $\Phi : V_x \rightrightarrows [0, 1]$ by $\Phi(y) := \{1\}$. Then the following holds:

(i) For each $y \in V_x$, $u^n(1) = 1 + \frac{1}{n} \geq \alpha + \varepsilon$.
(ii) For each $y \in V_x$, y does not belong to the convex hull of $\{z \in [0, 1] : u^n(z) \geq \alpha\} = \{1\}$.

On the other hand, $\mathfrak{G}^*_X$ subsumes the collection of continuous games. More precisely, we have the following result.[5]

Proposition 1 *The set of compact, metric, quasiconcave, continuous games* $(X_i, u_i)_{i\in N}$ *is contained in* $\mathfrak{G}^*_X$.

Proof Suppose that $(X_i, u_i)_{i\in N}$ is compact, metric, quasiconcave, and continuous. We first establish the following preliminary result: given i and $x \in X$, if $u_i(y_i, x_{-i}) > u_i(x)$ for some $y_i \in X_i$, then there exists a neighborhood V_x of x such that $(z_i, z_{-i}) \notin V_x$ for each $z_i \in \arg\max_{a\in X_i} u_i(a, y_{-i})$ and every $z \in V_x$. This follows from the fact that player i's best-reply correspondence $\Upsilon_i : X_{-i} \rightrightarrows X_i$, defined by $\Upsilon_i(x_{-i}) := \arg\max_{a\in X_i} u_i(a, x_{-i})$, has a closed graph (as a consequence of Berge's maximum theorem (see, e.g., Aliprantis and Border 2006, Theorem 17.31) and Theorem 17.11 of Aliprantis and Border 2006) (hence, the complement of the graph of B_i is open in $X_{-i} \times X_i$).[6]

For each n and each i, define $u^n_i : X \to \mathbb{R}$ by

$$u^n_i(x) := \begin{cases} u_i(x) + \frac{1}{n} & \text{if } x_i \in \arg\max_{a\in X_i} u_i(a, x_{-i}), \\ u_i(x) & \text{otherwise.} \end{cases}$$

Clearly, $u^n \to u$, so the proof will be complete if we show that $u^n \in \mathfrak{G}^{(c,\frac{1}{4n})}_X$ for each n. Fix n. The game u^n is clearly convex, compact, and metric, so it suffices to show that u^n is $\frac{1}{4n}$-continuously secure. We first note that the game u^n is quasiconcave.[7] Therefore, to show that u^n is $\frac{1}{4n}$-continuously secure, it suffices to prove the following: if $x \in X$ that is not a Nash equilibrium of u^n, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $u^n_i(z_i, y_{-i}) \geq \alpha_i + \frac{1}{4n}$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$, there exists i such that $u^n_i(y) < \alpha_i$.

[5] I would like to thank an anonymous referee for encouraging me to obtain Proposition 1.

[6] Thanks to an anonymous referee for suggesting this argument.

[7] Given i, $x_{-i} \in X_{-i}$, $\{x_i, y_i\} \subseteq X_i$, and $\lambda \in [0, 1]$, and putting $\overline{x}_i := \lambda x_i + (1 - \lambda)y_i$, it is routine to verify that $u^n_i(\overline{x}_i, x_{-i}) \geq \min\{u^n_i(x_i, x_{-i}), u^n_i(y_i, x_{-i})\}$.

Suppose that x is not a Nash equilibrium of u^n. It is easily seen that this implies that x is not a Nash equilibrium of u. Consequently, there exist j and $z_j \in \Upsilon_j(x_{-j})$ such that $u_j(z_j, x_{-j}) > u_j(x)$. Set

$$\alpha := \left(u_1(x), \ldots, u_{j-1}(x), u_j(x) + \tfrac{1}{2n}, u_{j+1}(x), \ldots, u_N(x)\right).$$

Because u is continuous, there is a neighborhood U_x of x such that for each i and every $y \in U_x$,

$$u_i(y) \in \left(u_i(x) - \tfrac{1}{4n}, u_i(x) + \tfrac{1}{4n}\right).$$

In addition, since $u_j(z_j, x_{-j}) > u_j(x)$, it follows from the result proven in the first paragraph that there exists a neighborhood N_x of x such that $(z_j, y_{-j}) \notin N_x$ for each $z_j \in \Upsilon_j(x_{-j})$ and every $y \in N_x$.

Set $V_x := U_x \cap N_x$, and for each i, define $\Phi_i : V_x \rightrightarrows X_i$ by $\Phi_i(x) := \Upsilon_i(x_{-i})$. By Berge's maximum theorem, Φ_i is nonempty-valued, compact-valued and upper hemicontinuous (hence closed), and Φ_i is in addition convex-valued thanks to the quasiconcavity of $(X_i, u_i)_{i \in N}$. Furthermore, the following holds:

- For every $y \in V_x$,

  $$u_j^n(z_j, y_{-j}) = u_j(z_j, y_{-j}) + \tfrac{1}{n} \geq u_j(y) + \tfrac{1}{n} \geq u_j(x) + \tfrac{3}{4n} = \alpha_j + \tfrac{1}{4n},$$

  for all $z_j \in \Phi_j(y)$.
- For each $i \neq j$ and every $y \in V_x$,

  $$u_i^n\,(z_i, y_{-i}) = u_i\,(z_i, y_{-i}) + \tfrac{1}{n} \geq u_i(y) + \tfrac{1}{n} \geq u_j(x) + \tfrac{3}{4n} = \alpha_i + \tfrac{3}{4n},$$

  for all $z_i \in \Phi_i(y)$.
- For each $y \in V_x$, we have

  $$u_j^n(y) = u_j(y) \leq u_j(x) + \tfrac{1}{4n} < \alpha_j.$$

This completes the proof. □

While we cannot guarantee that the members of $\mathfrak{G}_X^w$ (resp. $\mathfrak{G}_X^g$) belong to $\mathfrak{G}_X^*$, we can establish the essentiality of generic games for the union $\mathfrak{G}_X^* \cup \mathfrak{G}_X^w \cup \mathfrak{G}_X^g$.

Theorem 4 *Every member of a dense, residual subset of* $\mathfrak{G}_X^* \cup \mathfrak{G}_X^g \cup \mathfrak{G}_X^w$ *is essential.*

More generally, essential games can be shown to be generic within any union of the form $\mathfrak{G}_X^* \cup \mathrm{cl}(\mathfrak{G})$, where $\mathfrak{G}$ has certain properties (recall that $\mathrm{cl}(\mathfrak{G})$ represents the closure of $\mathfrak{G}$ within $B(X)^N$). More precisely, consider the following definitions and the ensuing theorem.

Definition 12 A metric game $G = (X_i, u_i)_{i \in N}$ satisfies *sequential* better-reply security* if the following condition is satisfied: if (u^n) is a sequence in $B(X)^N$ with $u^n \to u$, if (x^n) is a sequence in X with $x^n \to x \in X$, and if x is not a Nash equilibrium of G, then there exist an i, subsequences (u^k) of (u^n) and (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i^k(y_i^k, x_{-i}^k) > u_i^k(x^k)$.[8]

Definition 13 Given $\mathfrak{G} \subseteq B(X)^N$, a metric game $G = (X_i, u_i)_{i \in N}$ satisfies *sequential* better-reply security with respect to* $\mathfrak{G}$ if the following condition is satisfied: if (u^n) is a sequence in $\mathfrak{G}$ with $u^n \to u$, if (x^n) is a sequence in X with $x^n \to x \in X$, and if x is not a Nash equilibrium of G, then there exist an i, subsequences (u^k) of (u^n) and (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i^k(y_i^k, x_{-i}^k) > u_i^k(x^k)$.

For each $i \in N$, let X_i be an action space, and let $X := \times_{i \in N} X_i$. For $\mathfrak{G} \subseteq B(X)^N$, let $\mathfrak{G}_X^s(\mathfrak{G})$ be the set of compact, metric games $(X_i, u_i)_{i \in N}$ satisfying sequential* better-reply security with respect to $\mathfrak{G}$. Recall that $\mathfrak{G}_X$ denotes the set of compact, metric games $(X_i, u_i)_{i \in N}$ for which a pure-strategy Nash equilibrium exists.

Theorem 5 *Suppose that* $\mathfrak{G} \subseteq B(X)^N$ *satisfies* $\mathrm{cl}(\mathfrak{G}) \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s(\mathrm{cl}(\mathfrak{G}))$. *Then every member of a dense, residual subset of* $\mathfrak{G}_X^* \cup \mathrm{cl}(\mathfrak{G})$ *is essential.*

We conclude this section with two remarks. The first remark illustrates that the members of $\mathfrak{G}_X^* \setminus \mathfrak{G}_X^\infty$ may or may not possess Nash equilibria. The second remark discusses the relationship between the members of $\mathfrak{G}_X^*$ and generalized better-reply secure games.

Remark 2 Because each member of $\mathfrak{G}_X^{(c,\varepsilon)}$ is a convex, compact, metric game satisfying continuous security, it follows from Theorem 2.2 of Barelli and Meneghel (2013) that the elements of $\mathfrak{G}_X^{(c,\varepsilon)}$ have a Nash equilibrium. Consequently, each member of $\mathfrak{G}_X^\infty$ has a Nash equilibrium. The games in $\mathfrak{G}_X^* \setminus \mathfrak{G}_X^\infty$ may or may not possess Nash equilibria. To illustrate, the one-player game $([0,1], u)$ with $u(1) := 0$ and $u(x) := x$ for all $x \in [0, 1)$ has no Nash equilibrium, but can be approximated by a sequence of games in $\mathfrak{G}_{[0,1]}^\infty$. Indeed, let $u^n : [0,1] \to \mathbb{R}$ be defined by

$$u^n(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{n}), \\ \frac{1}{n} & \text{if } x \in [\frac{1}{n}, \frac{2}{n}), \\ \vdots & \quad \vdots \\ 1 & \text{if } x \in \left[\frac{n-1}{n}, \frac{1}{2}\left(\frac{n-1}{n}+1\right)\right], \\ \frac{n-1}{n} & \text{if } x \in \left(\frac{1}{2}\left(\frac{n-1}{n}+1\right), 1\right), \\ 0 & \text{if } x = 1. \end{cases}$$

Clearly, $u^n \to u$. To see that $u^n \in \mathfrak{G}_{[0,1]}^\infty$ for each n, fix n and suppose that x is not a Nash equilibrium of u^n. Then either $x > \frac{1}{2}(\frac{n-1}{n}+1)$ or $x < \frac{n-1}{n}$. Suppose that $x > \frac{1}{2}(\frac{n-1}{n}+1)$. Let $\alpha_n \in (\frac{n-1}{n}, 1)$ and choose a neighborhood V of 1 with $\inf V > \frac{1}{2}(\frac{n-1}{n}+1)$. Then the following holds:

8 Sequential* better-reply security is a variant of Carbonell-Nicolau and McLean's (2013) *sequential better-reply security*.

(i) $u^n(\frac{n-1}{n}) = 1 \geq \alpha_n + 1 - \alpha_n$.
(ii) For every $y \in V$, y does not belong to the convex hull of

$$\{z \in [0,1] : u^n(z) \geq \alpha_n\} = \left[\tfrac{n-1}{n}, \tfrac{1}{2}\left(\tfrac{n-1}{n} + 1\right)\right].$$

Thus, u^n is α_n-continuously secure. The case when $x < \frac{n-1}{n}$ can be handled similarly.

On the other hand, the one-player game $([0, 1], u)$ with $u(x) := x$ for all $x \in [0, 1]$ has a Nash equilibrium and belongs to $\mathfrak{G}^*_X \setminus \mathfrak{G}^\infty_X$.

Remark 3 Given a metric game $G = (X_i, u_i)_{i \in N}$, let Γ_G be the graph of the game's vector payoff function, i.e.,

$$\Gamma_G := \left\{(x, \alpha) \in X \times \mathbb{R}^N : \alpha = u(x)\right\}.$$

The closure of Γ_G is denoted by $\mathrm{cl}(\Gamma_G)$. The following definition is taken from Barelli and Meneghel (2013).

Definition 14 A metric game $G = (X_i, u_i)_{i \in N}$ is *generalized better-reply secure* if for every $(x, \alpha) \in \mathrm{cl}(\Gamma_G)$ such that x is not a Nash equilibrium of G, there exist a player i, $\beta \in \mathbb{R}$, a neighborhood V_x of x, and a nonempty-valued, closed correspondence $\Phi_i : V_x \rightrightarrows X_i$ such that

$$u_i(z_i, y_{-i}) \geq \beta > \alpha_i$$

for all (y, z_i) in the graph of Φ_i.

Let $\mathfrak{G}^b_X$ be the collection of compact, metric, quasiconcave games $(X_i, u_i)_{i \in N}$ satisfying generalized better-reply security. We have $\mathfrak{G}^*_X \setminus \mathfrak{G}^b_X \neq \emptyset$. Indeed, the first game in Remark 2 is a member of $\mathfrak{G}^*_X$, and since it has no Nash equilibria, it does not belong to $\mathfrak{G}^b_X$.[9]

On the other hand, $\mathfrak{G}^b_X \setminus \mathfrak{G}^\infty_X \neq \emptyset$.[10] For instance, the second example in Remark 2 is a member of $\mathfrak{G}^b_X \setminus \mathfrak{G}^\infty_X$.

4 Applications

The collection $\mathfrak{G}^\infty_X$ (hence $\mathfrak{G}^*_X$) contains economically meaningful games. To illustrate this, we first remark that continuously secure games with finite payoffs belong to $\mathfrak{G}^\infty_X$. We record this result without proof.

Proposition 2 *Suppose that $G = (X_i, u_i)_{i \in N}$ is continuously secure. Suppose further that u_i has finite range for each i. Then G is ε-continuously secure for some ε. Consequently, if G is continuously secure and u_i has finite range for each i, then $G \in \mathfrak{G}^\infty_X$.*

9 The members of $\mathfrak{G}^b_X$ possess Nash equilibria (cf. Barelli and Meneghel 2013, Proposition 2.4).

10 We have not been able to find an example demonstrating that $\mathfrak{G}^b_X \setminus \mathfrak{G}^*_X \neq \emptyset$.

Bagh and Jofre (2006, Example 1) consider a timing game with finite payoffs, and Barelli and Meneghel (2013, Example 3.1) present a continuously secure game of product quality competition between firms in which the ranges of the firms' payoff functions are finite. Barelli and Soza (2012) present a continuously secure location model in which the players' decision problem involves two simultaneous choices—location and price, quantity, or effort. The specialization of this game to a location model with a single variable—the location variable—is another instance of a game whose payoffs have finite range. Finally, Barelli and Soza (2009) consider an example of a multi-principal multi-agent game with finite payoffs. All these games lie in $\mathfrak{G}_X^\infty$ (for an appropriate choice of the set X).

To conclude this section, we present an economic game in the collection $\mathfrak{G}_X^*$ and illustrate how simple the task of verifying membership in $\mathfrak{G}_X^*$ can be.

Example 2 Consider a two-player game $([0, 1], [0, 1], u_1, u_2)$, where for each i,

$$u_i(t_i, t_{-i}) := \begin{cases} a(t_i) & \text{if } t_i < t_{-i}, \\ b(t_i) & \text{if } t_i = t_{-i}, \\ c(t_{-i}) & \text{if } t_i > t_{-i}, \end{cases}$$

where a, b, and c are bounded real-valued maps on $[0, 1]$. We make the following assumptions: a is lower semicontinuous on $[0, 1]$, b is upper semicontinuous on $[0, 1]$, c is continuous and nonincreasing on $[0, 1]$, and for every $t \in [0, 1]$, $a(t) > b(t) > c(t)$.[11]

We show that $u \in \mathfrak{G}^*_{[0,1]^2}$. For each n, consider the game $([0, 1], [0, 1], u_1^n, u_2^n)$, where for each i and every n,

$$u_i^n(t_i, t_{-i}) := \begin{cases} a(t_i) + \frac{1}{n} & \text{if } t_i < t_{-i}, \\ b(t_i) & \text{if } t_i = t_{-i}, \\ c(t_{-i}) - \frac{1}{n} & \text{if } t_i > t_{-i}. \end{cases}$$

Because $u^n \to u$, to see that $u \in \mathfrak{G}^*_{[0,1]^2}$, it suffices to show that for each n, u^n satisfies ϵ^n-continuous security, where (ϵ^n) is a sequence in $\mathbb{R}_{++}$.

Fix n and $t = (t_1, t_2) \in [0, 1]^2$, and suppose that t is not a Nash equilibrium of u^n. Then $t \neq (0, 0)$. Without loss of generality, suppose that $t_1 \leq t_2$.

If $0 \neq t_1 = t_2$, there exists $t^* \in [0, t_1)$ such that $u_i^n(t^*, t_{-i}) > b(t_1)$ for each i. Let V_{t_1} be a neighborhood of t_1 such that $t^* \notin V_{t_1}$ and

$$b(\tau) < b(t_1) + \frac{1}{2n}, \quad \text{for every } \tau \in V_{t_1}.$$

Then, for each i,

$$u_i^n(t^*, \tau_{-i}) = a(t^*) + \frac{1}{n} \geq b(t_i) + \frac{1}{n}, \quad \text{for all } \tau_{-i} \in V_{t_1}.$$

[11] Similar games have been used to study behavior in duels, R&D, and patent races (e.g. Karlin 1959; Pitchik 1982; Reinganum 1981a,b; Fudenberg and Tirole 1985).

In addition, for every $(\tau_1, \tau_2) \in V_{t_1} \times V_{t_1}$, and letting $\tau_2 = \max\{\tau_1, \tau_2\}$, we have, for every $x_2 \geq \tau_2$,

$$u_2^n(\tau_1, x_2) \leq b(\tau_1) < b(t_2) + \tfrac{1}{2n},$$

so τ_2 does not belong to the convex hull of $\{z_2 \in [0, 1] : u_2^n(\tau_1, z_2) \geq b(t_2) + \frac{1}{2n}\}$.

If $t_1 < t_2$, for each i, let V_{t_i} be a neighborhood of t_i such that $V_{t_1} \cap V_{t_2} = \emptyset$ and

$$c(t_1) - \tfrac{1}{2n} < c(\tau_1) < c(t_1) + \tfrac{1}{2n}, \quad \text{for all } \tau_1 \in \text{cl}(V_{t_1}).$$

Then, for every $(\tau_1, \tau_2) \in V_{t_1} \times V_{t_2}$,

$$u_1^n(t_1, \tau_2) = a(t_1) + \tfrac{1}{n}$$
$$\text{and} \quad u_2^n(\tau_1, \inf V_{t_1}) \geq a(\inf V_{t_1}) + \tfrac{1}{n} \geq c(\inf V_{t_1}) + \tfrac{1}{n} \geq c(t_1) + \tfrac{1}{2n},$$

and for every $x_2 \geq \tau_2$ we have

$$u_2^n(\tau_1, x_2) = c(\tau_1) - \tfrac{1}{n} < c(t_1) - \tfrac{1}{2n},$$

so τ_2 does not belong to the convex hull of $\{z_2 \in [0, 1] : u_2^n(\tau_1, z_2) \geq c(t_1) - \frac{1}{2n}\}$.

Consequently, u^n satisfies $\frac{1}{2n}$-continuous security.

We conclude that u is a member of $\mathfrak{G}^*_{[0,1]^2}$. In addition, in general u does not belong to $\mathfrak{G}^w_{[0,1]^2}$ or to $\mathfrak{G}^g_{[0,1]^2}$. Indeed, one can find maps a, b, and c such that the above game fails quasiconcavity and/or generalized positive quasitransfer continuity.

5 Proofs of the main results

This section contains the proofs of Theorem 1, Theorem 4, and Theorem 5.

5.1 Preliminaries

Definition 15 A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is *upper hemicontinuous at* $x \in A$ if the following condition is satisfied: for every neighborhood $V_{\Phi(x)}$ of $\Phi(x)$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \subseteq V_{\Phi(x)}$. Φ is *upper hemicontinuous* if it is upper hemicontinuous at every point in A.

Definition 16 A correspondence $\Phi : A \rightrightarrows B$ between topological spaces is *lower hemicontinuous at* $x \in A$ if the following condition is satisfied: for every open set $V \subseteq B$ with $V \cap \Phi(x) \neq \emptyset$ there is a neighborhood V_x of x such that $y \in V_x$ implies $\Phi(y) \cap V \neq \emptyset$. Φ is *lower hemicontinuous* if it is lower hemicontinuous at every point in A.

Remark 4 Suppose that $\mathfrak{G} \subseteq B(X)^N$. A game u in $\mathfrak{G}$ is essential relative to $\mathfrak{G}$ if and only if $\mathcal{E}_X|_{\mathfrak{G}}$ is lower hemicontinuous at u.

Recall that, for $\mathfrak{G} \subseteq B(X)^N$, cl$(\mathfrak{G})$ represents the closure of $\mathfrak{G}$ within $B(X)^N$.

For each $i \in N$, let X_i be an action space, and let $X := \times_{i\in N} X_i$. We recall the definitions of $\mathfrak{G}_X$ and $\mathfrak{G}^s(\mathfrak{G})$ and introduce new terminology:

- The set $\mathfrak{G}_X$ of compact, metric games $(X_i, u_i)_{i\in N}$ for which a pure-strategy Nash equilibrium exists.
- The set $\mathfrak{G}^s_X$ of compact, metric games $(X_i, u_i)_{i\in N}$ satisfying sequential* better-reply security.
- For $\mathfrak{G} \subseteq B(X)^N$, the set $\mathfrak{G}^s_X(\mathfrak{G})$ of compact, metric games $(X_i, u_i)_{i\in N}$ satisfying sequential* better-reply security with respect to $\mathfrak{G}$.
- The set $\mathfrak{G}^c_X$ of convex, compact, metric games $(X_i, u_i)_{i\in N}$ satisfying continuous security.
- The set $\mathfrak{G}^{(c,\varepsilon)}_X$ of convex, compact, metric games $(X_i, u_i)_{i\in N}$ satisfying ε-continuous security.
- $\mathfrak{G}^{\varepsilon}_X := \text{cl}\left(\mathfrak{G}^{(c,\varepsilon)}_X\right)$.
- $\overline{\mathfrak{G}}^{\infty}_X := \bigcup_{n=1}^{\infty} \mathfrak{G}^{\frac{1}{n}}_X$.
- $\overline{\mathfrak{G}}_X := \text{cl}\left(\overline{\mathfrak{G}}^{\infty}_X\right)$.

Remark 5 Clearly, $\mathfrak{G}^s_X \subseteq \mathfrak{G}^s_X(\mathfrak{G})$ for all $\mathfrak{G} \subseteq B(X)^N$.

Remark 6 We have $\mathfrak{G}^c_X \subseteq \mathfrak{G}_X$ (Barelli and Meneghel 2013, Theorem 2.2).

5.2 Preliminary results

Lemma 1 (Fort 1951, Theorem 2) *Suppose that X is a metric space and that Y is a topological space. Suppose that $F : Y \rightrightarrows X$ is a nonempty-valued, compact-valued, upper hemicontinuous correspondence. Then there exists a residual subset Q of Y such that F is lower hemicontinuous at every point in Q.*

Lemma 2 *Suppose that X is compact and metric. For $\mathfrak{G} \subseteq B(X)^N$, $\mathcal{E}_X|_{\mathfrak{G}}$ is compact-valued and upper hemicontinuous if, and only if, $\mathfrak{G} \subseteq \mathfrak{G}^s_X(\mathfrak{G})$.*[12]

Proof Since X is compact and metric, it suffices to show that $\mathcal{E}_X|_{\mathfrak{G}}$ has a closed graph if, and only if, $\mathfrak{G} \subseteq \mathfrak{G}^s_X(\mathfrak{G})$ (e.g., Aliprantis and Border 2006, Theorem 17.11).

Suppose that $\mathfrak{G} \subseteq \mathfrak{G}^s_X(\mathfrak{G})$. Take a sequence (u^n) in $\mathfrak{G}$, and take a sequence (x^n) such that x^n is a Nash equilibrium of u^n for each n. Suppose that

$$(x^n, u^n) \to (x, u),$$

for some $(x, u) \in X \times \mathfrak{G}$. We must show that x is a Nash equilibrium of u.

Suppose that x is not a Nash equilibrium of u. Then, since the game u, being a member of $\mathfrak{G}$, satisfies sequential* better-reply security with respect to $\mathfrak{G}$, there exist

[12] For one-player games, a characterization of a nonempty, compact-valued, and upper hemicontinuous Nash equilibrium correspondence can be furnished in terms of Tian and Zhou's (1995) *transfer upper continuity* and *quasitransfer upper continuity* (cf. Tian and Zhou 1995, Theorem 3). Sequential* better-reply security can be viewed as an extension of these conditions to n-person games.

an i, subsequences (u^k) of (u^n) and (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i^k(y_i^k, x_{-i}^k) > u_i^k(x^k)$, contradicting that x^n is a Nash equilibrium of u^n for each n.

Conversely, suppose that $\mathcal{E}_X|_{\mathfrak{G}}$ has a closed graph. Let (u^n) be a sequence in $\mathfrak{G}$ with $u^n \to u$, and let (x^n) be a sequence in X with $x^n \to x \in X$. Suppose that x is not a Nash equilibrium of u. Then, since $\mathcal{E}_X|_{\mathfrak{G}}$ has a closed graph, for infinitely many n, x^n is not a Nash equilibrium of u^n. Therefore, since the player set is finite, there exists i such that for infinitely many n, $u_i^n(y_i^n, x_{-i}^n) > u_i^n(x^n)$ for some $y_i^n \in X_i$. □

Proposition 3 *Suppose that* $\mathfrak{G} \subseteq B(X)^N$ *satisfies* $\text{cl}(\mathfrak{G}) \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s(\text{cl}(\mathfrak{G}))$. *Then every member of a dense, residual subset of* $\text{cl}(\mathfrak{G})$ *is essential.*

Proof Suppose that $\mathfrak{G} \subseteq B(X)^N$ satisfies $\text{cl}(\mathfrak{G}) \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s(\text{cl}(\mathfrak{G}))$. The correspondence $\mathcal{E}_X|_{\text{cl}(\mathfrak{G})}$ is clearly nonempty-valued, and $\mathcal{E}_X|_{\text{cl}(\mathfrak{G})}$ is compact-valued and upper hemicontinuous (Lemma 2). Consequently, Lemma 1 gives a residual subset $\mathfrak{Q}$ of $\text{cl}(\mathfrak{G})$ such that $\mathcal{E}_X|_{\text{cl}(\mathfrak{G})}$ is lower hemicontinuous at every point in $\mathfrak{Q}$, and it follows that for each $u \in \mathfrak{Q}$, any pure-strategy Nash equilibrium of u is essential relative to $\text{cl}(\mathfrak{G})$ (cf. Remark 4). To see that $\mathfrak{Q}$ is dense in $\text{cl}(\mathfrak{G})$, note that because $\text{cl}(\mathfrak{G})$ is a closed subset of $B(X)^N$, and since $B(X)^N$ is a complete, metric space, $\text{cl}(\mathfrak{G})$ is a complete, metric space. Therefore, $\text{cl}(\mathfrak{G})$ is a Baire space by the Baire category theorem. Consequently, $\mathfrak{Q}$, being a residual subset of a Baire space, is dense. □

The proof of the following lemma is relegated to Section 6.

Lemma 3 *For each* ε, $\text{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)} \cap \mathfrak{G}_X^s$ *for every* $\eta > 0$ *with* $\varepsilon - \eta > 0$.

Proposition 4 *For each* ε, *every member of a dense, residual subset of* $\mathfrak{G}_X^\varepsilon$ *is essential.*

Proof Fix $\varepsilon > 0$. By Lemma 3,

$$\text{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq \mathfrak{G}_X^c \cap \mathfrak{G}_X^s.$$

Therefore, in view of Remark 5 and Remark 6,

$$\text{cl}(\mathfrak{G}_X^\varepsilon) = \text{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s(\text{cl}(\mathfrak{G}_X^\varepsilon)).$$

Applying Proposition 3, we see that for every u in a dense, residual subset of $\mathfrak{G}_X^\varepsilon$, any pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}_X^\varepsilon$. □

The proof of the following lemma is given in Section 6.

Lemma 4 *For each* ε, *and for every* $\eta > 0$ *with* $\varepsilon - \eta > 0$, *there is an open subset* V *of* $B(X)^N$ *with* $\text{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq V \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)}$.

Lemma 5 *For each* ε, *there is an open subset* V *of* $B(X)^N$ *with* $\mathfrak{G}_X^\varepsilon \subseteq V \subseteq \mathfrak{G}_X^{\eta_\varepsilon}$ *for some* $\eta_\varepsilon < \varepsilon$.

Proof The statement follows immediately from Lemma 4. □

Lemma 6 $\overline{\mathfrak{G}}_X^{\infty} = \bigcup_{n=1}^{\infty} \mathfrak{G}_X^{(c,\frac{1}{n})}$, *and therefore* $\mathfrak{G}_X^{\infty} = \overline{\mathfrak{G}}_X^{\infty}$ *and* $\mathfrak{G}_X^{*} = \overline{\mathfrak{G}}_X$.

Proof Pick $u \in \overline{\mathfrak{G}}_X^{\infty}$. Then $u \in \mathrm{cl}(\mathfrak{G}_X^{(c,\frac{1}{\ell})})$ for some ℓ, and Lemma 4 implies that $u \in \mathfrak{G}_X^{(c,\frac{1}{m})}$ for some m. Hence, $\overline{\mathfrak{G}}_X^{\infty} \subseteq \bigcup_{n=1}^{\infty} \mathfrak{G}_X^{(c,\frac{1}{n})}$. The containment $\overline{\mathfrak{G}}_X^{\infty} \supseteq \bigcup_{n=1}^{\infty} \mathfrak{G}_X^{(c,\frac{1}{n})}$ is obvious. □

In the remainder of this subsection, we record, without proof, a number of well-known results on meager sets and Baire spaces that will be useful in the proofs of the main results. Their proofs can be found in standard topology manuals (e.g., Bourbaki 1989).

Lemma 7 *A subset of a topological space is residual if and only if it contains a countable intersection of open dense sets.*

Lemma 8 *Any subset of a nowhere dense (resp. meager) set is nowhere dense (resp. meager). Any superset of a residual set is residual. The union of countably many meager sets is meager. The intersection of countably many residual sets is residual.*

Lemma 9 *If* A *is a meager subset of* B *and* $B \subseteq C$, *then* A *is a meager subset of* C.

Lemma 10 *Every nonempty open subspace of a Baire space is a Baire space.*

5.3 Proof of Theorem 1

We begin by stating and proving the following result, which is used in the proof of Theorem 1.

Proposition 5 *Assume the following: (i)* A *is a complete metric space; (ii)* (A^n) *is an increasing sequence of closed subsets of* A*; (iii)* (B^n) *is a sequence with* B^n *a dense, residual subset of* A^n *for each* n*; and (iv)* (V^n) *is a sequence of open subsets of* A *with* $A^n \subseteq V^n \subseteq A^{n+1}$ *for each* n. *Then* $\bigcup_{n=1}^{\infty}(B^{n+1} \cap V^n)$ *is a dense, residual subset of* $\mathrm{cl}\left(\bigcup_{n=1}^{\infty} A^n\right)$.

Proof Define

$$R^n := B^{n+1} \cap V^n \quad \text{and} \quad R^{\infty} := \bigcup_{n=1}^{\infty} R^n.$$

Let

$$A^{\infty} := \bigcup_{n=1}^{\infty} A^n \quad \text{and} \quad \overline{A} := \mathrm{cl}\left(A^{\infty}\right).$$

Clearly, A^{∞} is dense in $\overline{A}$. Furthermore, since (V^n) is a sequence of open subsets of A with $A^n \subseteq V^n \subseteq A^{n+1}$ for each n, A^{∞} is open in $\overline{A}$. Thus, A^{∞} is open and dense in $\overline{A}$.

We begin by showing that the set $A^\infty \setminus R^\infty$ is a meager subset of A^∞. This flows from the following observations.

First, because B^{n+1} is a residual subset of A^{n+1}, B^{n+1} contains a countable intersection of open, dense sets (Lemma 7), i.e., there is a countable set $\{U^1, U^2, \ldots\}$ of open, dense subsets of A^{n+1} such that $B^{n+1} \supseteq \bigcap_{m=1}^\infty U^m$. This implies that $U^m \cap V^n$ is an open, dense subset of V^n for each m and

$$\bigcap_{m=1}^{\infty} [U^m \cap V^n] \subseteq B^{n+1} \cap V^n, \tag{1}$$

so that $R^n = B^{n+1} \cap V^n$ is a residual subset of V^n. The containment in (1) is easy to verify. To see that $U^m \cap V^n$ is open in V^n for each m, fix m and take $u \in U^m \cap V^n$. Then, since U^m is open in A^{n+1}, there exists a neighborhood V of u such that $V \cap A^{n+1} \subseteq U^m$. Therefore, because $V^n \subseteq A^{n+1}$, $V \cap V^n \subseteq U^m$, so $V \cap V^n \subseteq U^m \cap V^n$. To see that $U^m \cap V^n$ is a dense subset of V^n for each m, fix m and pick $u \in V^n$. Because $u \in V^n \subseteq A^{n+1}$, and since U^m is dense in A^{n+1} and V^n is open in A and contains u, there exists $f \in U^m \cap V^n$ arbitrarily close to u.

Because R^n is a residual subset of V^n, $V^n \setminus R^n$ is a meager subset of V^n. Consequently, since $V^n \subseteq A^{n+1} \subseteq A^\infty$, $V^n \setminus R^n$ is a meager subset of A^∞ (Lemma 9), and since $A^n \setminus R^n \subseteq V^n \setminus R^n$ (recall that $A^n \subseteq V^n$) it follows that $A^n \setminus R^n$ is a meager subset of A^∞ (Lemma 8). Therefore, $\bigcup_{n=1}^\infty (A^n \setminus R^n)$, being a countable union of meager subsets of A^∞, is a meager subset of A^∞ (Lemma 8).

Second,

$$A^\infty \setminus R^\infty \subseteq \bigcup_{n=1}^{\infty} (A^n \setminus R^n),$$

so $A^\infty \setminus R^\infty$, being a subset of a meager subset of A^∞, is a meager subset of A^∞ (Lemma 8).

Next, we show that the set $\overline{A} \setminus R^\infty$ is a meager subset of $\overline{A}$. The set $\overline{A} \setminus R^\infty$ can be expressed as

$$[(\overline{A} \setminus A^\infty) \setminus R^\infty] \cup \left[A^\infty \setminus R^\infty\right] \tag{2}$$

The right-hand side of this union can be written as a countable union of nowhere dense sets. In addition, since A^∞ is open and dense in $\overline{A}$, $\overline{A} \setminus A^\infty$ is nowhere dense, and so the left-hand side of the union in (2) is nowhere dense (Lemma 8). Consequently, the union in (2) can be expressed as a countable union of nowhere dense sets. Thus, the set $\overline{A} \setminus R^\infty$ is a meager subset of $\overline{A}$, implying that R^∞ is a residual subset of $\overline{A}$.

It only remains to show that R^∞ is dense in $\overline{A}$. First, observe that A^{n+1}, being a closed subspace of the complete, metric space A, is itself a complete, metric space, and hence (by the Baire category theorem) a Baire space. Consequently, since V^n is a nonempty, open subset of A^{n+1}, V^n is a Baire space (Lemma 10). To see that R^∞ is dense in $\overline{A}$, fix $u \in \overline{A}$, and let U be a neighborhood of u. Since A^∞ is dense in $\overline{A}$, there exists n such that $A^n \cap U \neq \emptyset$. Therefore, since R^n (being a residual subset of the Baire space V^n) is dense in V^n and $A^n \subseteq V^n$, we have $R^n \cap U \neq \emptyset$. □

We are now ready to prove Theorem 1.

Theorem 1 *Every member of a dense, residual subset of $\mathfrak{G}^*_X$ is essential.*

Proof By Lemma 6, it suffices to show that every member of a dense, residual subset of $\overline{\mathfrak{G}}_X$ is essential.

For each ε, and for every u in a dense, residual subset $\mathfrak{D}^\varepsilon_X$ of $\mathfrak{G}^\varepsilon_X$, any pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}^\varepsilon_X$ (Proposition 4). For each ε, Lemma 5 gives $\eta_\varepsilon < \varepsilon$ and an open subset N^ε of $B(X)^N$ such that $\mathfrak{G}^\varepsilon_X \subseteq N^\varepsilon \subseteq \mathfrak{G}^{\eta_\varepsilon}_X$.

Let $A := B(X)^N$, $A^n := \mathfrak{G}^{\frac{1}{n}}_X$, $B^n := \mathfrak{D}^{\frac{1}{n}}_X$, and $V^n := N^{\frac{1}{n}}$, and apply Proposition 5 to conclude that the set

$$\mathfrak{R}^\infty_X := \bigcup_{n=1}^{\infty} \left(\mathfrak{D}^{\frac{1}{n}}_X \cap N^{\frac{1}{n}} \right)$$

is a dense, residual subset of $\overline{\mathfrak{G}}_X$.

It only remains to show that if $u \in \mathfrak{R}^\infty_X$, then any pure-strategy Nash equilibrium of u is essential relative to $\overline{\mathfrak{G}}_X$. Fix $u \in \mathfrak{R}^\infty_X$, and let x be a pure-strategy Nash equilibrium of u. Because $u \in \mathfrak{R}^\infty_X$, $u \in \mathfrak{D}^{\eta_\varepsilon}_X \cap V^\varepsilon$ for some ε. Since $u \in \mathfrak{D}^{\eta_\varepsilon}_X$, every pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}^{\eta_\varepsilon}_X$. It follows that given a neighborhood V_x of x, there is a neighborhood V_u of u in $B(X)^N$ such that for every $f \in V_u \cap \mathfrak{G}^{\eta_\varepsilon}_X$, $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. Consequently, since N^ε is an open set containing u and $N^\varepsilon \subseteq \mathfrak{G}^{\eta_\varepsilon}_X$, the neighborhood $V_u \cap N^\varepsilon$ of u satisfies the following: $f \in V_u \cap N^\varepsilon \cap \overline{\mathfrak{G}}_X$ implies $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. It follows that x is essential relative to $\overline{\mathfrak{G}}_X$. □

Remark 7 Theorem 1 implies that every member of a dense, residual subset of

$$\mathfrak{G}^\infty_X = \bigcup_{n=1}^{\infty} \mathfrak{G}^{(c,\frac{1}{n})}_X$$

is essential. Indeed, let $\mathfrak{R}^\infty_X$ be a dense, residual subset of $\mathfrak{G}^*_X$, as given by Theorem 1. The proof of Theorem 1 makes it clear that $\mathfrak{R}^\infty_X$ may be chosen so that $\mathfrak{R}^\infty_X \subseteq \overline{\mathfrak{G}}^\infty_X = \mathfrak{G}^\infty_X$ (where the equality follows from Lemma 6). Given this containment, it is easily seen that $\mathfrak{R}^\infty_X$ is a dense, residual subset of $\mathfrak{G}^\infty_X$. Moreover, because every member of $\mathfrak{R}^\infty_X$ is essential relative to $\mathfrak{G}^*_X$, every member of $\mathfrak{R}^\infty_X$ is essential relative to $\mathfrak{G}^\infty_X$. These observations are formally stated as a corollary to Theorem 1.

Corollary 1 (to Theorem 1) *Every member of a dense, residual subset of $\mathfrak{G}^\infty_X$ is essential.*

5.4 Proof of Theorem 5

Prior to proving Theorem 5, we state and prove a refinement of Proposition 5 that will be used in the proof of Theorem 5.

Proposition 6 *Assume the following: (i) A is a complete metric space and A' is a closed subset of A; (ii) (A^n) is an increasing sequence of closed subsets of A; (iii) B is a dense, residual subset of A' and (B^n) is a sequence with B^n a dense, residual subset of A^n for each n; and (iv) (V^n) is a sequence of open subsets of A with $A^n \subseteq V^n \subseteq A^{n+1}$ for each n. Then the set*

$$\left[B\Big\backslash \mathrm{cl}\left(\bigcup_{n=1}^{\infty} A^n\right)\right] \cup \left[\bigcup_{n=1}^{\infty}(B^{n+1} \cap V^n)\right]$$

is a dense, residual subset of $A' \cup \left[\mathrm{cl}\left(\bigcup_{n=1}^{\infty} A^n\right)\right]$.

Proof Define

$$R^n := B^{n+1} \cap V^n \quad \text{and} \quad R^\infty := \bigcup_{n=1}^{\infty} R^n.$$

Let

$$A^\infty := \bigcup_{n=1}^{\infty} A^n \quad \text{and} \quad \overline{A} := \mathrm{cl}\left(A^\infty\right).$$

Thanks to Proposition 5, R^∞ is a dense, residual subset of $\overline{A}$.

Define $B' := B \setminus \overline{A}$. We need to show that $R^\infty \cup B'$ is a dense, residual subset of $\overline{A} \cup A'$.

To see that the set $R^\infty \cup B'$ is dense in $\overline{A} \cup A'$, take $u \in \overline{A} \cup A'$. If $u \in \overline{A}$, it is clear that there exists a member of R^∞ arbitrarily close to u. If, on the other hand, $u \in A' \setminus \overline{A}$, then, since $\overline{A}$ is closed in A, there exists a neighborhood V_u of u with $V_u \cap \overline{A} = \emptyset$. Because B is dense in A', there are members of B arbitrarily close to u. Hence, there exists $f \in B \cap V_u$ arbitrarily close to u, and since $V_u \cap \overline{A} = \emptyset$, we have $f \in B' = B \setminus \overline{A}$.

Next, we show that $B' = B \setminus \overline{A}$ is a residual subset of $A' \setminus \overline{A}$. Since B is a residual subset of A', B contains a countable intersection of open dense sets (Lemma 7), so there exists a countable set $\{U^1, U^2, \ldots\}$ of open and dense subsets of A' such that $B \supseteq \bigcap_{n=1}^{\infty} U^n$. This implies that $U^n \setminus \overline{A}$ is open and dense in $A' \setminus \overline{A}$ for each n and

$$\bigcap_{n=1}^{\infty}[U^n \setminus \overline{A}] \subseteq B \setminus \overline{A}, \tag{3}$$

from which it follows that $B' = B \setminus \overline{A}$ is a residual subset of $A' \setminus \overline{A}$. The containment in (3) is easy to verify. To see that $U^n \setminus \overline{A}$ is open in $A' \setminus \overline{A}$ for each n, fix n and take

$u \in U^n \setminus \overline{A}$. Since $u \in U^n$ and U^n is open in A', there is a neighborhood V of u such that $V \cap A' \subseteq U^n$. It follows that $V \cap [A' \setminus \overline{A}] \subseteq U^n \setminus \overline{A}$, and hence, $U^n \setminus \overline{A}$ is open in $A' \setminus \overline{A}$. To see that $U^n \setminus \overline{A}$ is dense in $A' \setminus \overline{A}$ for each n, fix n and take $u \in A' \setminus \overline{A}$. Because $\overline{A}$ is closed in A, there exists a neighborhood U of u such that $U \cap \overline{A} = \emptyset$, and since U^n is dense in A', there exists $f \in U^n \cap U$ arbitrarily close to u. Because $f \in U^n \cap U$ and $U \cap \overline{A} = \emptyset$, we have $f \in U^n \setminus \overline{A}$.

We are now ready to show that $R^\infty \cup B'$ is a residual subset of $\overline{A} \cup A'$. Since R^∞ (resp. B') is a residual subset of $\overline{A}$ (resp. $A' \setminus \overline{A}$), $[R^\infty \cup B'] \cap \overline{A}$ (resp. $[R^\infty \cup B'] \cap [A' \setminus \overline{A}]$) is a residual subset of $\overline{A}$ (resp. $A' \setminus \overline{A}$). It follows that $\overline{A} \setminus [[R^\infty \cup B'] \cap \overline{A}]$ (resp. $[A' \setminus \overline{A}] \setminus [[R^\infty \cup B'] \cap [A' \setminus \overline{A}]]$) is a meager subset of $\overline{A}$ (resp. $A' \setminus \overline{A}$), and since

$$\overline{A} \setminus [R^\infty \cup B'] \subseteq \overline{A} \setminus [[R^\infty \cup B'] \cap \overline{A}]$$

and

$$[A' \setminus \overline{A}] \setminus [R^\infty \cup B'] \subseteq [A' \setminus \overline{A}] \setminus [[R^\infty \cup B'] \cap [A' \setminus \overline{A}]],$$

we see that $\overline{A} \setminus [R^\infty \cup B']$ (resp. $[A' \setminus \overline{A}] \setminus [R^\infty \cup B']$) is a meager subset of $\overline{A}$ (resp. $A' \setminus \overline{A}$) (Lemma 8). Because $\overline{A} \setminus [R^\infty \cup B']$ is a meager subset of $\overline{A}$, $\overline{A} \setminus [R^\infty \cup B']$ is a meager subset of $\overline{A} \cup A'$ (Lemma 9). Similarly, $[A' \setminus \overline{A}] \setminus [R^\infty \cup B']$ is a meager subset of $\overline{A} \cup A'$. Therefore, the union

$$\left[\overline{A} \setminus [R^\infty \cup B']\right] \cup \left[[A' \setminus \overline{A}] \setminus [R^\infty \cup B']\right] = [\overline{A} \cup A'] \setminus [R^\infty \cup B']$$

is a meager subset of $\overline{A} \cup A'$, implying that $R^\infty \cup B'$ is a residual subset of $\overline{A} \cup A'$.

Theorem 5 *Suppose that* $\mathfrak{G} \subseteq B(X)^N$ *satisfies* $\text{cl}(\mathfrak{G}) \subseteq \mathfrak{G}_X \cap \mathfrak{G}^s_X(\text{cl}(\mathfrak{G}))$. *Then every member of a dense, residual subset of* $\mathfrak{G}^*_X \cup \text{cl}(\mathfrak{G})$ *is essential.*

Proof By Lemma 6, it suffices to show that every member of a dense, residual subset of $\overline{\mathfrak{G}}_X \cup \text{cl}(\mathfrak{G})$ is essential.

For each ε, and for every u in a dense, residual subset $\mathfrak{D}^\varepsilon_X$ of $\mathfrak{G}^\varepsilon_X$, any pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}^\varepsilon_X$ (Proposition 4). For each ε, Lemma 5 gives $\eta_\varepsilon < \varepsilon$ and an open subset N^ε of $B(X)^N$ such that $\mathfrak{G}^\varepsilon_X \subseteq N^\varepsilon \subseteq \mathfrak{G}^{\eta_\varepsilon}_X$. For every u in a dense, residual subset $\mathfrak{A}_X$ of $\text{cl}(\mathfrak{G})$, any pure-strategy Nash equilibrium of u is essential relative to $\text{cl}(\mathfrak{G})$ (Proposition 3).

Let $A := B(X)^N$, $A' := \text{cl}(\mathfrak{G})$, $A^n := \mathfrak{G}^{\frac{1}{n}}_X$, $B := \mathfrak{A}_X$, $B^n := \mathfrak{D}^{\frac{1}{n}}_X$, and $V^n := N^{\frac{1}{n}}$, and apply Proposition 6 to conclude that the set $\mathfrak{R}^\infty \cup \mathfrak{B}_X$, where

$$\mathfrak{R}^\infty_X := \bigcup_{n=1}^{\infty} \left(\mathfrak{D}^{\frac{1}{n}}_X \cap N^{\frac{1}{n}}\right) \quad \text{and} \quad \mathfrak{B}_X := \mathfrak{A}_X \setminus \overline{\mathfrak{G}}_X,$$

is a dense, residual subset of $\overline{\mathfrak{G}}_X \cup \text{cl}(\mathfrak{G})$.

It only remains to show that every member of $\mathfrak{R}^\infty \cup \mathfrak{B}_X$ is essential relative to $\overline{\mathfrak{G}}_X \cup \text{cl}(\mathfrak{G})$. Fix $u \in \mathfrak{R}^\infty \cup \mathfrak{B}_X$, and let x be a pure-strategy Nash equilibrium of

u. If $u \in \mathfrak{R}_X^\infty$, $u \in \mathfrak{D}_X^{\eta_\varepsilon} \cap V^\varepsilon$ for some ε. Since $u \in \mathfrak{D}_X^{\eta_\varepsilon}$, every pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}_X^{\eta_\varepsilon}$. It follows that given a neighborhood V_x of x, there is a neighborhood V_u of u in $B(X)^N$ such that for every $f \in V_u \cap \mathfrak{G}_X^{\eta_\varepsilon}$, $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. Consequently, since V^ε is an open set containing u and $V^\varepsilon \subseteq \mathfrak{G}_X^{\eta_\varepsilon}$, the neighborhood $V_u \cap V^\varepsilon$ of u satisfies the following: $f \in V_u \cap V^\varepsilon \cap [\overline{\mathfrak{G}}_X \cup \mathrm{cl}(\mathfrak{G})]$ implies $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. It follows that x is essential relative to $\overline{\mathfrak{G}}_X \cup \mathrm{cl}(\mathfrak{G})$. If, on the other hand, $u \in \mathfrak{B}_X$, then $u \in \mathfrak{A}_X$, and any pure-strategy Nash equilibrium of u is essential relative to $\mathrm{cl}(\mathfrak{G})$. That is, given a neighborhood V_x of x, there is a neighborhood N_u of u in $B(X)^N$ such that for every $f \in N_u \cap \mathrm{cl}(\mathfrak{G})$, $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. Because $u \in \mathfrak{A}_X \setminus \overline{\mathfrak{G}}_X$ and since $\overline{\mathfrak{G}}_X$ is closed in $B(X)^N$, there is a neighborhood U_u of u such that $U_u \cap \overline{\mathfrak{G}}_X = \emptyset$. It follows that for $f \in N_u \cap U_u \cap [\overline{\mathfrak{G}}_X \cup \mathrm{cl}(\mathfrak{G})]$, we have $V_x \cap \mathcal{E}_X(f) \neq \emptyset$. Hence, x is essential relative to $\overline{\mathfrak{G}}_X \cup \mathrm{cl}(\mathfrak{G})$. □

5.5 Proof of Theorem 4

The following lemmata are taken from Carbonell-Nicolau (2010).

Lemma 11 *$\mathfrak{G}_X^w$ is closed in $B(X)^N$.*

Lemma 12 *$\mathcal{E}_X|_{\mathfrak{G}_X^w}$ is compact-valued and upper hemicontinuous.*

The following lemmata are taken from Scalzo (2013).

Lemma 13 *$\mathfrak{G}_X^g$ is a complete subspace of the metric space $B(X)^N$.*

Lemma 14 *$\mathcal{E}_X|_{\mathfrak{G}_X^g}$ is compact-valued and upper hemicontinuous.*

Theorem 4 *Every member of a dense, residual subset of $\mathfrak{G}_X^* \cup \mathfrak{G}_X^g \cup \mathfrak{G}_X^w$ is essential.*

Proof $\mathcal{E}_X|_{\mathfrak{G}_X^g}$ and $\mathcal{E}_X|_{\mathfrak{G}_X^w}$ are compact-valued and upper hemicontinuous (Lemma 14 and Lemma 12). Therefore, $\mathcal{E}_X|_{\mathfrak{G}_X^g \cup \mathfrak{G}_X^w}$ is compact-valued and upper hemicontinuous, so by Lemma 2, $\mathfrak{G}_X^g \cup \mathfrak{G}_X^w \subseteq \mathfrak{G}_X^s(\mathfrak{G}_X^g \cup \mathfrak{G}_X^w)$. Consequently, since $\mathfrak{G}_X^g$, being a complete subspace of the complete metric space $B(X)^N$ (Lemma 13), is closed in $B(X)^N$, and $\mathfrak{G}_X^w$ is closed in $B(X)^N$ (Lemma 11), and because $\mathfrak{G}_X^g \cup \mathfrak{G}_X^w \subseteq \mathfrak{G}_X$ (from the definition of $\mathfrak{G}_X^g$ and Remark 1), we have

$$\mathrm{cl}(\mathfrak{G}_X^g \cup \mathfrak{G}_X^w) = \mathrm{cl}(\mathfrak{G}_X^g) \cup \mathrm{cl}(\mathfrak{G}_X^w) = \mathfrak{G}_X^g \cup \mathfrak{G}_X^w \subseteq \mathfrak{G}_X \cap \mathfrak{G}_X^s(\mathrm{cl}(\mathfrak{G}_X^g \cup \mathfrak{G}_X^w)).$$

Applying Theorem 5, we see that for every u in a dense, residual subset of $\mathfrak{G}_X^* \cup \mathfrak{G}_X^g \cup \mathfrak{G}_X^w$, any pure-strategy Nash equilibrium of u is essential relative to $\mathfrak{G}_X^* \cup \mathfrak{G}_X^g \cup \mathfrak{G}_X^w$. □

6 Auxiliary results

6.1 Proof of Lemma 3

We first prove two preparatory lemmas.

Lemma 15 *For each* ε, $\mathfrak{G}_X^{(c,\varepsilon)} \subseteq \mathfrak{G}_X^s$.

Proof Choose ε. Fix a game u in $\mathfrak{G}_X^{(c,\varepsilon)}$. Take a sequence (u^n) in $B(X)^N$ with $u^n \to u$ and a sequence (x^n) in X with $x^n \to x$ for some $x \in X$. Suppose that x is not a Nash equilibrium of u. Then, since u is ε-continuously secure, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $u_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$ there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i\}$.

Therefore, since $x^n \to x$, we have the following:

- For each i and each sufficiently large n, $u_i(z_i, x^n_{-i}) \geq \alpha_i + \varepsilon$ for all $z_i \in \Phi_i(x^n)$.
- For each sufficiently large n, there exists i such that $u_i(x^n) < \alpha_i$.

Because the player set is finite, it follows that there exist i and a subsequence (x^k) of (x^n) such that for each k,

$$u_i(z_i, x^k_{-i}) \geq \alpha_i + \varepsilon > \alpha_i > u_i(x^k), \quad \text{for all } z_i \in \Phi_i(x^k).$$

Therefore, since $u^n \to u$, for each large enough k we have $u^k_i(z_i, x^k_{-i}) > u^k_i(x^k)$ for all $z_i \in \Phi_i(x^k)$. It follows that $u \in \mathfrak{G}_X^s$. □

Lemma 16 *Suppose that* (u^n) *is a sequence in* $\mathfrak{G}_X^{(c,\varepsilon)}$ *such that* $u^n \to u$ *for some* $u \in B(X)^N$. *Then* $u \in \mathfrak{G}_X^{(c,\varepsilon-\eta)}$ *for every* $\eta > 0$ *with* $\varepsilon - \eta > 0$.

Proof Fix $x \in X$, and suppose that x is not a Nash equilibrium of u. Then, since $u^n \to u$, for any large enough n, x is not a Nash equilibrium of u^n. Hence, since $u^n \in \mathfrak{G}_X^{(c,\varepsilon)}$ for each n, for any large enough n there exist $\alpha^n \in \mathbb{R}^N$, a neighborhood V^n_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi^n : V^n_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V^n_x$, $u^n_i(z_i, y_{-i}) \geq \alpha^n_i + \varepsilon$ for all $z_i \in \Phi^n_i(y)$.
(ii) For each $y \in V^n_x$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u^n_i(z_i, y_{-i}) \geq \alpha^n_i\}$.

Passing to a subsequence if necessary, we have $\alpha^n \to \alpha$ for some $\alpha \in \mathbb{R}^N$. Fix $\eta > 0$ with $\varepsilon - \eta > 0$. Let $k := \frac{2\varepsilon}{\eta}$, so that

$$\varepsilon \left(1 - \tfrac{2}{k}\right) = \varepsilon - \eta. \tag{4}$$

Because $u^n \to u$, and since $\alpha^n \to \alpha$, for any large enough n we have, for each i and every $y \in V^n_x$,

$$u_i(z_i, y_{-i}) > u^n_i(z_i, y_{-i}) - \tfrac{\varepsilon}{2k}, \quad \text{for all } z_i \in \Phi^n_i(y),$$

and

$$\alpha_i^n > \alpha_i - \tfrac{\varepsilon}{2k}.$$

Consequently, for any large enough n, the following two conditions are satisfied:

- For each i and every $y \in V_x^n$, $u_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon - \frac{\varepsilon}{k}$ for all $z_i \in \Phi_i^n(y)$.
- For each $y \in V_x^n$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i^n(z_i, y_{-i}) \geq \alpha_i^n\}$.

Since $u^n \to u$ and $\alpha^n \to \alpha$, for each i and for any large enough n we have

$$\alpha_i^n + u_i(z) - u_i^n(z) < \alpha_i + \tfrac{\varepsilon}{k}, \quad \text{for all } z \in X.$$

Therefore, for each i and for any large enough n,

$$\begin{aligned}
&\left\{z_i \in X_i : u_i^n(z_i, y_{-i}) \geq \alpha_i^n\right\} \\
&\quad = \left\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i^n + u_i(z_i, y_{-i}) - u_i^n(z_i, y_{-i})\right\} \\
&\quad \supseteq \left\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i + \tfrac{\varepsilon}{k}\right\},
\end{aligned}$$

for all $y_{-i} \in X_{-i}$. Hence, since for any large enough n and for each $y \in V_x^n$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i^n(z_i, y_{-i}) \geq \alpha_i^n\}$, it follows that for any large enough n and for each $y \in V_x^n$, there exists i such that y_i does not belong to the convex hull of $\left\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i + \frac{\varepsilon}{k}\right\}$. We conclude that for any large enough n the following two conditions are satisfied:

- For each i and every $y \in V_x^n$, $u_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon - \frac{\varepsilon}{k}$ for all $z_i \in \Phi_i^n(y)$.
- For each $y \in V_x^n$, there exists i such that y_i does not belong to the convex hull of $\left\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i + \frac{\varepsilon}{k}\right\}$.

This implies that u satisfies $\varepsilon\left(1 - \frac{2}{k}\right)$-continuous security. Since $\varepsilon\left(1 - \frac{2}{k}\right) = \varepsilon - \eta$ by (4), the proof is complete. □

We are now ready to prove Lemma 3.

Lemma 3 *For each* ε, $\mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)} \cap \mathfrak{G}_X^s$ *for every* $\eta > 0$ *with* $\varepsilon - \eta > 0$.

Proof Fix ε. Let (u^n) be a sequence in $\mathfrak{G}_X^{(c,\varepsilon)}$. Suppose that $u^n \to u$ for some $u \in B(X)^N$. It suffices to show that $u \in \mathfrak{G}_X^{(c,\varepsilon-\eta)} \cap \mathfrak{G}_X^s$ for every $\eta > 0$ with $\varepsilon - \eta > 0$. But this follows immediately from Lemma 15 and Lemma 16.

6.2 Proof of Lemma 4

Lemma 4 *For each* ε *and for every* $\eta > 0$ *with* $\varepsilon - \eta > 0$, *there is an open subset* V *of* $B(X)^N$ *with* $\mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq V \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)}$.

Proof Fix $\varepsilon > 0$ and $\eta > 0$ with $\varepsilon - \eta > 0$. By Lemma 3, $\mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)}) \subseteq \mathfrak{G}_X^{(c,\varepsilon-\frac{\eta}{2})}$. Consequently, given $u \in \mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)})$, we have $u \in \mathfrak{G}_X^{(c,\varepsilon-\frac{\eta}{2})}$. Therefore, for every $x \in X$ that is not a Nash equilibrium of u, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $u_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon - \frac{\eta}{2}$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i\}$.

Choose $\beta > 0$ such that

$$\varepsilon - \frac{\eta}{2} - 2\beta = \varepsilon - \eta. \tag{5}$$

By (i) we have, for each $f \in N_\beta(u)$, i, and $y \in V_x$,

$$f_i(z_i, y_{-i}) \geq u_i(z_i, y_{-i}) - \beta \geq \alpha_i + \varepsilon - \frac{\eta}{2} - \beta, \quad \text{for all } z_i \in \Phi_i(y).$$

In addition, for each i and for every $f \in N_\beta(u)$ we have

$$\begin{aligned} &\{z_i \in X_i : u_i(z_i, y_{-i}) \geq \alpha_i\} \\ &\quad = \{z_i \in X_i : f_i(z_i, y_{-i}) \geq \alpha_i + f_i(z_i, y_{-i}) - u_i(z_i, y_{-i})\} \\ &\quad \supseteq \{z_i \in X_i : f_i(z_i, y_{-i}) \geq \alpha_i + \beta\}, \end{aligned}$$

for all $y_{-i} \in X_{-i}$. Therefore, since (ii) holds, for each $y \in V_x$ there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : f_i(z_i, y_{-i}) \geq \alpha_i + \beta\}$.

We conclude that if $f \in N_\beta(u)$, then, for every $x \in X$ that is not a Nash equilibrium of u, there exist $\alpha \in \mathbb{R}^N$, a neighborhood V_x of x, and a nonempty-valued, convex-valued, closed product correspondence $\Phi : V_x \rightrightarrows X$ such that the following two conditions are satisfied:

(i) For each i and every $y \in V_x$, $f_i(z_i, y_{-i}) \geq \alpha_i + \varepsilon - \frac{\eta}{2} - \beta$ for all $z_i \in \Phi_i(y)$.
(ii) For each $y \in V_x$, there exists i such that y_i does not belong to the convex hull of $\{z_i \in X_i : f_i(z_i, y_{-i}) \geq \alpha_i + \beta\}$.

Hence, $f \in \mathfrak{G}_X^{(c,\varepsilon-\frac{\eta}{2}-2\beta)} = \mathfrak{G}_X^{(c,\varepsilon-\eta)}$ (recall (5)).

We have shown that for every $u \in \mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)})$, there exists a neighborhood V_u of u such that $V_u \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)}$. Consequently,

$$\bigcup_{u \in \mathrm{cl}(\mathfrak{G}_X^{(c,\varepsilon)})} V_u \subseteq \mathfrak{G}_X^{(c,\varepsilon-\eta)},$$

and the proof is complete. □

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