Measuring Hierarchy with Multiple Supervisors

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Abstract

This paper develops an axiomatic framework for measuring hierarchical structures in modern organizations, where multiple supervisors and non-traditional arrangements, such as matrix structures, are common. Representing hierarchies as directed acyclic graphs (DAGs), we show that structural flattening in these organizations is path-dependent: determining hierarchical dominance generally requires explicitly tracing the sequence of subordination removals. We show that this complexity resolves in multipath-free hierarchies. When each ordered pair of individuals is connected by at most one directed path, the effects of removal order leave a static footprint, allowing hierarchical dominance to be verified by a single geometric condition rather than a step-by-step procedure. Our axiomatic framework is grounded in Anonymity, the Replication Principle, and two versions of the Subordination Removal postulate. The distinction between baseline and Strong Subordination Removal clarifies whether redundant formal reporting lines contribute to measured hierarchical weight or instead leave the effective chain of command unchanged. Using this foundation, we characterize the hierarchical pre-orders consistent with sequential subordination removal, including incomplete comparisons and complete extensions for organizations of varying sizes.

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JEL classifications: D23, L22.
Practitioner point: Axiomatic measurement of complex organizational hierarchies.

1. Introduction

Organizational hierarchies fundamentally shape the distribution of income and economic power. Consequently, systematically measuring these structures is essential for analyzing resource allocation and economic outcomes.

In mainstream organizational economics, standard models view these structures through the lens of technical efficiency. For instance, Garicano (2000) and Caliendo et al. (2015) model hierarchies as problem-solving architectures designed to optimize the allocation of knowledge. In their framework, firm growth requires the nonlinear addition of discrete management layers to handle complexity, which concentrates compensation at the top as a reflection of managerial productivity.

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Conversely, a separate institutional literature views the growth of hierarchies not as an optimization of technical efficiency, but as an apparatus of social power and resource extraction. Fix (2017) demonstrates that societies with higher energy consumption naturally develop larger, deeper institutional hierarchies. By examining this "energy-hierarchy-inequality" relationship, Fix (2019) traces the origins of macro-level inequality, arguing that expanding hierarchical structures serve primarily to concentrate power and resources at the top rather than merely maximize productive efficiency.

Yet, despite their ideological divide on why hierarchies grow, both the efficiency and power-based approaches associate institutional expansion with deeper organizational structures and greater concentration of rewards at higher ranks. Within firms, rank is a primary determinant of earnings, with compensation growing convexly across hierarchical levels (Baker et al., 1994). This internal scaling provides a mechanism for generating the power-law distributions observed among top incomes economy-wide (Fix, 2018, 2021).

From a theoretical perspective, Simon (1957) and Lydall (1959) pioneered models demonstrating how hierarchical structures generate power-law income distributions, which characterize economic inequality. In a stylized framework with a fixed ratio between the number of supervisors in each grade and the number of immediate subordinates, they showed that when two exponential trends—income growth with hierarchical rank and decreasing positions at higher ranks—intersect, a power-law distribution emerges.

As an initial step toward systematically exploring the theoretical connections between hierarchy and the determinants of power and resource distribution, Carbonell-Nicolau (2025) established a foundational framework for measuring hierarchy, specifically tailored to tree-like hierarchical structures. The present paper builds on this methodology, extending the analysis to encompass more complex hierarchical systems. Our work broadens the scope to include hierarchies where individuals may have multiple immediate supervisors, a characteristic prevalent in modern hierarchies such as matrix organizational structures.

Matrix organizations, widely adopted across various industries, including IT, consulting, aerospace, and defense, are characterized by employees reporting to both functional and project managers. According to Gallup (2017), 84% of U.S. employees work in organizations utilizing matrixed arrangements to varying degrees.

We represent hierarchies as *directed acyclic graphs* (DAGs), with nodes as individuals and directed edges as immediate subordination relations. This representation accommodates multiple supervisors while ruling out circular chains of command.

In the economic analysis of the firm, directed acyclic graphs serve as the natural mathematical representation of formal organizational charts, capturing the explicit distribution of authority and decision-making power.

While actual workplace dynamics may also feature cyclical feedback mechanisms or informal communication channels—what Simon (1981) termed "informal organizations," whose hierarchical tendencies have been quantified by Krackhardt (1994) (see also Everett and Krackhardt, 2012)—these fall outside the scope of our analysis. Our measurement approach focuses on the formal subordination structures that dictate institutional resource allocation.

By anchoring our analysis in these versatile acyclic structures, we develop an axiomatic framework for comparing organizational complexity. We evaluate these hierarchies through the lens of fundamental postulates: Anonymity, Subordination Removal, Strong Subordination Removal, and the Replication Principle. Anonymity and the Replication Principle were introduced in Carbonell-Nicolau (2025). Anonymity ensures label-independence, meaning that renaming individuals in a hierarchy leaves its structural properties unchanged. The Replication Principle maintains consistency across scales by asserting that creating "organizational clones"—exact replicas of a hierarchy—preserves its hierarchical character.

The core of our approach rests on the Subordination Removal postulate, which formalizes the intuitive principle that eliminating a reporting line flattens the organization. However, extending this concept to general DAGs introduces substantial complexity. In our framework, removing a formal subordination relation between a supervisor j and a subordinate i eliminates the edge while preserving the broader chain of command: if j has immediate superiors, those immediate superiors are connected directly to i, bypassing j, unless they already possess an alternative path to i that does not traverse j.

This bypass mechanism reveals a challenge unique to directed acyclic graphs: chronological interference. In general DAGs, overlapping matrix paths and redundant authority channels make the outcome of subordination removals depend on the order in which they are performed. The generation of bypass edges is evaluated in the current intermediate hierarchy, not merely in the initial graph. Consequently, determining whether one hierarchy can be reduced to another generally requires explicitly tracing the chronological sequence of subordination removals.

A central contribution of this paper is identifying when this path-dependent complexity resolves into a static topological condition. We demonstrate that by restricting the domain to *multipath-free hierarchies*—where matrix reporting is permitted but multiple directed paths between the same ordered pair of individuals are ruled out—chronological dependencies become statically visible. On this restricted domain, the existence of a valid sequence of subordination removals is equivalent to a clean, one-shot condition based on reachability containment and unique-path consumption.

Furthermore, the transition to matrix structures also motivates a distinction between two versions of Subordination Removal. By distinguishing between Subordination Removal and Strong Subordination Removal, our framework clarifies how different measures treat formal reporting redundancy. Baseline Subordination Removal evaluates hierarchy through the effective, transitive chain of command, allowing the removal of a redundant formal edge to leave hierarchical weight unchanged when the supervisory reachability relation is unaffected. Conversely, Strong Subordination Removal treats every formal reporting line as a contributor to hierarchical weight, demanding that the elimination of any edge flatten the hierarchy.

Using this axiomatic foundation, we characterize the class of hierarchical pre-orders that are consistent with sequential subordination removal. The analysis yields a hierarchy of measures ordered by their degree of completeness. These include an incomplete comparison based on the ordered preservation of supervisory relationships and a complete cardinal index based on the average number of supervisors per individual. Finally, the Replication Principle allows us to generalize our main characterizations from fixed-size organizations to hierarchies of varying sizes.

A substantial body of literature on hierarchy measurement spans multiple disciplines, offering diverse hierarchical indices that rank pairs of hierarchies. These measures range from intuitive indicators—such as the count of organizational layers from top executive to lowest employee or the "span of control" quantifying the average number of direct reports per manager—to more sophisticated statistical approaches that assess node asymmetries, directional relationships, and top-down reachability patterns in hierarchies and broader network structures (see, e.g., Trusina et al., 2004; Luo and Magee, 2011). Among these

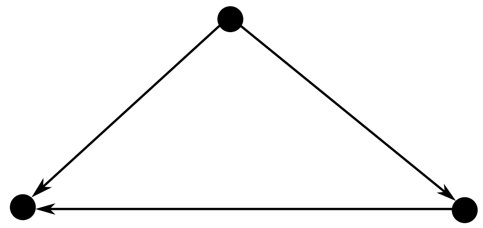

Figure 1: A hierarchy.

advanced measures is the "global reaching centrality" concept (Mones et al., 2012), which evaluates individuals' positional centrality within hierarchies based on their capacity to reach other nodes. As demonstrated in Carbonell-Nicolau (2025), this centrality-based approach diverges fundamentally from the axiomatic framework developed here.

Corominas-Murtra et al. (2013) map directed graphs, not necessarily acyclic, onto a concise set of quantitative dimensions—Treeness, Feedforwardness, and Orderability—each capturing distinct aspects of hierarchical structure. However, unlike our framework, which differentiates among acyclic structures, their methodology assigns identical hierarchical degrees to all acyclic graphs, treating them as equivalent. In contrast, Czégel and Palla (2015) developed an approach with discriminatory power among acyclic graphs by employing a simulation technique. Their method reverses traditional information flow patterns by simulating random walks from employees upward to managers, thereby tracing information to its origins. Within this framework, nodes accumulating numerous walkers occupy superior positions in the hierarchy, while those attracting fewer walkers reside in subordinate positions. Their hierarchy score quantifies the distributional inequality of these walkers—greater concentration at upper echelons signifies a more pronounced hierarchical structure.

Our approach diverges from these algorithmic methods by anchoring hierarchy measurement in precisely articulated theoretical principles. We identify a core incomplete pre-order that captures the unambiguous comparisons implied by the fundamental axioms. This core pre-order serves as the foundation for partial and complete extensions that preserve these essential properties. Greater completeness ranks more pairs of hierarchies, but requires judgments not implied by the basic axioms.

The paper is structured as follows. Section 2 introduces the foundational concept of directed acyclic graphs, providing the necessary background for understanding hierarchical structures. Section 3 establishes the mechanism of sequential subordination removals for fixed-size organizations, analyzes the geometric properties of the multipath-free domain, and presents the axiomatic characterizations of consistent pre-orders. The extension of this framework to hierarchies of varying sizes is detailed in Section 4. Finally, Section 5 concludes the paper, summarizing the main contributions and outlining directions for future research.

2. Hierarchies

We define a *hierarchy* as a finite *directed acyclic graph* (DAG). Thus, a hierarchy h consists of a finite set of nodes $V(h)$ and a set of directed edges $E(h) \subseteq V(h) \times V(h)$, with the property that no directed cycle exists. Acyclicity ensures that following directed edges can never lead from a node back to itself.

In our interpretation, nodes represent individuals, while directed edges represent immediate subordination relations, capturing the formal lines of authority and communication within a firm. An edge $j \to i$ means that individual j is an immediate supervisor of individual i, or equivalently that i is an immediate subordinate of j. More generally, j is a supervisor of i if there exists a directed path of positive length from j to i. We write $j \rightsquigarrow i$ when such a path exists. If this path has length one, then j is an immediate supervisor of i; otherwise, j is an indirect supervisor of i.

For each individual $i \in V(h)$, let S_i denote the set of all supervisors of i, both immediate and indirect. Equivalently, S_i is the set of ancestors of i in h. Acyclicity implies a useful hierarchical constraint: if individual i has k supervisors, then every supervisor of i has strictly fewer than k supervisors.

Indeed, suppose that $j \in S_i$. Since the supervisory relation is transitive, every supervisor of j is also a supervisor of i, so $S_j \subseteq S_i$. Moreover, $j \in S_i$ but $j \notin S_j$, because acyclicity rules out self-supervision. Hence $S_j \subsetneq S_i$, and therefore

$$|S_j| < |S_i| = k.$$

Figure 1 illustrates a simple hierarchy comprising three individuals connected by three subordination relations.

This directed-acyclic-graph formulation is flexible enough to represent complex reporting structures in which an individual may have multiple immediate supervisors. It therefore accommodates matrix organizational structures and other non-traditional hierarchical arrangements while preserving the acyclicity of formal authority relations.

This approach differs from the hierarchy definition presented in Carbonell-Nicolau (2025), which constrained each individual to have at most one immediate supervisor and explicitly partitioned nodes into distinct ranks or levels. The present conceptualization is more general: it relies only on subordination relationships between individuals and does not impose predefined rank structures. Nevertheless, rank-based analysis remains compatible with the framework and can be incorporated when formal hierarchical levels are available.

For each individual i in a hierarchy h, we define the *sub-hierarchy* $h(i)$ as the subgraph of h induced by i and all of its direct and indirect subordinates. That is, $h(i)$ contains i, every node reachable from i by a directed path, and all directed edges among these nodes that are present in h.

This sub-hierarchy preserves the supervisory relationships below i and constitutes a hierarchy in its own right. It represents individual i's domain of authority.

Notationally, a hierarchy comprising $m \in \{1, 2, \ldots\}$ independent hierarchies $h_1, \ldots, h_m$ is denoted by $(h_1, \ldots, h_m)$. Here, *independence* means that the hierarchies are pairwise disconnected: for any $r \neq s$, no directed edge exists between any node in h_r and any node in h_s.

Finally, we define the *size* of a hierarchy h as the cardinality of its node set, $|V(h)|$; that is, the total number of individuals in the hierarchy.

3. Hierarchical dominance for fixed-size organizations

We initially restrict our analysis to hierarchies of a fixed size n, deferring the case of variable-sized hierarchies to a subsequent section.

For a hierarchy h, let $V(h)$ and $E(h)$ denote its node set and edge set, respectively. Let $\mathscr{H}_n$ denote the set of all hierarchies h with $|V(h)| = n$.

A hierarchical pre-order $\succcurlyeq$ is defined as a reflexive and transitive binary relation on the set $\mathscr{H}_n$. These pre-orders represent potentially incomplete comparative assessments between pairs of hierarchies, where the notation $h \succcurlyeq h'$ carries the interpretation that "hierarchy h exhibits at least as much hierarchical structure as hierarchy h'."

Definition 1. A *hierarchical pre-order* on $\mathscr{H}_n$ is a reflexive and transitive binary relation on $\mathscr{H}_n$.

The symmetric and asymmetric parts of a hierarchical pre-order $\succcurlyeq$ are denoted by $\sim$ and $\succ$, respectively.

The following property of a hierarchy, called Anonymity, was introduced in Carbonell-Nicolau (2025).

A hierarchy h' is said to be a *relabeling* of another hierarchy h if h' is obtained from h by relabeling the individuals in h. Formally, there is a bijection $\phi : V(h) \to V(h')$ preserving immediate subordination relations: for every pair of nodes $i, j \in V(h)$, j is an immediate supervisor of i in h if and only if $\phi(j)$ is an immediate supervisor of $\phi(i)$ in h'.

Anonymity (A). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies A if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \sim h'$ whenever h' is a relabeling of h.

This axiom emphasizes that the essential structure of a hierarchy remains invariant under relabeling and is independent of the specific identities of the employees assigned to the hierarchy's nodes.

The concept of subordination relation removal, originally introduced by Carbonell-Nicolau (2025), is extended in this work to align with our more flexible definition of a hierarchy.

Definition 2. We say that hierarchy h' is obtained from hierarchy h by *removing a subordination relation* if there exist a subordinate $i \in V(h)$ and an immediate supervisor j of i in h such that h' is obtained as follows:

- First delete the edge $j \to i$.
- If j has no immediate supervisors in h, no bypass edge is added.
- If j has at least one immediate supervisor in h, then for each immediate supervisor p of j in h, add the edge $p \to i$ unless there already exists in h a directed path from p to i that does not traverse j, i.e., a path on which j is not a node.
- All other edges are preserved.

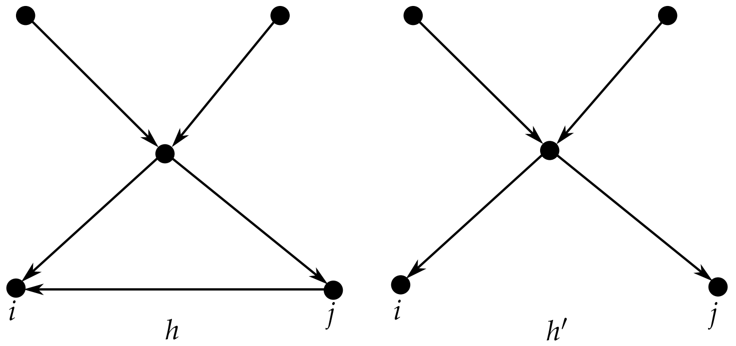

Figure 2: Subordination removal without a bypass edge.

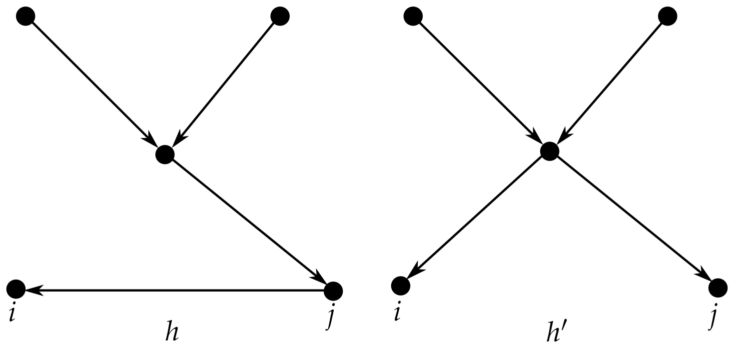

Figure 3: Subordination removal with a bypass edge.

This definition formalizes subordination removal in two cases. When the supervisor j has no immediate supervisors, the link $j \to i$ is simply deleted. When j has immediate supervisors, each immediate supervisor p of j is connected directly to i, bypassing j, unless p already has an alternative route to i that does not pass through j. Thus, a bypass is added precisely when p's authority over i would otherwise be mediated only through j. Each bypass $p \to i$ short-circuits a pre-existing path $p \to j \to i$, so the operation preserves acyclicity. Hence the resulting graph is again a hierarchy.

To illustrate this definition, consider the hierarchies from Figure 2, where individual i has multiple direct supervisors, including j. We focus on the direct subordination relation $j \to i$. In this instance, h' is obtained from h by removing $j \to i$. No bypass edge is added for any immediate supervisor of j that already has a path to i not traversing j.

Figure 3 presents a case where h' is also obtained from h by removing $j \to i$, but now a bypass edge must be added. After deleting $j \to i$, an immediate supervisor p of j would otherwise have no path to i except through j. Hence the edge $p \to i$ is added.

In Figure 4, we remove the subordination relation $j' \to i$ by deleting the edge from j' to i. In this case, no bypass edge is added from the supervisors of j' to i, since those supervisors already have alternative paths to i that do not traverse j'.

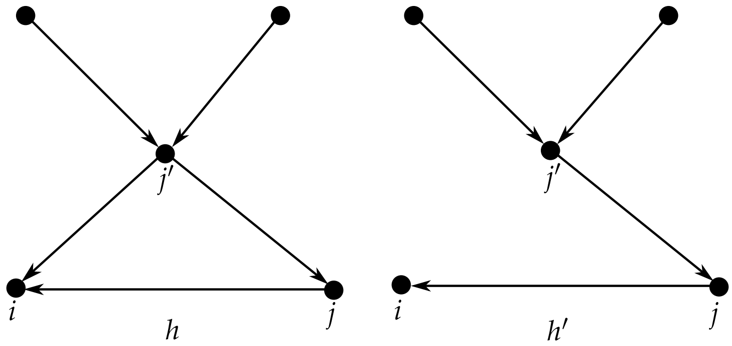

Figure 4: Subordination removal when alternative paths already exist.

The previous figures illustrate the three relevant possibilities under Definition 2: simple deletion when no bypass is required, deletion with a new bypass edge, and deletion without bypasses because independent alternative paths already exist.

An edge $j \to i$ is *redundant* if there exists a directed path from j to i distinct from $j \to i$ itself. A redundant edge represents a dual line of authority.

We maintain the principle, originally introduced by Carbonell-Nicolau (2025), that the removal of a subordination relation results in a less hierarchical structure.

Subordination Removal (SR). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies SR if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \succcurlyeq h'$ whenever h' is obtained from h by removing a subordination relation. Moreover, $h \succ h'$ whenever the removed edge is non-redundant.

Thus, Subordination Removal requires weak dominance for every removal, but strict dominance whenever a removal alters the transitive supervisory relation. Weak dominance accounts for the fact that deleting a redundant formal edge may leave the underlying reachability intact. Even when such a deletion generates bypass edges to preserve upstream authority (Definition 2), no new reachability relations are created, meaning the effective chain of command remains unchanged.

In some contexts, it is natural to view *any* formal reporting line—even a redundant one—as contributing to the hierarchical weight of an organization. For instance, in matrix structures, redundant edges may represent dual reporting obligations, additional oversight, or bureaucratic rigidity. From this perspective, eliminating a reporting line should always strictly flatten the organizational structure, regardless of whether an alternative indirect chain of command remains intact. This motivates a stronger version of the Subordination Removal postulate.

Strong Subordination Removal (SSR). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ satisfies SSR if for any two hierarchies h and h' in $\mathscr{H}_n$, $h \succ h'$ whenever h' is obtained from h by removing a subordination relation.

3.1. Sequential subordination removal and reduction traces

A central objective of our axiomatic framework is to identify a structural property of hierarchies that captures sequential subordination removals. In the tree-based domain of

Carbonell-Nicolau (2025), this was possible through a clean, one-shot geometric condition based on supervisory ranks and path containment.

For general DAGs, chronological interference prevents the type of simple one-shot characterization available on the tree domain. In a general DAG, matrix reporting allows for multiple paths between nodes. Consequently, removing one reporting line may alter the path structure relevant for determining whether a later removal generates a bypass edge. Because bypass generation is evaluated in the *current* intermediate graph, the transformation is path-dependent: the order of removals may determine the final topology.

We therefore define dominance through explicit *sequences of subordination removals*. For formal purposes, we record these sequences explicitly using a *reduction trace*—a step-by-step record of the removed edges together with the corresponding intermediate edge sets. The trace is not a new conceptual primitive; it is a bookkeeping device that makes chronological dependence explicit and will be useful below when identifying the boundary between general DAG complexity and the multipath-free case.

The formalization begins by aligning the node sets of the two hierarchies being compared. Since individuals in h and h' may be labeled differently, we use a bijection ϕ to translate the target graph h' back onto the nodes of the original graph h. We denote this "pulled-back" target edge set by E^*_ϕ.

To verify that h' is obtainable from h by sequential subordination removals, one records the removed edges together with the intermediate topological "snapshots" of the organization.

Definition 3 (reduction trace)**.** Let $h, h' \in \mathscr{H}_n$, and let $\phi : V(h) \to V(h')$ be a bijection. Define the target edge set pulled back to h by

$$E^*_\phi = \{(u, v) : (\phi(u), \phi(v)) \in E(h')\}.$$

A *reduction trace* from h to h' via ϕ consists of a finite list of edges $e_1, \dots, e_L$ and a finite list of edge sets $E^0, E^1, \dots, E^L$ such that:

(i) $E^0 = E(h)$;

(ii) for each $\ell \in \{1, \dots, L\}$, the edge $e_\ell = j_\ell \to i_\ell$ belongs to $E^{\ell-1}$;

(iii) E^ℓ is obtained from $E^{\ell-1}$ by deleting e_ℓ and adding exactly the bypass edges prescribed by Definition 2 applied to the intermediate graph $(V(h), E^{\ell-1})$;

(iv) $E^L = E^*_\phi$.

Thus, a reduction trace is simply a precise record of a finite sequence of subordination removals after relabeling. Condition (i) fixes the starting hierarchy. Condition (ii) guarantees that an edge can only be removed if it is present in the current intermediate graph. Condition (iii) enforces that every step obeys Definition 2 relative to the *current* intermediate graph, thereby correctly tracking chronology-dependent bypass generation. Finally, condition (iv) requires that the resulting edge set coincide exactly with the pulled-back target hierarchy.

We now define the hierarchical dominance relation for general DAGs.

Definition 4 (DAG hierarchical dominance)**.** For any two hierarchies $h, h' \in \mathscr{H}_n$, we write $h \succcurlyeq_H h'$ if and only if there exists a bijection $\phi : V(h) \to V(h')$ and a reduction trace from h to h' via ϕ.

Equivalently, $h \succcurlyeq_H h'$ holds if and only if, after pulling back labels through some bijection ϕ, the target hierarchy can be obtained from h by a finite sequence of DAG subordination removals.

As is standard for pre-orders, let $\succ_H$ and $\sim_H$ denote the asymmetric and symmetric parts of $\succcurlyeq_H$, respectively.

Two hierarchies h and h' are considered isomorphic, denoted $h \cong h'$, if there exists a bijection $\phi : V(h) \to V(h')$ such that for all individuals $i, j \in V(h)$,

$$j \to i \in E(h) \Leftrightarrow \phi(j) \to \phi(i) \in E(h').$$

We begin by verifying that the trace-based relation $\succcurlyeq_H$ is itself a well-defined hierarchical pre-order and that it respects relabeling.

Lemma 1. *The binary relation $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive, and satisfies A.*

The proof of this lemma is given in Appendix A.1. The following lemma establishes that a single edge removal strictly decreases the hierarchy under $\succ_H$, even when the operation triggers the addition of bypass edges (Definition 2).

Lemma 2. *For $h, h' \in \mathscr{H}_n$, if h' is obtained from h by a single subordination removal in the sense of Definition 2, then $h \succ_H h'$.*

Lemma 2 gives the one-step strictness implication. Its proof is relegated to Appendix A.3.

While Lemma 2 ensures that an isolated subordination removal flattens the hierarchy, actual organizational reforms typically involve multiple reporting line changes. To evaluate these broader structural transformations, the global dominance relation $\succ_H$ must order hierarchies whenever one is obtainable from another through a finite sequence of removals.

To show that this multi-step global relation is well-behaved, we must ensure that sequential flattening is strictly unidirectional and cannot result in cyclic comparisons. We establish this by showing that each removal step permanently reduces the number of supervisors overseeing the active managers.

For any hierarchy g, let $s_g(x)$ denote the depth of node x, representing the number of supervisors above individual x. Define a multiset, $M_{\mathrm{src}}(g)$, that lists the depth of the supervisor for every active reporting line in the hierarchy:

$$M_{\mathrm{src}}(g) = \{\, s_g(x) : (x, y) \in E(g) \,\}.$$

To illustrate, consider an organization where individual a (depth 0) directly supervises b and c, while b (depth 1) also supervises c. The reporting lines are (a, b), (a, c), and (b, c). The multiset collects the depth of the supervisor for each of these three lines, yielding $M_{\mathrm{src}}(g) = \{0, 0, 1\}$.

We compare these multisets using the strict multiset ordering, denoted by $<$. A feature of this ordering is that a multiset strictly decreases when a single element is removed, even if it is replaced by *multiple* strictly smaller elements. This property mirrors the bypass mechanism: eliminating a single reporting line removes one supervisor's depth from the tally, but it may trigger the creation of *several* new bypass edges. Because these new bypass edges originate from higher-level managers (Definition 2), every newly added depth is strictly smaller than the depth that was removed.

The following lemma establishes that every step of sequential subordination removal strictly decreases this multiset of supervisor depths.

Lemma 3. *If there is a nonempty reduction trace from h to h', then $M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h)$.*

The proof is deferred to Appendix A.2.

We can now characterize strict global dominance ($\succ_H$) through sequential subordination removals.

Proposition 1. *For hierarchies $h, h' \in \mathscr{H}_n$, $h \succ_H h'$ if and only if h' can be obtained from some relabeling of h by successive DAG subordination removals of one or more subordination relations.*

Proof. Sufficiency. Suppose that h' is obtained from a relabeling $\widehat{h}$ of h by a sequence of one or more subordination removals. Let $\rho : V(h) \to V(\widehat{h})$ be the relabeling bijection.

Transporting each removed edge and each intermediate edge set in this sequence back along ρ^{-1}, and using the fact that Definition 2 commutes with relabeling, yields a reduction trace from h to h' via the corresponding target-labeling bijection. Hence $h \succcurlyeq_H h'$.

Because the sequence contains one or more removals, the corresponding trace is nonempty. By Lemma 3, $M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h)$.

We claim that $h' \not\succcurlyeq_H h$. Suppose, toward a contradiction, that $h' \succcurlyeq_H h$.

If the trace from h' to h is nonempty, then Lemma 3 gives $M_{\mathrm{src}}(h) < M_{\mathrm{src}}(h')$, contradicting the asymmetry of the strict multiset order.

If the trace from h' to h is empty, then h' is a relabeling of h. Relabeling preserves ancestor relations and hence source depths, so $M_{\mathrm{src}}(h') = M_{\mathrm{src}}(h)$, again contradicting $M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h)$.

Therefore $h' \not\succcurlyeq_H h$. Since $h \succcurlyeq_H h'$ and $h' \not\succcurlyeq_H h$, we conclude that $h \succ_H h'$.

Necessity. Suppose $h \succ_H h'$. Then $h \succcurlyeq_H h'$. By Definition 4, there exists a reduction trace from h to h' via some bijection $\phi : V(h) \to V(h')$. The explicit list $e_1, \ldots, e_L$ and transitions $E^{\ell-1} \to E^{\ell}$ realize a valid finite sequence of subordination removals from h to the hierarchy $\phi^{-1}(h')$ defined on the node set $V(h)$. Because Definition 2 commutes with relabeling, applying ϕ to every intermediate hierarchy in this sequence yields a valid sequence of subordination removals from the relabeling $\phi(h)$ of h exactly to h'.

It remains to show that $L > 0$. If $L = 0$, then

$$E(h) = \phi^{-1}(E(h')),$$

so h' is a relabeling of h. Hence the inverse relabeling gives an empty trace from h' to h, implying $h' \succcurlyeq_H h$. This contradicts $h \succ_H h'$, the asymmetric part of $\succcurlyeq_H$. Therefore $L > 0$.

Hence h' is obtained from a relabeling of h by successive removals of one or more subordination relations. ∎

3.1.1. The limits of static topological characterizations

Because the reduction trace in Definition 3 relies on a chronological sequence of intermediate subgraphs, it diverges from the purely static, one-shot geometric characterization of hierarchical dominance established for trees in Carbonell-Nicolau (2025). To understand why this sequential formalization is necessary for general DAGs, it is useful to first examine a single, isolated subordination removal.

When restricted to exactly one operation, the subordination removal from Definition 2 can be perfectly captured by a static topological condition. This one-step formulation mirrors the geometric aesthetic of the H-dominance condition in Carbonell-Nicolau (2025), adapted to accommodate multiple intersecting reporting lines.

Definition 5 (one-step reduction)**.** Let $h, h' \in \mathscr{H}_n$. We say that h' is a *one-step reduction* of h if there exists a bijection $\phi : V(h) \to V(h')$ and an immediate subordination relation $j \to i \in E(h)$ satisfying the following conditions:

(i) For each individual $x \in V(h) \setminus \{i\}$, an individual y' is an immediate supervisor of $\phi(x)$ in h' if and only if $\phi^{-1}(y')$ is an immediate supervisor of x in h.

(ii) For the individual i, an individual y' is an immediate supervisor of $\phi(i)$ in h' if and only if $\phi^{-1}(y')$ is either:

   (a) an immediate supervisor of i in h distinct from j, or

   (b) an immediate supervisor of j in h such that every directed path from $\phi^{-1}(y')$ to i in h traverses j.

This definition evaluates the removal of a reporting line by statically comparing the "before" (h) and "after" (h') hierarchies. The mapping ϕ^{-1} simply aligns the identities of employees across the two structures.

The two conditions ensure the restructuring is strictly local:

Condition (i) guarantees that for every employee other than the target subordinate, i, reporting lines remain unchanged.

Condition (ii) determines i's new supervisors:

- Employee i keeps all prior direct supervisors except j.
- An immediate supervisor of j gains direct authority over i if and only if their *only* path to i previously went through j. If they already had an independent path to i, no bypass is created.

Proposition 2. *For hierarchies $h, h' \in \mathscr{H}_n$, h' is a one-step reduction of h if and only if h' is a relabeling of a hierarchy obtained from h by a single subordination removal in the sense of Definition 2.*

Proof. Necessity. Suppose h' is a one-step reduction of h via the bijection ϕ, witnessed by the edge $j \to i \in E(h)$. Let $\widehat{h}$ be the hierarchy obtained from h by applying Definition 2 to remove $j \to i$. We show that ϕ is an isomorphism from $\widehat{h}$ to h'.

By Definition 2, the only possible changes from h to $\widehat{h}$ are among edges entering i: the edge $j \to i$ is deleted, all other edges whose target is not i are preserved, and bypass edges $p \to i$ are added exactly for immediate supervisors p of j such that there is no directed path from p to i in h avoiding j.

Condition (i) in Definition 5 states precisely that, for every node $x \neq i$, the immediate supervisors of $\phi(x)$ in h' are exactly the ϕ-images of the immediate supervisors of x in h. Hence all edges not entering i are preserved under ϕ.

For the target node i, Condition (ii)(a) says that every immediate supervisor of i in h other than j remains an immediate supervisor of $\phi(i)$ in h'. The deleted edge $j \to i$ is absent from h', since j is neither an immediate supervisor of i distinct from j, nor an immediate supervisor of j, by acyclicity.

Finally, Condition (ii)(b) says that an edge $\phi(p) \to \phi(i)$ is present in h' exactly when p is an immediate supervisor of j in h and every directed path from p to i in h traverses j. This is equivalent to saying that no directed path from p to i avoids j, which is exactly the bypass condition in Definition 2. Therefore the incoming edges of $\phi(i)$ in h' are exactly the ϕ-images of the incoming edges of i in $\widehat{h}$.

Thus ϕ preserves and reflects every edge of $\widehat{h}$, so h' is a relabeling of $\widehat{h}$. Hence h' is a relabeling of a hierarchy obtained from h by a single subordination removal.

Sufficiency. Suppose h' is a relabeling of a hierarchy $\widehat{h}$ obtained from h by a single subordination removal of the edge $j \to i$. Since $\widehat{h}$ is obtained from h by modifying only the edge set, we identify $V(\widehat{h})$ with $V(h)$. Let $\phi : V(h) \to V(h')$ be the relabeling bijection from $\widehat{h}$ to h'.

By Definition 2, no edges are modified except edges entering i: the edge $j \to i$ is deleted, all pre-existing edges $x \to i$ with $x \neq j$ are preserved, and bypass edges $p \to i$ are added exactly for immediate supervisors p of j such that every directed path from p to i in h traverses j.

Therefore, for every $x \neq i$, the immediate supervisors of $\phi(x)$ in h' are exactly the ϕ-images of the immediate supervisors of x in h, which gives Condition (i). For i, the immediate supervisors of $\phi(i)$ in h' are exactly the ϕ-images of nodes satisfying either Condition (ii)(a) or Condition (ii)(b). Hence h' is a one-step reduction of h. ■

Non-commutativity and path redundancy. Sequential subordination removal is a finite composition of the one-step reduction characterized in Definition 5. A useful static characterization would determine directly from h and h' whether some valid sequence of removals transforms h into h', without examining the intermediate hierarchies. Two features complicate such a characterization on the full domain of directed acyclic graphs. First, subordination removals need not commute. Second, overlapping paths make bypass generation depend on the current intermediate hierarchy. These problems are distinct but reinforcing.

To see that non-commutativity arises even in the absence of path redundancy, consider the four-person chain

$$a \to b \to c \to d.$$

Suppose that the two original reporting lines $b \to c$ and $c \to d$ are to be removed.

If $b \to c$ is removed first, the deletion generates the bypass $a \to c$, producing the intermediate hierarchy

$$\{a \to b,\ a \to c,\ c \to d\}.$$

Removing $c \to d$ next generates the bypass $a \to d$, because a is then the immediate supervisor of c. The resulting hierarchy h' has edge set

$$E(h') = \{a \to b,\ a \to c,\ a \to d\}.$$

If the order is reversed, removing $c \to d$ first generates the bypass $b \to d$, producing the intermediate hierarchy

$$\{a \to b,\ b \to c,\ b \to d\}.$$

Removing $b \to c$ next generates the bypass $a \to c$. The resulting hierarchy h'' has edge set

$$E(h'') = \{a \to b,\ a \to c,\ b \to d\}.$$

Thus, $h' \neq h''$, even though the same two original reporting lines are removed in both sequences. The initial hierarchy is multipath-free, so this order dependence does not arise from overlapping paths. It arises because an earlier removal changes which supervisors are available to generate later bypasses. In this sense, removing $b \to c$ first "burns the bridge" from b to c before the removal of $c \to d$ can promote b to a direct supervisor of d.

Path redundancy introduces an additional source of chronological interference by activating or suppressing bypasses. Consider a matrix hierarchy with edge set

$$\{a \to b,\ a \to x,\ b \to c,\ x \to c,\ c \to d\}.$$

Thus, c has two immediate supervisors, b and x, both of whom report to a, and c supervises d. Suppose that the two original edges $b \to c$ and $c \to d$ are to be removed.

- If $b \to c$ is removed first, no bypass $a \to c$ is generated, because the alternative path $a \to x \to c$ avoids b. After this removal, x is the sole immediate supervisor of c. Consequently, removing $c \to d$ generates the bypass $x \to d$.

- If $c \to d$ is removed first, both immediate supervisors of c, namely b and x, receive bypasses to d. Removing $b \to c$ afterward does not generate $a \to c$, because the path $a \to x \to c$ still avoids b.

The two resulting hierarchies differ by the edge $b \to d$. In the first sequence, removing $b \to c$ prevents b from receiving a bypass to d when $c \to d$ is subsequently removed. In the second sequence, the bypass $b \to d$ is generated before b's connection to c is severed and therefore survives in the final hierarchy. Hence, the bypasses generated by a removal cannot, in general, be determined from the initial graph and the set of original edges selected for removal alone. They depend on which reporting paths remain when the removal is performed.

These examples do not prove that a static characterization is impossible. One could always formulate a condition that searches over all possible intermediate hierarchies, but this would merely restate the reduction-trace definition. The relevant question is whether there is a simple one-shot condition that avoids such a search. In general DAGs, any exact condition must account for the order in which paths disappear and bypasses arise.

3.2. Static reduction in multipath-free DAGs

The preceding examples identify two features of chronological interference: the order-dependence of sequential removals and the ambiguity introduced by overlapping alternative paths. It is tempting to assume that ruling out multiple paths restores commutativity. This is not the case. As the $a \to b \to c \to d$ example shows, removals fail to commute even in a simple multipath-free chain: removing $b \to c$ before $c \to d$ yields a different target hierarchy than removing $c \to d$ before $b \to c$.

However, path uniqueness facilitates the verification of transitions between initial and target hierarchies.

Definition 6 (multipath-free hierarchy)**.** A hierarchy $h \in \mathscr{H}_n$ is *multipath-free* if, for any two individuals $u, v \in V(h)$, there exists at most one directed path from u to v.

In a multipath-free DAG, lateral matrix reporting is permitted, but for any ordered pair of individuals there is at most one chain of command connecting the first to the second. Consequently, chronological dependencies are not obscured by competing paths. Their effects can be detected by comparing the unique paths in the initial hierarchy with the edges in the target hierarchy, without reconstructing the sequence of removals.

To build intuition for how these chronological dependencies become visible in the target graph, consider again the four-person chain h consisting of $a \to b \to c \to d$. Recall that removing the edges $\{b \to c, c \to d\}$ yielded two distinct target hierarchies depending on the sequence of operations:

- $h' = \{a \to b, a \to c, a \to d\}$, when $b \to c$ was removed first.
- $h'' = \{a \to b, a \to c, b \to d\}$, when $c \to d$ was removed first.

In a general DAG, alternative paths make this comparison more difficult. A bypass may be absent because an earlier removal eliminated the path needed to generate it, or because an alternative path suppressed its generation. Thus, the initial and target hierarchies may not yield a simple test for whether some valid sequence of removals connects them.

The situation is different when paths are unique. Because h is multipath-free, the path

$$a \to b \to c \to d$$

is the only path from a to d. In h'', the edge $b \to d$ shows that $c \to d$ was removed while b still directly supervised c. By contrast, generating the bypass $a \to d$ requires $b \to c$ to be removed first, so that a directly supervises c when $c \to d$ is removed, as in h'. Thus, a valid target cannot contain both $b \to d$ and $a \to d$.

This example suggests that, when paths are unique, the initial and target hierarchies may contain enough information to recognize valid reductions.

Organizational restructuring may involve any finite sequence of removals. To apply this unique-path reasoning throughout such a sequence, we must ensure that every intermediate hierarchy remains multipath-free. The following lemma establishes that the multipath-free domain is closed under subordination removal.

Lemma 4 (closure of multipath-free DAGs)**.** *If $h \in \mathscr{H}_n$ is multipath-free and h^+ is obtained from h by one subordination removal, then h^+ is multipath-free.*

Proof. Suppose h^+ is obtained from h by removing $j \to i$. Any bypass added by Definition 2 has the form $p \to i$, where $p \to j \in E(h)$. Since $p \to j \to i$ is already a path from p to i in h, the multipath-free property implies that no path from p to i avoiding j exists. Thus any added bypass replaces the unique path segment $p \to j \to i$.

Acyclicity is preserved: if a bypass $p \to i$ created a directed cycle in h^+, then h^+ would contain a path $i \rightsquigarrow p$. Replacing each bypass on that path by its original two-edge segment produces a path from i to p in h, which together with $p \to j \to i$ would form a cycle in h. Hence h^+ is acyclic.

Now suppose h^+ contains two distinct directed paths from some u to some v. Because every bypass edge added during the removal of $j \to i$ shares the same target node i, and because h^+ is acyclic, any directed path in h^+ can contain at most one bypass edge. Indeed, once a path enters i via one bypass edge, any second bypass edge would require the path to leave i and later return to i, creating a directed cycle through i.

If neither path contains a bypass, then both are distinct directed paths in h, contradicting multipath-freeness. Suppose instead that one or both paths contain a bypass $p \to i$. Expand each such bypass into the segment $p \to j \to i$. This expansion does not repeat any vertices. Indeed, the node j cannot appear earlier on the path, since the subpath $j \rightsquigarrow p$ together with $p \to j$ would form a cycle in h. Similarly, j cannot appear later on the path, since the subpath $i \rightsquigarrow j$ together with $j \to i$ would form a cycle in h. Therefore each expanded path is a simple directed path in h.

Expansion is injective on directed paths. Indeed, every bypass edge in h^+ has the unique form $p \to i$, and its expansion replaces that edge by the two-edge segment $p \to j \to i$, while no ordinary edge is altered. Hence the expanded path uniquely determines the original path, obtained by replacing each occurrence of $p \to j \to i$ with the corresponding bypass edge $p \to i$. Therefore two distinct paths in h^+ cannot expand to the same directed path in h.

Thus h would contain two distinct directed paths from u to v, contradicting multipath-freeness. Therefore h^+ is multipath-free. ■

The multipath-free restriction allows us to recognize sequential subordination removal directly from the initial and target hierarchies, without examining the intermediate steps.

Definition 7 (static reduction)**.** Let $h, h' \in \mathscr{H}_n$ be multipath-free hierarchies. We say that h' is a *static reduction* of h if there exists a bijection $\phi : V(h) \to V(h')$ such that:

(i) **Reachability Containment.** For all $u', v' \in V(h')$, if $u' \rightsquigarrow v'$ in h', then

$$\phi^{-1}(u') \rightsquigarrow \phi^{-1}(v')$$

in h.

(ii) **Unique-Path Consumption.** For every edge $(u', v') \in E(h')$, let

$$u = \phi^{-1}(u'), \qquad v = \phi^{-1}(v').$$

If $(u, v) \notin E(h)$, then Reachability Containment implies that $u \rightsquigarrow v$ in h. Since h is multipath-free, this path is unique. Moreover, because $(u, v) \notin E(h)$, it has length at least two:

$$u = d_0 \to d_1 \to \cdots \to d_m = v, \qquad m \geq 2.$$

For every internal node d_r, $1 \leq r \leq m - 1$,

$$(\phi(d_r), v') \notin E(h').$$

The definition compares the initial and target hierarchies without reconstructing the sequence of removals. Reachability Containment requires every supervisory relation in h' to have already existed, directly or indirectly, in h. Thus, removals may shorten or eliminate chains of command, but they cannot create authority between previously unrelated individuals.

Unique-Path Consumption governs the new edges in h'. Suppose $u' \to v'$ is not the image of an edge in h. It must then represent a bypass of the unique path

$$u = d_0 \to d_1 \to \cdots \to d_m = v$$

in h. The condition requires that none of the images $\phi(d_1), \ldots, \phi(d_{m-1})$ be a direct supervisor of v' in the target hierarchy.

The condition is easy to apply. Consider again the initial hierarchy

$$h = \{a \to b,\ b \to c,\ c \to d\}$$

and the target

$$h' = \{a \to b,\ a \to c,\ a \to d\}.$$

Reachability Containment holds because every relation in h' already holds in h: a reaches b, c, and d in the initial chain. For Unique-Path Consumption, only the new edges need to be checked. The edge $a \to c$ bypasses the unique path

$$a \to b \to c,$$

and the internal node b is not a direct supervisor of c in h'. Likewise, $a \to d$ bypasses

$$a \to b \to c \to d,$$

and neither internal node b nor c is a direct supervisor of d in h'. Hence h' is a static reduction of h.

The second possible target,

$$h'' = \{a \to b,\ a \to c,\ b \to d\},$$

also satisfies the two conditions. The new edge $a \to c$ consumes the internal node b on the path $a \to b \to c$, while $b \to d$ consumes c on the path $b \to c \to d$. Thus, the static condition recognizes both valid targets without searching for the removal orders that generate them.

By contrast, consider a target containing both $a \to d$ and $b \to d$. Although both target edges preserve reachability from h, the target violates Unique-Path Consumption. Because $a \to d$ bypasses the unique path

$$a \to b \to c \to d,$$

the internal node b cannot remain a direct supervisor of d. Reachability Containment alone would not detect this problem; Unique-Path Consumption rules it out.

The two conditions capture complementary restrictions. Reachability Containment rules out new authority relations, while Unique-Path Consumption rules out incompatible bypasses along the original unique paths. The next theorem shows that together they exactly characterize the targets obtainable by sequential subordination removal.

Theorem 1. *Let $h, h' \in \mathscr{H}_n$ be multipath-free hierarchies. Then $h \succcurlyeq_H h'$ if and only if h' is a static reduction of h.*

Proof. Necessity. Suppose $h \succcurlyeq_H h'$. Then there exists a reduction trace from h to h' via some bijection $\phi : V(h) \to V(h')$. Let $\bar{h}'$ denote the pullback of h' to the node set $V(h)$, so that

$$E(\bar{h}') = E^*_\phi = \{(u, v) : (\phi(u), \phi(v)) \in E(h')\}.$$

By Lemma 4, every intermediate graph appearing in the trace is multipath-free.

We prove by induction on the trace length L that $\bar{h}'$ is a static reduction of h via the identity map on $V(h)$; equivalently, h' is a static reduction of h via ϕ.

If $L = 0$, then $E(h) = E(\bar{h}')$, so h' is a relabeling of h, and both conditions of Definition 7 hold immediately.

Assume the claim holds for traces of length $L-1$. Let g be the penultimate intermediate hierarchy, and suppose the final transition removes the edge $j \to i \in E(g)$, producing $\bar{h}'$. By the induction hypothesis, g is a static reduction of h. The final step deletes $j \to i$ and adds only bypasses $p \to i$, where $p \to j \in E(g)$ and there is no path from p to i in g avoiding j.

Reachability Containment is preserved. Indeed, every newly added bypass $p \to i$ replaces the pre-existing path

$$p \to j \to i$$

in g. Therefore every path in $\bar{h}'$ can be expanded into a path in g. Since g is, by the induction hypothesis, reachability-contained in h, every reachability relation in $\bar{h}'$ is also present in h.

It remains to verify Unique-Path Consumption. Let $(u, v) \in E(\bar{h}')$ be an edge such that $(u, v) \notin E(h)$. Since h is multipath-free, there is a unique path

$$u = d_0 \to d_1 \to \cdots \to d_m = v, \qquad m \geq 2,$$

from u to v in h. We must show that no internal node d_r, $1 \leq r \leq m-1$, is an immediate supervisor of v in $\bar{h}'$.

There are two cases.

First, suppose that $(u, v) \in E(g)$, so the edge was already present before the final removal. By the induction hypothesis, no internal node on the unique u-to-v path in h is an immediate supervisor of v in g. The final transition only adds new incoming edges to the node i. Hence, if $v \neq i$, no new immediate supervisor of v is created, and the desired condition remains true.

Now suppose $v = i$. If some internal node d_r became an immediate supervisor of i in $\bar{h}'$ during the final transition, then $d_r \to i$ must be one of the newly added bypasses. Hence d_r is an immediate supervisor of j in g. Since d_r lies on the unique path from u to i in h, and $d_r \to j$ in g, Reachability Containment for g implies that $d_r \rightsquigarrow j \rightsquigarrow i$ is represented along the unique u-to-i path in h. Thus j is also an internal node on the unique path from u to i in h. But $j \to i \in E(g)$, so j was an immediate supervisor of i in g, contradicting the induction hypothesis applied to the edge $(u, i) \in E(g)$. Therefore no internal node of the unique u-to-i path in h becomes an immediate supervisor of i in $\bar{h}'$.

Second, suppose that $(u, v) \notin E(g)$. Then (u, v) must be newly created in the final transition. Hence $v = i$, $u = p$ for some immediate supervisor p of j in g, and the new edge $p \to i$ replaces the segment

$$p \to j \to i$$

in g. If $p \to j \notin E(h)$, the induction hypothesis applied to the edge $p \to j$ in g implies that no internal node on the unique path from p to j in h is an immediate supervisor of j in g; if $p \to j \in E(h)$, there are no such internal nodes. By the same reasoning applied to the edge $j \to i$ in g, no internal node on the unique path from j to i in h is an immediate supervisor of i in g.

In the final transition, the node j is explicitly removed as an immediate supervisor of i. The only new incoming edges to i originate from immediate supervisors of j in g. Since no internal node on the unique path from p to j in h is an immediate supervisor of j in g, none of those internal nodes is promoted to an immediate supervisor of i. Similarly, no internal node on the unique path from j to i in h was an immediate supervisor of i in g, and the final step does not add incoming edges from such nodes. Finally, j itself is not an immediate supervisor of i in $\bar{h}'$, since $j \to i$ is the edge removed.

Thus no internal node on the combined unique path

$$p \rightsquigarrow j \rightsquigarrow i$$

in h is an immediate supervisor of i in $\bar{h}'$. This is exactly the Unique-Path Consumption condition for the newly created bypass edge $p \to i$.

Therefore $\bar{h}'$ is a static reduction of h, and hence h' is a static reduction of h via ϕ.

Sufficiency. Suppose h' is a static reduction of h via ϕ. We construct a reduction trace from h to h' via ϕ. Work on the node set $V(h)$, and write

$$E^*_\phi = \{(u, v) : (\phi(u), \phi(v)) \in E(h')\}.$$

Let g be the current intermediate hierarchy, initially $g = h$. We maintain the invariant that h' is a static reduction of g via the same bijection ϕ.

For any intermediate hierarchy g, define the surplus edge set

$$D(g) = E(g) \setminus E^*_\phi$$

and the associated multiset of depths

$$M_D(g) = \{\, s_g(x) : (x, y) \in D(g) \,\},$$

where $s_g(x)$ denotes the number of ancestors of x in g.

First suppose there exists a target edge $(u', v') \in E(h')$ whose preimage

$$(u, v) = (\phi^{-1}(u'), \phi^{-1}(v'))$$

is not an edge of g. Choose such an edge with v of maximal depth in g among all missing target preimages. Since g is multipath-free, Reachability Containment gives a unique path

$$u = d_0 \to d_1 \to \cdots \to d_m = v, \qquad m \geq 2.$$

Each displayed arrow is an edge of g; in particular,

$$d_{m-2} \to d_{m-1} \in E(g),$$

so d_{m-2} is an immediate supervisor of d_{m-1} in g. By Unique-Path Consumption,

$$(\phi(d_{m-1}), v') \notin E(h'),$$

so the terminal edge $d_{m-1} \to v$ belongs to $D(g)$. In particular, it is a surplus edge. Remove $d_{m-1} \to v$. By Lemma 4, the resulting intermediate hierarchy remains multipath-free. By

Definition 2, a bypass $r \to v$ is added for each immediate supervisor r of d_{m-1} in g. In particular, the bypass

$$d_{m-2} \to v$$

is added, so the preimage path for (u', v') shortens by one edge.

We now verify that the static-reduction invariant is preserved. To establish Reachability Containment, it suffices to show that every target edge $(a', b') \in E(h')$ retains a representing path in the new intermediate graph. Let $(a, b) = (\phi^{-1}(a'), \phi^{-1}(b'))$. By the invariant, there is a representing path P from a to b in g. If P avoids the removed edge $d_{m-1} \to v$, it survives intact. Suppose P uses $d_{m-1} \to v$. Because g is multipath-free, P is the unique path from a to b in g.

Could $d_{m-1} \to v$ be the first edge of P? If so, $a = d_{m-1}$. Since $d_{m-1} \to v \in D(g)$, it is not the preimage of any target edge, so $v \neq b$. Thus v is an ancestor of b in g, which means that b has strictly greater depth than v in g. If (a, b) were a missing target preimage, this would contradict our maximal-depth choice of v. Hence (a, b) must already be a edge in g. But since g is multipath-free, its unique path is exactly the length-one edge $d_{m-1} \to b$. Since $v \neq b$, this path avoids $d_{m-1} \to v$, a contradiction.

Therefore, $d_{m-1} \to v$ cannot be the first edge of P. It must be preceded by some immediate supervisor r on the path. After the removal, the segment

$$r \to d_{m-1} \to v$$

is replaced by the bypass

$$r \to v.$$

Thus the preimage path is shortened rather than destroyed. Since every target edge retains a representing path, any target reachability relation $x' \rightsquigarrow y'$ in h' remains represented by concatenating the representing paths of its constituent edges.

For Unique-Path Consumption, consider any target edge $(a', b') \in E(h')$ whose preimage is not direct after the removal. If its preimage path avoids $d_{m-1} \to v$, the condition is unchanged. If its preimage path uses $d_{m-1} \to v$, then the segment $r \to d_{m-1} \to v$ is replaced by $r \to v$. The removed node d_{m-1} no longer appears on the shortened path. Any remaining internal node was already internal on the old path and, by the invariant, did not map to an immediate supervisor of b' in h'. Thus no forbidden internal supervisor is introduced, and Unique-Path Consumption remains valid.

If every edge of h' already has a direct preimage in g, but $D(g) \neq \emptyset$, choose an edge $x \to y \in D(g)$ with y of maximal depth in g among targets of surplus edges, and remove it. Again, Lemma 4 ensures that the resulting intermediate hierarchy remains multipath-free. By definition, the removed edge $x \to y$ is not the preimage of any target edge. Any bypass produced either coincides with a target preimage, in which case it preserves a required edge, or belongs to the surplus set and will be removed later. Since every target edge already has a direct preimage in g, no target path depends on a non-target bypass for its representation. Hence any newly created bypass that is not itself a target preimage is genuinely surplus. Because no direct preimage of a target edge is destroyed, every target edge continues to have a direct preimage in the new graph. Consequently, Reachability Containment is preserved, and Unique-Path Consumption is vacuous because there are no target edges lacking a direct preimage. Thus the static-reduction invariant is fully preserved.

In both cases above, the explicitly removed edge belongs to the surplus set $D(g)$, so its source depth is removed from $M_D(g)$. Every newly generated bypass that does not correspond to a target edge enters the surplus set. The source of any such bypass is an immediate supervisor of the removed edge's source, and therefore has a strictly smaller depth. Any generated bypass that does coincide with a target preimage simply drops out of $D(g)$ altogether. Moreover, the operation creates no new reachability relations. Hence, for every node x, every ancestor of x in the new graph was already an ancestor of x in g, so the ancestor set of x can only shrink. Therefore all surviving source-depth values weakly decrease. By the definition of the strict multiset extension of $<$ on $\mathbb{N}$— that is, replacing one or more entries by finitely many strictly smaller entries—replacing one occurrence of the removed source depth by any finite multiset of strictly smaller source depths, while weakly decreasing all remaining entries, yields a strict decrease in the multiset ordering. Hence each step strictly decreases $M_D(g)$, and termination follows.

At termination, no target edge lacks a direct preimage and no surplus edge remains. Hence the terminal edge set is exactly E^*_ϕ. The sequence of removals constructed above is therefore a reduction trace from h to h' via ϕ. Thus $h \succcurlyeq_H h'$. ■

3.3. Axiomatic characterization of fixed-size pre-orders

Having developed the trace-based dominance relation for general DAGs and established its static characterization on the multipath-free domain, we now characterize the hierarchical pre-orders that are compatible with the core relation $\succcurlyeq_H$. We seek extensions of $\succcurlyeq_H$ that preserve its fundamental ordering properties while potentially introducing additional comparability between hierarchies.

We define two levels of consistency with the core pre-order $\succcurlyeq_H$, reflecting the distinction between SR and SSR.

Definition 8 ($\succcurlyeq_H$-consistency)**.** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ is $\succcurlyeq_H$*-consistent* if it satisfies the following conditions for all $h, h' \in \mathscr{H}_n$:

(i) $h \sim_H h' \Rightarrow h \sim h'$.

(ii) $h \succ_H h' \Rightarrow h \succcurlyeq h'$.

(iii) $h \succ h'$ whenever h' can be obtained from h by removing a non-redundant subordination relation.

Definition 9 (strong $\succcurlyeq_H$-consistency)**.** A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}_n$ is *strongly* $\succcurlyeq_H$*-consistent* if it satisfies the following conditions for all $h, h' \in \mathscr{H}_n$:

(i) $h \sim_H h' \Rightarrow h \sim h'$.

(ii) $h \succ_H h' \Rightarrow h \succ h'$.

A strongly $\succcurlyeq_H$-consistent pre-order preserves all strict comparisons and indifference classes of $\succcurlyeq_H$. The notion of $\succcurlyeq_H$-consistency is more permissive: it allows a pre-order to register indifference when a removal is redundant and leaves the effective supervisory relation unchanged. Thus, $\succcurlyeq_H$-consistency treats hierarchy primarily through the effective chain of command rather than through formal organizational wiring.

We first identify the indifference classes of the core pre-order. The following lemma shows that hierarchies indifferent under the core pre-order are relabelings of one another.

Lemma 5. *For any hierarchies $h, h' \in \mathcal{H}_n$, $h \sim_H h'$ implies that h is a relabeling of h'.*

The proof of Lemma 5 is deferred to Appendix A.4. With this lemma and Proposition 1, which established that $h \succ_H h'$ precisely characterizes sequential removals, we can now demonstrate the exact equivalence between our two levels of axiomatic strictness, SR and SSR, and our two levels of consistency.

Theorem 2. *Let $\succcurlyeq$ be a hierarchical pre-order on $\mathcal{H}_n$.*

(i) $\succcurlyeq$ satisfies A and SR if and only if it is $\succcurlyeq_H$-consistent.

(ii) $\succcurlyeq$ satisfies A and SSR if and only if it is strongly $\succcurlyeq_H$-consistent.

Proof. Proof of (i): Sufficiency. Suppose $\succcurlyeq$ is $\succcurlyeq_H$-consistent. If h' is a relabeling of h, then $h \sim_H h'$, implying $h \succcurlyeq_H h'$ and $h' \succcurlyeq_H h$. By consistency, $h \sim h'$, satisfying A.

If h' is obtained from h by removing a subordination relation, Lemma 2 establishes $h \succ_H h'$, which by consistency yields $h \succcurlyeq h'$. Furthermore, if the removed edge is non-redundant, condition (iii) of Definition 8 guarantees $h \succ h'$. Thus, SR is satisfied.

Necessity. Suppose $\succcurlyeq$ satisfies A and SR. We must establish $\succcurlyeq_H$-consistency.

First, if $h \sim_H h'$, Lemma 5 establishes that h is a relabeling of h'. Since $\succcurlyeq$ satisfies A, $h \sim h'$.

Second, suppose $h \succ_H h'$. By Proposition 1, h' can be obtained from some relabeling h^* of h by a finite sequence of subordination removals. Because $\succcurlyeq$ satisfies SR, each removal yields a weak dominance, giving the chain

$$h^* \succcurlyeq h_L \succcurlyeq \cdots \succcurlyeq h_1 \succcurlyeq h'.$$

Since h^* is a relabeling of h and $\succcurlyeq$ satisfies A, $h \sim h^*$. By transitivity, $h \succcurlyeq h'$.

Finally, SR explicitly dictates $h \succ h'$ for non-redundant removals. Thus, $\succcurlyeq$ is $\succcurlyeq_H$-consistent.

Proof of (ii): Sufficiency. Suppose $\succcurlyeq$ is strongly $\succcurlyeq_H$-consistent. If h' is a relabeling of h, then $h \sim_H h'$, and strong consistency implies $h \sim h'$, satisfying A.

If h' is obtained from h by removing any subordination relation, Lemma 2 establishes $h \succ_H h'$, and strong consistency enforces $h \succ h'$, satisfying SSR.

Necessity. Suppose $\succcurlyeq$ satisfies A and SSR. The proof for $h \sim_H h' \Rightarrow h \sim h'$ follows identically from A and Lemma 5.

Suppose $h \succ_H h'$. By Proposition 1, h' is obtained from a relabeling h^* of h by one or more removals. Because $\succcurlyeq$ satisfies SSR, every single removal yields a strict dominance. Thus,

$$h^* \succ h_L \succ \cdots \succ h_1 \succ h'.$$

Since $h \sim h^*$ by Anonymity, transitivity implies $h \succ h'$. Thus, $\succcurlyeq$ is strongly $\succcurlyeq_H$-consistent. ∎

The distinction between SR and SSR—and consequently between consistency and strong consistency—disappears on the multipath-free domain. Indeed, multipath-freeness rules out redundant edges. Hence every subordination removal in a multipath-free hierarchy is non-redundant, and SR already requires a strict decrease in hierarchy. In particular, this applies to trees, which form a special case of multipath-free hierarchies. The distinction between SR and SSR becomes substantive only on the full domain of directed acyclic graphs, where matrix-style reporting structures may contain redundant edges.

We now turn to cardinal measures. A hierarchical index on $\mathscr{H}_n$ is a function $I : \mathscr{H}_n \to \mathbb{R}$ that assigns a hierarchical degree $I(h)$ to every hierarchy. Each index I induces a complete hierarchical order $\succcurlyeq_I$ on $\mathscr{H}_n$ defined by

$$h \succcurlyeq_I h' \Leftrightarrow I(h) \geq I(h').$$

For any hierarchy $h \in \mathscr{H}_n$, let $s_h(i)$ denote the number of ancestors, direct and indirect supervisors, of individual i in h. We define the supervisor index I_S as

$$I_S(h) = \frac{1}{n} \sum_{i \in V(h)} s_h(i).$$

The complete order $\succcurlyeq_{I_S}$ evaluates hierarchy entirely through reachability. If h' is obtained from h by a non-redundant subordination removal, then the transitive chain of command is reduced, and therefore $I_S(h) > I_S(h')$. If instead the removed edge is redundant, then the transitive supervisory relation is unchanged, so $I_S(h) = I_S(h')$. Thus $\succcurlyeq_{I_S}$ satisfies SR but need not satisfy SSR when redundant edges are present in the domain. Since I_S depends only on the numbers of ancestors, it is invariant under relabeling and therefore satisfies A.

Proposition 3. *The hierarchical order $\succcurlyeq_{I_S}$ defined on $\mathscr{H}_n$ is complete, reflexive, and transitive, and satisfies A and SR.*

By Proposition 3 and Theorem 2(i), $\succcurlyeq_{I_S}$ is $\succcurlyeq_H$-consistent. On any fixed-size domain $\mathscr{H}_n$ containing redundant edges, however, $\succcurlyeq_{I_S}$ fails SSR, because removing a redundant edge leaves I_S unchanged.

We next show that the core pre-order admits a complete strongly $\succcurlyeq_H$-consistent extension. For $d \in \{0, \dots, n-1\}$, define

$$\omega_d(h) = \left|\{(x, y) \in E(h) : s_h(x) = d\}\right|,$$

the number of reporting lines whose source has exactly d ancestors. Define the formal wiring profile

$$\Omega_n(h) = \big(\omega_{n-1}(h), \omega_{n-2}(h), \dots, \omega_0(h)\big).$$

Thus, the profile records reporting lines by source depth, beginning with the greatest depth.

For pairs $(r, \boldsymbol{\omega}), (r', \boldsymbol{\omega}') \in \mathbb{R} \times \mathbb{N}^n$, write

$$(r, \boldsymbol{\omega}) \geq_{\text{lex}} (r', \boldsymbol{\omega}')$$

if either $r > r'$, or $r = r'$ and ω is lexicographically weakly greater than ω'. The coordinates of ω and ω' are compared from left to right. The corresponding strict relation is denoted by $>_{\text{lex}}$.

Define the hierarchical order $\succcurlyeq_{SSR}$ on $\mathscr{H}_n$ by

$$h \succcurlyeq_{SSR} h' \Leftrightarrow \big(I_S(h), \Omega_n(h)\big) \geq_{\text{lex}} \big(I_S(h'), \Omega_n(h')\big).$$

Suppose that h' is obtained from h by a subordination removal. If the removed edge is non-redundant, the supervisory relation is strictly reduced, so $I_S(h) > I_S(h')$. Hence $h \succ_{SSR} h'$.

Now suppose that the removed edge $j \to i$ is redundant. In this case, the supervisory relation is unchanged, so $I_S(h) = I_S(h')$. It remains to compare the formal wiring profiles. Removing $j \to i$ deletes an edge whose source has depth $s_h(j)$. Every bypass $p \to i$ has a source p satisfying $p \to j$, and therefore $s_h(p) < s_h(j)$. Moreover, the removal creates no new supervisory relation, so the depth of each surviving edge source can only weakly decrease. It follows that, at the highest depth where the two profiles differ, the coordinate of $\Omega_n(h)$ is strictly larger than that of $\Omega_n(h')$. Hence

$$\Omega_n(h) >_{\text{lex}} \Omega_n(h').$$

Therefore $h \succ_{SSR} h'$ for every subordination removal.

The order $\succcurlyeq_{SSR}$ is complete, reflexive, and transitive, and it satisfies A and SSR. By Theorem 2(ii), it is therefore a complete strongly $\succcurlyeq_H$-consistent extension on $\mathscr{H}_n$.

4. Comparing hierarchies of varying sizes

We extend the analysis to hierarchies of different sizes using the Replication Principle of Carbonell-Nicolau (2025). This principle allows us to compare hierarchical structures across different organizational scales while preserving the essential properties of the hierarchical orders. The formal procedure involves creating comparable replicated structures and then applying our fixed-size results to these standardized hierarchies.

The set of all hierarchies of any positive size is defined as

$$\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n,$$

where $\mathscr{H}_n$ is the set of all n-person hierarchies.

Definition 10. A *hierarchical pre-order* on $\mathscr{H}$ is a reflexive and transitive binary relation on $\mathscr{H}$.

For a positive integer k, let h^k denote the hierarchy consisting of k independent copies of h:

$$h^k = \underbrace{(h, \ldots, h)}_{k \text{ times}}.$$

We call h^k a *replication* of h. By convention, $h^1 = h$.

The Replication Principle asserts that replicating a hierarchy does not change its hierarchical degree.

Replication Principle (RP). A hierarchical pre-order $\succcurlyeq$ on $\mathscr{H}$ satisfies RP if for any two hierarchies h and h' in $\mathscr{H}$, $h' \sim h$ whenever h' is a replication of h.

We extend the fixed-size core relation $\succcurlyeq_H$ to $\mathscr{H}$ by comparing common-size replications. If $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$, define

$$h \succcurlyeq_H h' \Leftrightarrow h^n \succcurlyeq_H (h')^m$$

under the fixed-size relation on $\mathscr{H}_{mn}$.

The following observation records that the fixed-size relation $\succcurlyeq_H$ is compatible with replication.

Lemma 6. *Let $g, g' \in \mathscr{H}_N$. If $g \succcurlyeq_H g'$ on $\mathscr{H}_N$, then for every positive integer r, $g^r \succcurlyeq_H (g')^r$ on $\mathscr{H}_{rN}$.*

Proof. Suppose $g \succcurlyeq_H g'$. Then there exists a bijection $\phi : V(g) \to V(g')$ and a reduction trace from g to g' via ϕ. Replicate this trace independently on each of the r copies of g, using the replicated bijection ϕ^r that acts as ϕ on each copy. Since the copies are pairwise disconnected, each removal step acts within a single copy and does not affect the others. Hence the replicated sequence is a reduction trace from g^r to $(g')^r$ via ϕ^r. Therefore $g^r \succcurlyeq_H (g')^r$. ∎

Lemma 7. *The hierarchical pre-order $\succcurlyeq_H$ defined on $\mathscr{H}$ is reflexive and transitive and satisfies A, SR, SSR, and RP.*

Proof. Reflexivity. If $h \in \mathscr{H}_m$, then $h^m \succcurlyeq_H h^m$ by reflexivity of the fixed-size relation on $\mathscr{H}_{m^2}$. Hence $h \succcurlyeq_H h$ on $\mathscr{H}$.

Transitivity. Let $h \in \mathscr{H}_m$, $h' \in \mathscr{H}_n$, and $h'' \in \mathscr{H}_p$, and suppose

$$h \succcurlyeq_H h' \quad \text{and} \quad h' \succcurlyeq_H h''.$$

By definition,

$$h^n \succcurlyeq_H (h')^m \quad \text{on } \mathscr{H}_{mn},$$

and

$$(h')^p \succcurlyeq_H (h'')^n \quad \text{on } \mathscr{H}_{np}.$$

By Lemma 6, replicating the first relation p times and the second relation m times gives

$$h^{np} \succcurlyeq_H (h')^{mp} \quad \text{and} \quad (h')^{pm} \succcurlyeq_H (h'')^{nm}$$

on $\mathscr{H}_{mnp}$. Since $(h')^{mp} = (h')^{pm}$, transitivity of the fixed-size relation yields

$$h^{np} \succcurlyeq_H (h'')^{nm}$$

on $\mathscr{H}_{mnp}$. Therefore $h \succcurlyeq_H h''$ on $\mathscr{H}$.

Anonymity. If h' is a relabeling of h and both belong to $\mathscr{H}_m$, then $(h')^m$ is a relabeling of h^m. By Anonymity of the fixed-size core relation,

$$h^m \sim_H (h')^m$$

on $\mathscr{H}_{m^2}$. Hence $h \sim_H h'$ on $\mathscr{H}$, so A holds.

SR *and* SSR. Suppose h' is obtained from $h \in \mathscr{H}_m$ by removing a single subordination relation. Then $(h')^m$ is obtained from h^m by performing the same removal independently in each copy. By Proposition 1, this yields $h^m \succ_H (h')^m$ on $\mathscr{H}_{m^2}$. Hence $h \succ_H h'$ on $\mathscr{H}$. In particular, the weak inequality required by SR holds for every removal, and the strict inequality required by SSR also holds for every removal.

RP. If $h' = h^k$ is a replication of $h \in \mathscr{H}_m$, then $h' \in \mathscr{H}_{km}$, and by definition

$$h \succcurlyeq_H h' \Leftrightarrow h^{km} \succcurlyeq_H (h^k)^m.$$

But both h^{km} and $(h^k)^m$ consist of km independent copies of h, so they are relabelings of one another. Hence

$$h^{km} \sim_H (h^k)^m,$$

which gives $h \succcurlyeq_H h'$. The same argument in reverse gives $h' \succcurlyeq_H h$. Therefore $h \sim_H h'$, establishing RP. ∎

For hierarchical pre-orders defined on the extended domain $\mathscr{H}$, we use the same two consistency notions as in Definitions 8 and 9. That is, $\succcurlyeq_H$-consistency and strong $\succcurlyeq_H$-consistency are defined by the same implications, with $\mathscr{H}_n$ replaced by $\mathscr{H}$, so that all quantifiers now range over $h, h' \in \mathscr{H}$.

The following result extends Theorem 2 to the domain $\mathscr{H}$.

Theorem 3. *Let $\succcurlyeq$ be a hierarchical pre-order on $\mathscr{H}$.*

(i) $\succcurlyeq$ satisfies A, SR, and RP if and only if it is $\succcurlyeq_H$-consistent.

(ii) $\succcurlyeq$ satisfies A, SSR, and RP if and only if it is strongly $\succcurlyeq_H$-consistent.

Proof. The argument parallels the proof of Theorem 2. For the necessity directions, we pass to common-size replications and apply the fixed-size results.

Proof of (i): Sufficiency. Suppose $\succcurlyeq$ is $\succcurlyeq_H$-consistent. If h' is a relabeling of h, then $h \sim_H h'$ by Lemma 7. By consistency, $h \sim h'$, so A holds. If h' is obtained from h by removing a subordination relation, then Lemma 7 gives $h \succ_H h'$, and consistency therefore yields $h \succcurlyeq h'$. If the removed edge is non-redundant, condition (iii) of Definition 8 gives $h \succ h'$. Hence SR holds. Finally, if $h' = h^k$ is a replication of h, then $h \sim_H h'$ by Lemma 7, so consistency gives $h \sim h'$. Thus RP holds.

Necessity. Suppose $\succcurlyeq$ satisfies A, SR, and RP. We must prove $\succcurlyeq_H$-consistency.

Fix $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$, and define

$$h_r = h^n \quad \text{and} \quad h'_r = (h')^m.$$

Then $h_r, h'_r \in \mathscr{H}_{mn}$. By RP for $\succcurlyeq$,

$$h \sim h_r \quad \text{and} \quad h' \sim h'_r.$$

If $h \sim_H h'$, then by definition of the extended core relation, $h_r \sim_H h'_r$ on $\mathscr{H}_{mn}$. Since $h_r, h'_r \in \mathscr{H}_{mn}$, Lemma 5 implies that h_r is a relabeling of h'_r. By A for $\succcurlyeq$, $h_r \sim h'_r$. Together with $h \sim h_r$ and $h' \sim h'_r$, transitivity yields $h \sim h'$.

If $h \succ_H h'$, then by definition of the extended core relation, $h_r \succ_H h'_r$ on $\mathscr{H}_{mn}$. Since $h_r, h'_r \in \mathscr{H}_{mn}$, Proposition 1 implies that h'_r can be obtained from some relabeling h^*_r of h_r by one or more subordination removals. Because $\succcurlyeq$ satisfies SR, each removal yields a weak dominance, giving

$$h^*_r \succcurlyeq h_L \succcurlyeq \cdots \succcurlyeq h_1 \succcurlyeq h'_r.$$

Since h^*_r is a relabeling of h_r and $\succcurlyeq$ satisfies A, we have $h_r \sim h^*_r$, hence $h_r \succcurlyeq h'_r$. Using $h \sim h_r$ and $h'_r \sim h'$, transitivity yields $h \succcurlyeq h'$.

Finally, if h' is obtained from h by removing a non-redundant subordination relation, then SR explicitly gives $h \succ h'$. Therefore $\succcurlyeq$ is $\succcurlyeq_H$-consistent.

Proof of (ii): Sufficiency. Suppose $\succcurlyeq$ is strongly $\succcurlyeq_H$-consistent. If h' is a relabeling of h, then $h \sim_H h'$ by Lemma 7, and strong consistency implies $h \sim h'$. Thus A holds. If h' is obtained from h by removing any subordination relation, then Lemma 7 gives $h \succ_H h'$, and strong consistency yields $h \succ h'$. Thus SSR holds. If $h' = h^k$ is a replication of h, then $h \sim_H h'$ by Lemma 7, and strong consistency gives $h \sim h'$. Hence RP holds.

Necessity. Suppose $\succcurlyeq$ satisfies A, SSR, and RP. We must prove strong $\succcurlyeq_H$-consistency.

The argument for

$$h \sim_H h' \Rightarrow h \sim h'$$

is identical to part (i).

Now suppose $h \succ_H h'$. By definition of the extended core relation, $h_r \succ_H h'_r$ on $\mathscr{H}_{mn}$. Since $h_r, h'_r \in \mathscr{H}_{mn}$, Proposition 1 implies that h'_r can be obtained from a relabeling h^*_r of h_r by one or more subordination removals. Because $\succcurlyeq$ satisfies SSR, every single removal yields a strict dominance. Thus

$$h^*_r \succ h_L \succ \cdots \succ h_1 \succ h'_r.$$

Since h^*_r is a relabeling of h_r and $\succcurlyeq$ satisfies A, we have $h_r \sim h^*_r$, and hence $h_r \succ h'_r$. Using $h \sim h_r$ and $h'_r \sim h'$, we obtain

$$h \succcurlyeq h_r \succ h'_r \succcurlyeq h',$$

so $h \succcurlyeq h'$. If $h' \succcurlyeq h$, then $h'_r \succcurlyeq h_r$ by $h' \sim h'_r$, $h \sim h_r$, and transitivity, contradicting $h_r \succ h'_r$. Therefore $h \succ h'$. This proves strong $\succcurlyeq_H$-consistency. ∎

The hierarchical pre-order $\succcurlyeq_S$ introduced in Carbonell-Nicolau (2025) can be extended to $\mathscr{H}$ analogously. For $h \in \mathscr{H}_m$ and $h' \in \mathscr{H}_n$, define

$$h \succcurlyeq_S h' \Leftrightarrow h^n \succcurlyeq_S (h')^m$$

under the fixed-size relation on $\mathscr{H}_{mn}$. By the same replication argument used above, this extension is reflexive and transitive and satisfies A, SR, and RP. Therefore, by Theorem 3(i), $\succcurlyeq_S$ is $\succcurlyeq_H$-consistent. However, exactly as in the fixed-size domain, $\succcurlyeq_S$ evaluates hierarchy entirely through reachability and fails SSR. Therefore it is not strongly $\succcurlyeq_H$-consistent.

As demonstrated in Carbonell-Nicolau (2025), $\succcurlyeq_S$ is a partial completion of $\succcurlyeq_H$: it preserves the comparisons implied by $\succcurlyeq_H$ while ranking some hierarchies that $\succcurlyeq_H$ leaves incomparable.

The hierarchical index I_S defined previously can be naturally extended to $\mathscr{H}$. For any hierarchy $h \in \mathscr{H}$ with $n = |V(h)|$ individuals, define

$$I_S(h) = \frac{1}{n} \sum_{i \in V(h)} s_h(i),$$

where $s_h(i)$ denotes the number of supervisors of individual i in h. The hierarchical order induced by I_S on $\mathscr{H}$ is defined as

$$h \succcurlyeq_{I_S} h' \Leftrightarrow I_S(h) \geq I_S(h').$$

Proposition 4. *The hierarchical order $\succcurlyeq_{I_S}$ defined on $\mathscr{H}$ is complete, reflexive, and transitive, and satisfies axioms A, SR, and RP.*

Proof. Completeness, reflexivity, and transitivity are immediate because $\succcurlyeq_{I_S}$ is induced by the real-valued index I_S.

To verify A, observe that relabeling preserves the number of ancestors of every node, and hence preserves the average value $I_S(h)$. Therefore relabelings are indifferent under $\succcurlyeq_{I_S}$.

To verify RP, let h^k be a replication of h. Replicating h multiplies both the total number of supervisors and the number of individuals by k, so $I_S(h^k) = I_S(h)$. Hence $h^k \sim_{I_S} h$.

To verify SR, let h' be obtained from h by removing a subordination relation. This operation creates no new ancestor-descendant reachability relation. Hence

$$s_{h'}(i) \leq s_h(i) \quad \text{for all } i \in V(h).$$

If the removed edge is non-redundant, at least one reachability relation is lost, so the inequality is strict for some node, and therefore $I_S(h) > I_S(h')$. If the removed edge is redundant, the transitive supervisory relation is unchanged, so $I_S(h) = I_S(h')$. Thus $\succcurlyeq_{I_S}$ satisfies SR. ∎

By Proposition 4 and Theorem 3(i), $\succcurlyeq_{I_S}$ is $\succcurlyeq_H$-consistent. Since $\mathscr{H}$ contains hierarchies with redundant edges, $\succcurlyeq_{I_S}$ fails SSR and is not strongly $\succcurlyeq_H$-consistent.

Remark 1. The complete strongly $\succcurlyeq_H$-consistent extension constructed on the fixed-size domain can be extended to $\mathscr{H}$ by normalizing the formal wiring profile by population size. Specifically, for $h \in \mathscr{H}$, define

$$\bar{\omega}_d(h) = \frac{1}{|V(h)|} \left|\{(x, y) \in E(h) : s_h(x) = d\}\right|, \quad d \geq 0,$$

and let

$$\bar{\Omega}(h) = (\ldots, \bar{\omega}_2(h), \bar{\omega}_1(h), \bar{\omega}_0(h)).$$

The lexicographic order induced by $\left(I_S(h), \bar{\Omega}(h)\right)$ is complete. The fixed-size argument shows that every subordination removal strictly decreases this lexicographic pair: I_S decreases when the removed edge is non-redundant, while $\bar{\Omega}$ decreases when it is redundant. Relabeling preserves both components, and normalization ensures replication invariance. Hence the order satisfies A, SSR, and RP. By Theorem 3(ii), it is a complete strongly $\succcurlyeq_H$-consistent extension on $\mathscr{H}$.

5. Concluding remarks

Viewing hierarchy as an organizational channel through which economic power and resources are allocated centers our focus on concrete institutional structures that can be formally represented, measured, and compared.

We propose an axiomatic approach to measuring hierarchy in directed acyclic graphs, rooted in fundamental principles: Anonymity, Subordination Removal, and the Replication Principle. The transition from traditional organizational trees to modern matrix structures introduces substantial topological complexity. We demonstrate that in general directed acyclic graphs, the structural flattening of an organization is path-dependent due to chronological interference. Because of the failure of commutativity and the ambiguity introduced by path redundancy, determining whether one hierarchy can be reduced to another requires tracing the explicit chronological sequence of subordination removals.

Our analysis establishes the domain on which chronological complexity resolves into a static condition. In multipath-free hierarchies, the initial and target graphs contain enough information to determine whether a valid removal sequence exists. Thus, the existence of a valid sequence of removals is equivalent to a static condition based on Reachability Containment and Unique-Path Consumption.

Furthermore, the transition to matrix structures motivates the distinction between Subordination Removal and Strong Subordination Removal. By distinguishing between these two concepts, our framework clarifies how different measures treat formal reporting redundancy. Subordination Removal evaluates hierarchy through the effective, transitive chain of command, while Strong Subordination Removal treats even redundant formal reporting lines as contributors to hierarchical weight.

The core hierarchical pre-order developed in this paper, $\succcurlyeq_H$, captures the unambiguous comparisons implied by sequential subordination removal. This pre-order establishes a minimal consistency threshold, enabling the development of more complete rankings, such as those derived from the average number of supervisors. The framework then extends naturally from fixed-size hierarchies to hierarchies of varying sizes through the Replication Principle.

The link between this general framework and earnings inequality remains a promising area for future exploration. By integrating our generalized measurement approach with the well-established field of inequality measurement, future work can investigate how complex hierarchical structures influence income distribution. This perspective opens a path for connecting axiomatic hierarchy measurement with broader questions about organizational design, compensation structures, and systemic economic inequality.

A. Appendix

A.1. Proof of Lemma 1

Lemma 1. *The binary relation $\succcurlyeq_H$ defined on $\mathscr{H}_n$ is reflexive and transitive, and satisfies A.*

Proof. Reflexivity. Set $L = 0$, $E^0 = E(h)$, and let ϕ be the identity bijection. Then the conditions of Definition 3 are trivially satisfied, so $h \succcurlyeq_H h$.

Anonymity. If h' is a relabeling of h, let $\phi : V(h) \to V(h')$ be the relabeling bijection. Taking $L = 0$ gives

$$E^0 = E(h) = E^*_\phi,$$

so $h \succcurlyeq_H h'$. Similarly, $h' \succcurlyeq_H h$.

Transitivity. Suppose $h \succcurlyeq_H h'$ via $\phi_1 : V(h) \to V(h')$, and $h' \succcurlyeq_H h''$ via $\phi_2 : V(h') \to V(h'')$.

For any edge set $A \subseteq V(h') \times V(h')$, we use the standard pullback notation:

$$\phi_1^{-1}(A) = \{(\phi_1^{-1}(a), \phi_1^{-1}(b)) : (a, b) \in A\}.$$

Let

$$(e_1, \ldots, e_{L_1}; E^0, \ldots, E^{L_1})$$

be a reduction trace from h to h' via ϕ_1, so

$$E^0 = E(h), \qquad E^{L_1} = \phi_1^{-1}(E(h')).$$

Let

$$(f_1, \ldots, f_{L_2}; F^0, \ldots, F^{L_2})$$

be a reduction trace from h' to h'' via ϕ_2, so

$$F^0 = E(h'), \qquad F^{L_2} = \phi_2^{-1}(E(h'')).$$

For each $\ell = 0, \ldots, L_2$, define the pulled-back edge set

$$\widetilde{F}^\ell = \phi_1^{-1}(F^\ell),$$

and for each $\ell = 1, \ldots, L_2$, if $f_\ell = (a_\ell, b_\ell)$, define the pulled-back removed edge

$$\widetilde{f}_\ell = (\phi_1^{-1}(a_\ell), \phi_1^{-1}(b_\ell)).$$

Because relabeling preserves immediate-supervisor relations and preserves the existence of directed paths avoiding a specified node, Definition 2 commutes with relabeling. Hence for each ℓ, $\widetilde{f}_\ell \in \widetilde{F}^{\ell-1}$, and $\widetilde{F}^\ell$ is obtained from $\widetilde{F}^{\ell-1}$ by removing $\widetilde{f}_\ell$ and adding exactly the bypass edges prescribed by Definition 2. Thus

$$(\widetilde{f}_1, \ldots, \widetilde{f}_{L_2}; \widetilde{F}^0, \ldots, \widetilde{F}^{L_2})$$

is a reduction trace on $V(h)$. Moreover,

$$\widetilde{F}^0 = \phi_1^{-1}(F^0) = \phi_1^{-1}(E(h')) = E^{L_1},$$

so it begins exactly where the first trace ends. Its terminal edge set is

$$\widetilde{F}^{L_2} = \phi_1^{-1}(F^{L_2}) = \phi_1^{-1}\big(\phi_2^{-1}(E(h''))\big) = (\phi_2 \circ \phi_1)^{-1}(E(h'')).$$

Therefore, concatenating the sequence of edges

$$e_1, \ldots, e_{L_1}, \widetilde{f}_1, \ldots, \widetilde{f}_{L_2}$$

and the sequence of edge sets

$$E^0, \ldots, E^{L_1} = \widetilde{F}^0, \widetilde{F}^1, \ldots, \widetilde{F}^{L_2}$$

yields a reduction trace from h to h'' via $\phi_2 \circ \phi_1$. Hence $h \succcurlyeq_H h''$. ■

A.2. Proof of Lemma 3

Lemma 3. *If there is a nonempty reduction trace from h to h', then $M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h)$.*

Proof. It suffices to show that a single subordination removal strictly decreases the source-depth multiset. Let g^+ be obtained from an intermediate hierarchy g by removing the edge $j \to i$. Removing $j \to i$ deletes one instance of $s_g(j)$ from $M_{\mathrm{src}}(g)$.

Every bypass edge added by Definition 2 has the form $p \to i$, where $p \to j \in E(g)$. Thus p is an ancestor of j in g, and hence

$$s_g(p) < s_g(j).$$

Moreover, each bypass edge $p \to i$ replaces the pre-existing directed path

$$p \to j \to i.$$

Hence the operation creates no new ancestor-descendant reachability relation: indeed, any directed path in g^+ that uses a bypass edge $p \to i$ can be expanded in g by replacing $p \to i$ with the path $p \to j \to i$, while every other edge of g^+ is already an edge of g. Therefore every reachability relation in g^+ was already present in g. Consequently,

$$s_{g^+}(v) \leq s_g(v) \quad \text{for all } v \in V(g).$$

Therefore, the removed source-depth value $s_g(j)$ is replaced only by values

$$s_{g^+}(p) \leq s_g(p) < s_g(j),$$

while all surviving source-depth values weakly decrease. By the definition of the strict multiset extension of $<$ on $\mathbb{N}$, replacing one occurrence of $s_g(j)$ by any finite multiset of strictly smaller values, while weakly decreasing all remaining entries, yields a strict decrease in the multiset ordering. Hence $M_{\mathrm{src}}(g^+) < M_{\mathrm{src}}(g)$.

Iterating this strict descent over the nonempty reduction trace and using transitivity of the multiset order gives $M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h)$. ■

A.3. Proof of Lemma 2

Lemma 2. *For $h, h' \in \mathcal{H}_n$, if h' is obtained from h by a single subordination removal in the sense of Definition 2, then $h \succ_H h'$.*

Proof. Suppose h' is obtained from h by removing the subordination relation $j \to i$. Then, via the identity bijection on $V(h)$, this single removal yields a reduction trace of length $L = 1$, with removed-edge list $(j \to i)$ and edge sets

$$E^0 = E(h) \quad \text{and} \quad E^1 = E(h').$$

Hence $h \succcurlyeq_H h'$.

By Lemma 3, this nonempty reduction trace implies

$$M_{\mathrm{src}}(h') < M_{\mathrm{src}}(h).$$

We claim that $h' \not\succcurlyeq_H h$. Suppose, toward a contradiction, that $h' \succcurlyeq_H h$.

If the reduction trace from h' to h is nonempty, then Lemma 3 gives

$$M_{\mathrm{src}}(h) < M_{\mathrm{src}}(h'),$$

contradicting the asymmetry of the strict multiset order.

If the reduction trace from h' to h is empty, then h' is a relabeling of h. Relabeling preserves ancestor relations and hence source depths, so

$$M_{\text{src}}(h') = M_{\text{src}}(h),$$

again contradicting

$$M_{\text{src}}(h') < M_{\text{src}}(h).$$

Therefore $h' \not\succcurlyeq_H h$. Since $h \succcurlyeq_H h'$ and $h' \not\succcurlyeq_H h$, we conclude that $h \succ_H h'$. ∎

A.4. Proof of Lemma 5

Lemma 5. *For any hierarchies $h, h' \in \mathscr{H}_n$, $h \sim_H h'$ implies that h is a relabeling of h'.*

Proof. Suppose $h \sim_H h'$. By definition,

$$h \succcurlyeq_H h' \quad \text{and} \quad h' \succcurlyeq_H h.$$

Thus, there exists a reduction trace from h to h', and a reduction trace from h' to h.

Assume, for contradiction, that the reduction trace from h to h' is nonempty. By Lemma 3,

$$M_{\text{src}}(h') < M_{\text{src}}(h).$$

Now consider the reduction trace from h' to h. If this reduction trace is nonempty, then Lemma 3 gives

$$M_{\text{src}}(h) < M_{\text{src}}(h'),$$

contradicting the asymmetry of the strict multiset ordering. If this reduction trace is empty, then h is a relabeling of h', so

$$M_{\text{src}}(h) = M_{\text{src}}(h'),$$

again contradicting $M_{\text{src}}(h') < M_{\text{src}}(h)$.

Therefore, the reduction trace from h to h' must be empty. By Definition 3, an empty reduction trace, $L = 0$, implies that

$$E(h) = E^0 = E^*_\phi = \{(u, v) : (\phi(u), \phi(v)) \in E(h')\}.$$

Equivalently,

$$(u, v) \in E(h) \Leftrightarrow (\phi(u), \phi(v)) \in E(h').$$

Thus h' is a relabeling of h via the bijection ϕ. ∎

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