

Note

# Essential equilibria in normal-form games

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## Abstract

A Nash equilibrium  $x$  of a normal-form game  $G$  is essential if any perturbation of  $G$  has an equilibrium close to  $x$ . Using payoff perturbations, we show that for games that are generic in the set of compact, quasiconcave, and generalized payoff secure games with upper semicontinuous sum of payoffs, all equilibria are essential. Some variants of this result are also established.

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## 1. Introduction

Given a normal-form game, we say that a Nash equilibrium  $x$  is *essential* if neighboring games with slightly perturbed payoffs have equilibria close to  $x$ . Similar definitions have been used elsewhere (e.g. Wu and Jiang [17] and Yu [16]).

Consider the class of normal-form games with the following properties: (1) the action space of each player is a nonempty, convex, compact subset of a metrizable topological vector space; (2) each player's payoff is concave in his own strategy; (3) the sum of payoffs is upper semicontinuous in all players' strategies; and (4) each player's payoff is lower semicontinuous in the other players' strategies. Yu [16] shows that for games that are generic within this class, all pure-strategy equilibria are essential.

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There is a recent literature that proves the existence of Nash equilibrium via conditions that are significantly weaker than (2)–(4).<sup>2</sup> It is thus natural to ask if the existence of standard refinements of the Nash equilibrium concept can be established under weaker conditions. In this note, we focus on the notion of essentiality and seek a genericity result along the lines of Yu [16].<sup>3</sup> Specifically, we show that conditions (4) and (2) can be weakened: it suffices that the game be generalized payoff secure (Barelli and Soza [4]) and that each player's payoff be quasiconcave in his own strategy.<sup>4</sup> When each  $u_i$  is upper semicontinuous, generalized payoff security can be weakened to weak payoff security (Carmona [8]).

Our proofs rely on a theorem of Fort [11], which states that a correspondence  $F$  from a topological space to a metric space is lower hemicontinuous at a residual subset of its domain if  $F$  is nonempty-valued, compact-valued, and upper hemicontinuous. When the domain of  $F$  is a Baire space, the said residual set is dense  $\mathcal{G}_\delta$ . Our use of Fort's theorem is standard: we identify a Baire space of games and show that the associated Nash equilibrium correspondence has the desired properties—specifically, it is nonempty-valued with a closed graph. While nonemptiness and graph closeness can be established under relatively weak conditions (e.g., better-reply security (Reny [13]) or generalized better-reply security (Barelli and Soza [4])), the requirement that the class of games be a Baire space is tighter: we show that the set of (compact, quasiconcave, and) better-reply secure games is not complete, so that one cannot invoke the Baire category theorem to conclude that the space of games is Baire.<sup>5</sup>

A corollary of the main result states that a suitable strengthening of Reny's [13] payoff security (for the payoffs of the original game), along with the rest of the assumptions, ensures that for generic games all mixed-strategy equilibria are essential.

## 2. Preliminaries

We consider normal-form games  $G = (X_i, u_i)_{i=1}^N$ , where  $N$  is a finite number of players, each  $X_i$  is a nonempty set, and each  $u_i : X \rightarrow \mathbb{R}$  is bounded, with  $X := \times_{i=1}^N X_i$ . If each  $X_i$  is a nonempty compact subset of a metric space, we say that  $G$  is a *compact metric game*. All the normal-form games considered in this paper are compact metric games.

We use the symbol  $X_{-i}$  to designate the set  $\times_{j \neq i} X_j$ , and, given a player  $i$  and  $(x_i, x_{-i}) \in X_i \times X_{-i}$ , we slightly abuse notation and represent the point  $(x_1, \dots, x_N) \in \times_j X_j$  as  $(x_i, x_{-i})$ .

If each  $X_i$  is a convex subset of a topological vector space and for each  $i$  and every  $x_{-i} \in X_{-i}$   $u_i(\cdot, x_{-i})$  is quasiconcave on  $X_i$ , we say that  $G$  is *quasiconcave*.

The *graph* of  $G$  is the set

$$\Gamma_G := \{(x, u) \in X \times \mathbb{R}^N : u_i(x) = u_i, \text{ all } i\}.$$

The closure of  $\Gamma_G$  is denoted  $\bar{\Gamma}_G$ .

<sup>2</sup> See, for instance, Reny [13], Bagh and Jofre [3], Barelli and Soza [4], and Carmona [8,9].

<sup>3</sup> Other refinement specifications are studied in Simon and Stinchcombe [14], Al-Najjar [2], and Carbonell-Nicolau [5–7].

<sup>4</sup> Our requirement that each player's payoff be quasiconcave in his own strategy serves the sole purpose of applying available results (on the existence of Nash equilibria in discontinuous games) to obtain the nonemptiness of the Nash equilibrium correspondence. In this regard, Yu's [16] concavity assumption can be relaxed by direct application of the said existence results.

<sup>5</sup> A similar statement is true about other classes of games considered in the literature on the existence of Nash equilibrium. This is discussed in Section 4.

**Definition 1.** A strategy profile  $x \in X$  is a *pure-strategy Nash equilibrium* of  $G$  if  $u_i(x) \geq u_i(y_i, x_{-i})$  for each  $y_i \in X_i$  and every  $i$ .

We now define Reny’s [13] better-reply security and payoff security.

**Definition 2.** The game  $G$  is *better-reply secure* if, for every  $(x, u) \in \bar{\Gamma}_G$  such that  $x$  is not a (pure-strategy) Nash equilibrium of  $G$ , there exist  $i$  and  $y_i \in X_i$  such that  $u_i(y_i, O_{x_{-i}}) \geq \alpha > u_i$ , some  $\alpha \in \mathbb{R}$  and some neighborhood  $O_{x_{-i}}$  of  $x_{-i}$ .<sup>6</sup>

**Definition 3.** The game  $G$  is *payoff secure* if for each  $\varepsilon > 0$ ,  $x \in X$ , and  $i$ , there exists  $y_i \in X_i$  such that  $u_i(y_i, O_{x_{-i}}) > u_i(x) - \varepsilon$  for some neighborhood  $O_{x_{-i}}$  of  $x_{-i}$ .

The following definitions appear in Barelli and Soza [4].

**Definition 4.** The game  $G$  is *generalized better-reply secure* if, for every  $(x, u) \in \bar{\Gamma}_G$  such that  $x$  is not a (pure-strategy) Nash equilibrium of  $G$ , there exist a player  $i$ , a neighborhood  $O_x$  of  $x$ , and a correspondence  $\Phi_i : O_x \rightrightarrows X_i$  such that  $u_i(\Phi_i(y), y_{-i}) \geq \alpha > u_i$  for every  $y \in O_x$ , some  $\alpha \in \mathbb{R}$ , and  $\Phi_i$  is nonempty, convex-valued, compact-valued, and upper hemicontinuous.<sup>7</sup>

**Definition 5.** The game  $G$  is *generalized payoff secure* if for each  $\varepsilon > 0$ ,  $x \in X$ , and  $i$ , one can find a neighborhood  $O_x$  of  $x$  and a correspondence  $\Phi_i : O_x \rightrightarrows X_i$  such that  $u_i(\Phi_i(y), y_{-i}) > u_i(x) - \varepsilon$  for every  $y \in O_x$ , and  $\Phi_i$  is nonempty, convex-valued, compact-valued, and upper hemicontinuous.

Carmona [8] presents the following variant of payoff security, which is weaker than generalized payoff security.

**Definition 6.** The game  $G$  is *weakly payoff secure* if for each  $\varepsilon > 0$ ,  $x \in X$ , and  $i$ , there exists a neighborhood  $O_{x_{-i}}$  of  $x_{-i}$  for which  $y_{-i} \in O_{x_{-i}}$  implies  $u_i(y_i, y_{-i}) > u_i(x) - \varepsilon$ , some  $y_i \in X_i$ .<sup>8</sup>

The following condition, which is stronger than payoff security, appears in Monteiro and Page [12].

**Definition 7.** The game  $G$  is *uniformly payoff secure* if for each  $i$ ,  $\varepsilon > 0$ , and  $x_i \in X_i$ , there exists  $y_i \in X_i$  such that for every  $y_{-i} \in X_{-i}$ , there is a neighborhood  $O_{y_{-i}}$  of  $y_{-i}$  such that  $u_i(y_i, O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$ .

The following implications are immediate:

<sup>6</sup> By a slight abuse of notation, we use the equation  $u_i(y_i, O_{x_{-i}}) \geq \alpha > u_i$  to represent the following inequalities:  $u_i(y_i, y_{-i}) \geq \alpha > u_i$  for all  $y_{-i} \in O_{x_{-i}}$ .

<sup>7</sup> The statement ‘ $u_i(\Phi_i(y), y_{-i}) \geq \alpha > u_i$  for every  $y \in O_x$ ’ means ‘ $u_i(z_i, y_{-i}) \geq \alpha > u_i$  for each  $z_i \in \Phi_i(y)$  and every  $y \in O_x$ .’

<sup>8</sup> Carmona’s notion was introduced by Dasgupta and Maskin [10] via the following equivalent formulation: Given  $i$ , define  $v_i : X_{-i} \rightarrow \mathbb{R}$  by  $v_i(x_{-i}) := \sup_{x_i \in X_i} u_i(x_i, x_{-i})$ . The game  $G$  is weakly payoff secure if and only if  $v_i$  is lower semicontinuous for each  $i$ .

uniform payoff security  $\Rightarrow$  payoff security  
 $\Rightarrow$  generalized payoff security  
 $\Rightarrow$  weak payoff security.

**Definition 8.** A correspondence  $\Phi : A \rightrightarrows B$  between topological spaces is **upper hemicontinuous at**  $x \in A$  if the following is true: for every neighborhood  $O_{\Phi(x)}$  of  $\Phi(x)$  there is a neighborhood  $O_x$  of  $x$  such that  $y \in O_x$  implies  $\Phi(y) \subseteq O_{\Phi(x)}$ .  $\Phi$  is **upper hemicontinuous** if it is upper hemicontinuous at every  $x \in A$ .

A correspondence  $\Phi : A \rightrightarrows B$  between topological spaces is **lower hemicontinuous at**  $x \in A$  if the following is true: for every open set  $O \subseteq B$  with  $O \cap \Phi(x) \neq \emptyset$  there is a neighborhood  $O_x$  of  $x$  such that  $y \in O_x$  implies  $\Phi(y) \cap O \neq \emptyset$ .  $\Phi$  is **lower hemicontinuous** if it is lower hemicontinuous at every  $x \in A$ .

### 3. Essential equilibria

For each player  $i$ , fix  $X_i$ , and let  $X := \times_i X_i$ . We shall consider the following classes of games:

- The set  $\tilde{\mathfrak{g}}_X$  of games  $(X_i, u_i)_{i=1}^N$  that are compact, metric, quasiconcave, and generalized better-reply secure.
- The set  $\mathfrak{g}_X$  of games  $(X_i, u_i)_{i=1}^N$  that are compact, metric, quasiconcave, and generalized payoff secure, with  $\sum_i u_i$  upper semicontinuous (usc).
- The set  $\mathfrak{g}_X^w$  of games  $(X_i, u_i)_{i=1}^N$  that are compact, metric, quasiconcave, and weakly payoff secure, with each  $u_i$  usc.
- The set  $\mathfrak{g}_X^u$  of games  $(X_i, u_i)_{i=1}^N$  that are compact, metric, and uniformly payoff secure, with  $\sum_i u_i$  usc and each  $u_i$  Borel measurable.

We view  $\tilde{\mathfrak{g}}_X, \mathfrak{g}_X, \mathfrak{g}_X^w$ , and  $\mathfrak{g}_X^u$  as subsets of the metric space  $(B(X)^N, \rho_X)$ , where  $B(X)$  stands for the set of bounded maps  $f : X \rightarrow \mathbb{R}$ , and the associated metric  $\rho_X : B(X)^N \times B(X)^N \rightarrow \mathbb{R}$  is defined by

$$\rho_X((u_1, \dots, u_N), (f_1, \dots, f_N)) := \sum_{i=1}^N \sup_{x \in X} |u_i(x) - f_i(x)|.$$

Let  $\mathcal{N}_X : B(X)^N \rightrightarrows X$  be the Nash equilibrium correspondence, which assigns the subset of (pure-strategy) Nash equilibria of  $G, \mathcal{N}_X(G)$ , to each game  $G$  in  $B(X)^N$ . Given  $\mathfrak{g} \subseteq B(X)^N$ , the restriction of  $\mathcal{N}_X$  to  $\mathfrak{g}$  is denoted as  $\mathcal{N}_X|_{\mathfrak{g}}$ .

**Definition 9.** Given a class of games  $\mathfrak{g} \subseteq B(X)^N$ , a pure-strategy Nash equilibrium  $x$  of  $G \in \mathfrak{g}$  is an **essential equilibrium of  $G$  relative to  $\mathfrak{g}$**  if for every open neighborhood  $O_x$  of  $x$  there is an open neighborhood  $O_G$  of  $G$  such that for every  $g \in O_G \cap \mathfrak{g}$  there exists  $y$  in  $O_x \cap \mathcal{N}_X(g)$ .

**Remark 1.** For the classes of games considered in this paper, essentiality is equivalent to the lower hemicontinuity of the Nash equilibrium correspondence. (See, for instance, Lemma 4 below.)

This paper studies certain classes of games whose generic members have only essential equilibria. We adopt the standard topological notion of genericity, i.e., a property holds *generically* in a topological space  $Y$  if it holds for all the elements of a dense  $\mathcal{G}_\delta$  (Definition 12) subset of  $Y$ .<sup>9</sup>

**Lemma 1.** *Suppose that  $B(X) \ni f^n \rightarrow f \in B(X)$ . If each  $f^n$  is upper semicontinuous—respectively, quasiconcave—then  $f$  is upper semicontinuous—respectively, quasiconcave.*

**Proof.** Suppose that  $B(X) \ni f^n \rightarrow f \in B(X)$ , where each  $f^n$  is usc—respectively, quasiconcave. Fix  $\alpha \in \mathbb{R}$ . Then the set  $\{x: f(x) \geq \alpha\}$  can be written as  $\bigcap_n \{x: f(x) \geq \alpha - \sup_{x \in X} |f^n(x) - f(x)|\}$ , a countable intersection of closed—respectively, convex—sets. It follows that  $\{x: f(x) \geq \alpha\}$  is closed—respectively, convex—or, equivalently, that  $f$  is usc—respectively, quasiconcave.<sup>10</sup>  $\square$

**Lemma 2.** *The sets  $\mathfrak{g}_X$ ,  $\mathfrak{g}_X^w$ , and  $\mathfrak{g}_X^u$  are closed in  $B(X)$ .*

**Proof.** We only prove the statement for the set  $\mathfrak{g}_X$  (the other sets can be handled similarly). Suppose that  $(u^n) = (u_1^n, \dots, u_N^n)$  is a sequence in  $\mathfrak{g}_X$  with  $u^n \rightarrow u = (u_1, \dots, u_N)$  for some  $u \in B(X)^N$ . It suffices to show that  $u$  lies in  $\mathfrak{g}_X$ . Clearly,  $(X_i, u_i)$  is compact. By Lemma 1,  $(X_i, u_i)$  is quasiconcave, with  $\sum_i u_i$  usc (since each  $\sum_i u_i^n$  is usc and  $\sum_i u_i^n \rightarrow \sum_i u_i$ ). We show that  $(X_i, u_i)$  is generalized payoff secure.

Fix  $\varepsilon > 0$ ,  $x \in X$ , and  $i$ . We have to find a neighborhood  $O_x$  of  $x$  and a correspondence  $\Phi_i: O_x \rightrightarrows X_i$  such that  $u_i(\Phi_i(y), y_{-i}) > u_i(x) - \varepsilon$  for every  $y \in O_x$ , and  $\Phi_i$  is nonempty, convex-valued, compact-valued, and upper hemicontinuous. Since each  $(X_i, u_i^n)$  is generalized payoff secure, given  $n$  and  $\alpha > 0$  there exist  $O_x^n$  and  $\Phi_i^n: O_x^n \rightrightarrows X_i$  such that  $u_i^n(\Phi_i^n(y), y_{-i}) > u_i^n(x) - \alpha$  for every  $y \in O_x^n$ , and  $\Phi_i^n$  is nonempty, convex-valued, compact-valued, and upper hemicontinuous. Hence, because  $u^n \rightarrow u$ , if  $\alpha$  is small enough and  $n$  is large enough,  $u_i^n(x) - \alpha$  is close enough to  $u_i(x)$  to ensure that  $u_i^n(\Phi_i^n(y), y_{-i}) > u_i(x) - \frac{\varepsilon}{2}$  for every  $y \in O_x^n$ . Now, since  $u^n \rightarrow u$  we obtain, for  $n$  sufficiently large,  $u_i(\Phi_i^n(y), y_{-i}) > u_i(x) - \varepsilon$  for every  $y \in O_x^n$ .  $\square$

**Lemma 3.**  $\mathcal{N}_X(G) \neq \emptyset$  for every  $G \in \tilde{\mathfrak{g}}_X \cup \mathfrak{g}_X^w$ .

**Proof.** Given  $G \in \tilde{\mathfrak{g}}_X$ ,  $G$  is compact, quasiconcave, and generalized better-reply secure. It follows from Corollary 4.15 of Barelli and Soza [4] that  $G$  has a pure-strategy Nash equilibrium.

Given  $G \in \mathfrak{g}_X^w$ ,  $G$  is compact, metric, quasiconcave, usc, and weakly payoff secure. Therefore, Corollary 2 of Carmona [8] gives a pure-strategy Nash equilibrium of  $G$ .  $\square$

**Lemma 4.**  $\mathcal{N}_X|_{\mathfrak{g}}$  is compact-valued and upper hemicontinuous for each  $\mathfrak{g} \in \{\mathfrak{g}_X, \mathfrak{g}_X^w\}$ .

**Proof.** We only consider the case when  $\mathfrak{g} = \mathfrak{g}_X$ , since the other case is analogous. Since  $X$  is compact, it suffices to show that  $\mathcal{N}_X|_{\mathfrak{g}_X}$  has a closed graph (e.g. Theorem 17.11 of Aliprantis and Border [1]). Let  $(G^n = (X_i, u_i^n), x^n)$  be a sequence in the graph of  $\mathcal{N}_X|_{\mathfrak{g}_X}$ , i.e., each  $G^n$  lies in  $\mathfrak{g}_X$  and each  $x^n$  is a Nash equilibrium of  $G^n$ . Suppose that  $(G^n, x^n) \rightarrow (G, x)$  for some

<sup>9</sup> The measure-theoretic notion of genericity says that a property holds generically in a probability space if it holds “almost everywhere,” or with probability one with respect to some probability measure. For the relationship between the two approaches to genericity, see, for instance, Zindulka [18] and references therein.

<sup>10</sup> We thank an anonymous referee for suggesting this proof.

$(G, x)$  in  $\mathfrak{g}_X \times X$ . We wish to show that  $x$  is a Nash equilibrium of  $G$ . We assume that  $x$  is not an equilibrium and derive a contradiction.

Let  $G = (X_i, u_i)$ . Because  $x^n \rightarrow x$  and each  $u_i$  is bounded, we may write (passing to a subsequence if necessary)

$$(x^n, (u_1(x^n), \dots, u_N(x^n))) \rightarrow (x, (u_1, \dots, u_N)) \tag{1}$$

for some  $u := (u_1, \dots, u_N)$ . Therefore,  $(x, u)$  belongs to  $\overline{F}_G$ . Hence, if  $x$  is not a Nash equilibrium of  $G$ , by generalized better-reply security of  $G$  we see that

$$u_i(\Phi_i(y), y_{-i}) \geq u_i + \alpha, \quad \text{all } y \in O_x,$$

some  $i, \alpha > 0$ , some neighborhood  $O_x$  of  $x$ , and some nonempty  $\Phi_i : O_x \rightrightarrows X_i$ . This, together with (1), gives

$$u_i(\Phi_i(x^n), x^n_{-i}) > \beta > u_i(x^n),$$

for some  $\beta \in \mathbb{R}$  and for any large enough  $n$ . Consequently, since  $u_i^n \rightarrow u_i$ , we obtain, for some  $y_i \in X_i$ ,

$$u_i^n(y_i, x^n_{-i}) > u_i^n(x^n),$$

for some large enough  $n$ , thereby contradicting that  $x^n$  is a Nash equilibrium in  $G^n$ .  $\square$

**Definition 10.** A subset of a topological space  $Y$  is *nowhere dense* if it is not dense in any open subset of  $Y$ . A subset  $A$  of a topological space is *meager* if it is a countable union of nowhere dense sets. The set  $A$  is *residual* if it is the complement of a meager set, or, equivalently, if it is a countable intersection of open dense sets.

**Definition 11.** A *Baire space* is a topological space for which every countable intersection of open dense sets is also dense.

**Definition 12.** A subset of a topological space is a  $\mathcal{G}_\delta$ -set, or simply a  $\mathcal{G}_\delta$ , if it is a countable intersection of open sets.

**Lemma 5** (Fort [11], Theorem 2). *Suppose that  $X$  is a metric space and  $Y$  a topological space. Suppose further that  $F : Y \rightrightarrows X$  is a compact-valued and upper hemicontinuous correspondence with  $F(y) \neq \emptyset$  for all  $y \in Y$ . Then there exists a residual subset  $Q$  of  $Y$  such that  $F$  is lower hemicontinuous at every point in  $Q$ .<sup>11</sup>*

For completeness we furnish a proof of Lemma 5 in Section 5.

Lemma 5, along with Lemma 3 and Lemma 4, gives the following:

**Theorem 1.** *For any  $G$  in a residual subset of  $\tilde{\mathfrak{g}}_X$ , any pure-strategy Nash equilibrium of  $G$  is essential.*

**Remark 2.** Since we are after a genericity result, we are interested in the case where the residual set given by Theorem 1 is dense  $\mathcal{G}_\delta$  in  $\tilde{\mathfrak{g}}_X$  (or some subset of  $\tilde{\mathfrak{g}}_X$ ). If  $\tilde{\mathfrak{g}}_X$  were a Baire space,

<sup>11</sup> We thank an anonymous referee for pointing out the differences between this statement (Fort’s Theorem 2) and the version reported in Yu ([16], Lemma 2.1) (Lemma 6 below) and Tan et al. ([15], Lemma 2.3).

the said residual set would be dense in  $\tilde{g}_X$ . Unfortunately,  $\tilde{g}_X$  fails to be complete (Example 1, Section 4) or locally compact, and therefore the Baire category theorem cannot be invoked to show that  $\tilde{g}_X$  is Baire. As shown below, a genericity result can be derived by restricting attention to the sets  $g_X$  and  $g_X^w$ .

The following variant of Lemma 5 can be proven.

**Lemma 6.** *Suppose that  $X$  is a metric space and  $Y$  a Baire space. Suppose further that  $F : Y \rightrightarrows X$  is a compact-valued and upper hemicontinuous correspondence with  $F(y) \neq \emptyset$  for all  $y \in Y$ . Then there exists a dense  $\mathcal{G}_\delta$  subset  $Q$  of  $Y$  such that  $F$  is lower hemicontinuous at every point in  $Q$ .*

**Proof.** The proof of Lemma 5 (Section 5) makes it clear that the residual set  $Q$  given by Lemma 5 is a  $\mathcal{G}_\delta$  set. We thus obtain a residual  $\mathcal{G}_\delta$  subset  $Q$  of  $Y$  such that  $F$  is lower hemicontinuous at every point in  $Q$ . It only remains to observe that  $Q$ , being a residual subset of a Baire space, is dense in  $Y$ .  $\square$

**Lemma 7.** *There exists a dense  $\mathcal{G}_\delta$  subset  $q$  of  $g_X$  such that  $\mathcal{N}_X$  is lower hemicontinuous at every point in  $q$ .*

**Proof.** By Lemma 2,  $g_X$  is closed in  $B(X)$ . Since  $B(X)$  is a complete metric space, it follows that  $g_X$  is complete and metric, and hence (by the Baire category theorem) a Baire space. Further, by Lemma 3 and Lemma 4,  $\mathcal{N}_X|_{g_X}$  is nonempty-valued, compact-valued, and upper hemicontinuous. Therefore, Lemma 6 gives a dense  $\mathcal{G}_\delta$  subset  $q$  of  $g_X$  such that  $\mathcal{N}_X$  is lower hemicontinuous at every point in  $q$ .  $\square$

The preceding lemmata give the following:

**Theorem 2.** *For any  $G$  in a dense  $\mathcal{G}_\delta$  subset of  $g_X$ , any pure-strategy Nash equilibrium of  $G$  is essential.*

When each  $u_i$  is usc, generalized payoff security can be weakened. The proof of the following lemma is analogous to that of Lemma 7.

**Lemma 7'.** *There exists a dense  $\mathcal{G}_\delta$  subset  $q$  of  $g_X^w$  such that  $\mathcal{N}_X$  is lower hemicontinuous at every point in  $q$ .*

Lemma 7', together with Lemma 4, gives the following variant of Theorem 2:

**Theorem 2'.** *For any  $G$  in a dense  $\mathcal{G}_\delta$  subset of  $g_X^w$ , any pure-strategy Nash equilibrium of  $G$  is essential.*

### 3.1. Mixed-strategy equilibria

For the members  $G = (X_i, u_i)$  of  $\mathfrak{g}_X^u$ , we can define the *mixed extension of  $G$*  as the game  $\overline{G} = (M_i, U_i)$ , where  $M_i$  stands for the set of Borel probability measures on  $X_i$ , and  $U_i : M \rightarrow \mathbb{R}$  is defined by

$$U_i(\mu) := \int_X u_i d\mu,$$

where  $M := \times_i M_i$ .<sup>12</sup>

**Definition 13.** A strategy profile  $\mu \in M$  is a *mixed-strategy Nash equilibrium* of  $G$  if it is a pure-strategy Nash equilibrium of  $\overline{G}$ .

Consider the following redefinition of  $\mathcal{N}_X : \mathfrak{g}_X^u \Rightarrow M$  assigns the subset of (pure-strategy) Nash equilibria of  $\overline{G}$ ,  $\mathcal{N}_X(G)$ , to each game  $G$  in  $\mathfrak{g}_X^u$ . As usual, given  $\mathfrak{g} \subseteq B(X)^N$ , the restriction of  $\mathcal{N}_X$  to  $\mathfrak{g}$  is denoted as  $\mathcal{N}_X|_{\mathfrak{g}}$ .

**Definition 14.** Given a class of games  $\mathfrak{g} \subseteq B(X)^N$ , a mixed-strategy Nash equilibrium  $\mu$  of  $G \in \mathfrak{g}$  is an *essential equilibrium of  $G$  relative to  $\mathfrak{g}$*  if for every open neighborhood  $O_\mu$  of  $\mu$  there is an open neighborhood  $O_G$  of  $G$  such that for every  $g \in O_G \cap \mathfrak{g}$  there exists  $\nu$  in  $O_\mu \cap \mathcal{N}_X(g)$ .

We have the following corollary to Theorem 2.

**Corollary 1.** *For any  $G$  in a dense  $\mathcal{G}_\delta$  subset of  $\mathfrak{g}_X^u$ , any mixed-strategy Nash equilibrium of  $G$  is essential.*

The proof of the corollary is analogous to that of Theorem 2: one can use the following result in place of Lemma 3, together with the other lemmata, with ‘ $\mathfrak{g}_X^u$ ’ replacing ‘ $\mathfrak{g}_X$ .’

**Lemma 3’.**  $\mathcal{N}_X(G) \neq \emptyset$  for every  $G \in \mathfrak{g}_X^u$ .

**Proof.** Given  $G \in \mathfrak{g}_X^u$ ,  $\overline{G}$  is compact, payoff secure (by Theorem 1 of [12]), with  $\sum_i U_i$  usc (by Proposition 5.1 of [13], since  $\sum_i u_i$  is usc). Therefore, Corollary 5.2 of Reny [13] gives a Nash equilibrium of  $\overline{G}$ .  $\square$

## 4. Discussion

Deriving Theorem 2 amounted to identifying a class of games,  $\mathfrak{g}_X$ , with the following properties: (1)  $\mathfrak{g}_X$  is a Baire space; (2) all the games in  $\mathfrak{g}_X$  possess a Nash equilibrium; and (3) the Nash equilibrium correspondence, defined on  $\mathfrak{g}_X$ , has a closed graph. Items (1)–(3), together with Fort’s theorem, gave Theorem 2.

The proofs of Lemma 3 and Lemma 4 reveal that items (2) and (3) can be established under conditions that are weaker than those imposed on the members of  $\mathfrak{g}_X$ .<sup>13</sup> Thus, it is natural to ask

<sup>12</sup> Since  $G \in \mathfrak{g}_X^u$  is compact and metric, each  $M_i$  is compact in the weak\* topology.

<sup>13</sup> Lemma 4 can also be proven for the class of games satisfying Bagh and Jofre’s [4] weak reciprocal upper semicontinuity or the class of games satisfying weak payoff security.



if a stronger version of Theorem 2 can be obtained by identifying an expanded class of games that can replace  $\mathfrak{g}_X$  in the statement of the theorem.

We argue that the proposed generalization poses difficulties. Let  $\mathfrak{g}_X^b$  stand for the class of games  $G = (X_i, u_i)$  that are compact, metric, quasiconcave, and better-reply secure (so that  $\mathfrak{g}_X \subsetneq \mathfrak{g}_X^b$ ). The following example demonstrates that  $\mathfrak{g}_X^b$  is not a complete space, implying that for the class  $\mathfrak{g}_X^b$  one cannot rely on the Baire category theorem to derive the analogue of Lemma 7, with ‘ $\mathfrak{g}_X^b$ ’ replacing ‘ $\mathfrak{g}_X$ .’<sup>14</sup> Thus, a strengthening of Theorem 2 in the proposed direction, with ‘ $\mathfrak{g}_X^b$ ’ replacing ‘ $\mathfrak{g}_X$ ,’ would require a suitable generalization of some aspect of either the Baire category theorem or Fort’s theorem.<sup>15</sup>

**Example 1.** Consider the two-player game  $G = ([0, 1], [0, 1], u_1, u_2)$ , where  $u_2(0, 0) := 1$ ,  $u_2 := 0$  elsewhere, and  $u_1(0, 0) := 0$ ,  $u_1 := 1$  elsewhere. It is easy to verify that  $G$  is quasiconcave with  $\sum_i u_i$  usc. Further,  $G$  is not better-reply secure, but can be approximated by a sequence of compact, quasiconcave, and better-reply secure games, with usc sum of payoffs. Indeed, it suffices to take  $G^n := ([0, 1], [0, 1], u_1^n, u_2)$ , where  $u_1^n(0, 0) := 0$  and  $u_1^n := \frac{1}{n}f + (1 - \frac{1}{n})u_1$  elsewhere, and  $f : [0, 1]^2 \rightarrow \mathbb{R}$  is defined by  $f(x_1, x_2) := 1 + x_1$ . Consequently,  $\mathfrak{g}_X^b$  is not closed, and therefore  $\mathfrak{g}_X^b$  is incomplete.

Similar arguments apply to other supersets of  $\mathfrak{g}_X$  studied in the literature on the existence of Nash equilibria. Let  $\widehat{G} := ([0, 1], [0, 1], \widehat{u}_1, \widehat{u}_2)$  be the following variant of  $G$ :  $\widehat{u}_2 := 0$ , and  $\widehat{u}_1 := u_1$ . The game  $\widehat{G}$  fails Carmona’s [8] weak upper semicontinuity, while the members of the sequence  $(\widehat{G}^n := ([0, 1], [0, 1], \widehat{u}_1^n, \widehat{u}_2))$  with  $\widehat{u}_1^n := \frac{1}{n}f + (1 - \frac{1}{n})\widehat{u}_1$  ( $f$  defined as before) satisfy weak usc. Hence, the class of games that are compact, metric, quasiconcave, weakly usc, and weakly payoff secure (a class for which the Nash equilibrium correspondence is nonempty-valued (Carmona [8])) is not complete. Similarly,  $G$  fails Barelli and Soza’s [4] generalized better-reply security, unlike the sequence  $(G^n)$  defined in the example. Furthermore,  $\widehat{G}$  fails Carmona’s [9] weak better-reply security and Bagh and Jofre’s [3] weak reciprocal upper semicontinuity, unlike the sequence  $(\widehat{G}^n)$ .

### 5. Proof of Lemma 5

**Lemma 5** (Fort [11], Theorem 2). *Suppose that  $X$  is a metric space and  $Y$  a topological space. Suppose further that  $F : Y \rightrightarrows X$  is a compact-valued and upper hemicontinuous correspondence with  $F(y) \neq \emptyset$  for all  $y \in Y$ . Then there exists a residual subset  $Q$  of  $Y$  such that  $F$  is lower hemicontinuous at every point in  $Q$ .*

**Proof.** Given  $\varepsilon > 0$ , let  $C(\varepsilon)$  be the set of all points  $y \in Y$  such that for each  $\alpha \in (0, 3\varepsilon)$  and every neighborhood  $O_y$  of  $y$  there exists  $z \in O_y$  with  $F(y) \not\subseteq N_\alpha(F(z))$ .<sup>16</sup>

We first show that the interior of the closure of  $C(\varepsilon)$  is empty, implying that  $C(\varepsilon)$  is nowhere dense. To this end, we first prove that  $C(\varepsilon)$  is closed. Let  $y$  be a point in the closure of  $C(\varepsilon)$  and choose any  $\alpha \in (0, 3\varepsilon)$ . Given  $\beta \in (\alpha, 3\varepsilon)$ , since  $F$  is upper hemicontinuous at  $y$ ,

<sup>14</sup> The example shows that more is true: the class of games in  $\mathfrak{g}_X^b$  whose sum of payoffs is usc is not complete.

<sup>15</sup> Locally compact Hausdorff (hence possibly incomplete) spaces are Baire spaces. Unfortunately,  $\mathfrak{g}_X^b$  and other supersets of  $\mathfrak{g}_X$  fail to be locally compact.

<sup>16</sup> Here  $N_\alpha(F(z))$  stands for the  $\alpha$ -neighborhood of  $F(z)$ , i.e.,  $N_\alpha(F(z)) := \bigcup_{x \in F(z)} N_\alpha(x)$ , where  $N_\alpha(x)$  is the  $\alpha$ -neighborhood of  $x$ .

there exists a neighborhood  $O_y$  of  $y$  such that  $z \in O_y$  implies  $F(z) \subseteq N_{\beta-\alpha}(F(y))$ . Since  $y$  lies in the closure of  $C(\varepsilon)$ , we may choose  $z \in O_y \cap C(\varepsilon)$ , so that there exists  $w \in O_y$  such that  $F(z) \not\subseteq N_\beta(F(w))$ . We have  $F(y) \not\subseteq N_\alpha(F(w))$  (in fact,  $F(y) \subseteq N_\alpha(F(w))$  would imply  $N_{\beta-\alpha}(F(y)) \subseteq N_\beta(F(w))$ , which, combined with the inclusion  $F(z) \subseteq N_{\beta-\alpha}(F(y))$  would give  $F(z) \subseteq N_\beta(F(w))$ , a contradiction), and therefore  $y \in C(\varepsilon)$ . Thus,  $C(\varepsilon)$  is closed.

We now show that  $C(\varepsilon)$  cannot contain a nonempty open set. For  $y \in Y$  and  $\varepsilon > 0$ , let  $n(F(y), \varepsilon)$  stand for the smallest integer  $n$  such that  $F(y)$  can be covered by  $n$   $\varepsilon$ -neighborhoods in  $X$  (note that  $n(F(y), \varepsilon)$  is well-defined since  $F$  is compact-valued). Observe that to prove that  $C(\varepsilon)$  cannot contain a nonempty open set it suffices to show that  $y \in C(\varepsilon)$  implies the existence of points  $z$  arbitrarily close to  $y$  for which  $n(F(z), \varepsilon) \leq n(F(y), \varepsilon) - 1$  (for then any nonempty open set in  $C(\varepsilon)$  would contain infinitely many points, some of which would be points  $y$  with  $n(F(y), \varepsilon) < 0$ , which is impossible). Fix  $y \in C(\varepsilon)$  and let  $n = n(F(y), \varepsilon)$ . Let  $O_1, \dots, O_n$  be  $n$   $\varepsilon$ -neighborhoods that cover  $F(y)$ . Because  $F$  is upper hemicontinuous, there is a neighborhood  $O_y$  of  $y$  such that  $F(z) \subseteq \bigcup_{k=1}^n O_k$  for every  $z \in O_y$ . Moreover, since  $y \in C(\varepsilon)$ , there exists  $w \in O_y$  such that  $F(y) \not\subseteq N_{2\varepsilon}(F(w))$ . We have  $F(w) \cap O_k = \emptyset$  for some  $k$ , for  $F(w) \cap O_k \neq \emptyset$  for all  $k$  implies  $F(y) \subseteq N_{2\varepsilon}(F(w))$ . Hence,  $F(w)$  can be covered by  $n - 1$   $\varepsilon$ -neighborhoods, so that  $n(F(w), \varepsilon) \leq n - 1$ .

Now define

$$C := \bigcup_{\varepsilon \text{ rational}} C(\varepsilon).$$

Because  $C$  is a countable union of nowhere dense sets,  $C$  is meager, and so  $Y \setminus C$  is residual. It remains to show that  $F$  is lower hemicontinuous at any point in  $Y \setminus C$ . Fix  $y \in Y \setminus C$  and let  $O$  be an open set in  $X$  with  $O \cap F(y) \neq \emptyset$ . Since  $y \notin C$ , for each  $\alpha > 0$  there is a neighborhood  $O_y$  of  $y$  such that  $F(y) \subseteq N_\alpha(F(z))$  for every  $z \in O_y$ . Consequently, for  $\alpha$  sufficiently small, there exists a neighborhood  $O_y$  of  $y$  such that for every  $z \in O_y$  we have  $F(z) \cap O \neq \emptyset$ .  $\square$

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