Journal of Mathematical Economics 117 (2025) 103098

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

On progressive tax systems with heterogeneous preferences☆

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A R T I C L E I N F O

Manuscript handled by Editor A. Toda

JEL classification:
D31
D63
D71

Keywords:
Progressive income taxation
Income inequality
Income bipolarization
Endogenous income

A B S T R A C T

The properties of progressive income tax systems vis-à-vis standard measures of inequality and polarization have been studied elsewhere, both for economies with exogenous and endogenous income. In the case of endogenous income, preferences are assumed to be identical across consumers. This paper relaxes the preference homogeneity assumption. Using the relative Lorenz inequality order and the relative Foster–Wolfson bipolarization order, we show that income tax systems reduce both inequality and polarization — no matter what the economy's initial conditions are — only if they are progressive. Furthermore, we identify specific conditions related to heterogeneous consumer preferences under which progressive tax systems effectively mitigate inequality and polarization.

1. Introduction

Normatively, progressive income tax systems can be viewed as essential mechanisms for the reduction of "market-driven" income inequality. The theoretical literature on the foundations of progressive taxation goes back to the seminal result on the equivalence between tax progressivity — in the sense of increasing average tax rates on income — and the inequality-reducing property (see Jakobsson, 1976; Fellman, 1976; Kakwani, 1977).

This result, which is couched in terms of exogenous income, has been extended in several directions (see, e.g., Hemming and Keen, 1983; Eichhorn et al., 1984; Liu, 1985; Formby et al., 1986; Thon, 1987; Latham, 1988; Thistle, 1988; Moyes, 1988, 1989, 1994; Le Breton et al., 1996; Ebert and Moyes, 2000; Ju and Moreno-Ternero, 2008; Zoli, 2018; Kakwani and Son, 2021; Carbonell-Nicolau, 2019, 2024). For the most part, these extensions maintain the exogenous income framework. The dynamics of endogenous income introduce nuanced complexities that distinguish it from the exogenous income scenario. In the exogenous case, where gross incomes remain fixed, the mapping between gross and net incomes directly characterizes the redistributive impact of income taxation. Conversely, the endogenous framework introduces a critical additional dimension: taxation's potential influence on gross income generation itself. This interaction can potentially counteract the equalizing transition from gross incomes to net incomes, depending on the magnitude and distributional incidence of the elasticity of gross income with respect to nontaxed income. Notably, there are instances where progressive tax schedules can paradoxically amplify, rather than reduce, income inequality, as demonstrated by research from Allingham (1979), Ebert and Moyes (2003, 2007).

The seminal Jakobsson–Fellman–Kakwani result has been extended to scenarios with endogenous income by Carbonell-Nicolau and Llavador (2018, 2021a). Employing the relative Lorenz inequality pre-order, these studies demonstrate that inequality-reducing tax schedules must be progressive, characterized by increasing marginal tax rates across income levels, and precisely delineate the necessary and sufficient conditions on consumer preferences that determine whether specific progressive tax schedules will effectively reduce income inequality across different wage and ability distributions.

Building on prior research, Carbonell-Nicolau and Llavador (2021b) establish a pivotal equivalence between inequality reduction and the Foster–Wolfson relative bipolarization order (Foster and Wolfson, 2010; Wang and Tsui, 2000; Chakravarty, 2009, 2015) — a metric widely employed in the literature to quantify middle-class dynamics (see, e.g., Foster and Wolfson, 2010; Wolfson, 1994). This equivalence reveals that tax schedules are inequality-reducing if and only if they are bipolarization-reducing, a finding that extends the conceptual and analytical scope of the Jakobsson–Fellman–Kakwani result and its variants.[1]

☆ Valuable comments and suggestions by the referees and the associate editor are gratefully acknowledged.
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[1] Further extensions can be found in Carbonell-Nicolau (2019, 2024). Carbonell-Nicolau (2019) examines the inequality-reducing effects of commodity tax systems in both exogenous and endogenous income contexts. Carbonell-Nicolau (2024) extends this analysis to mixed tax systems, which integrate income and commodity taxation, evaluating their capacity to mitigate both income inequality and bipolarization.

https://doi.org/10.1016/j.jmateco.2025.103098
Received 9 August 2024; Received in revised form 18 December 2024; Accepted 27 January 2025
Available online 6 February 2025

The present study extends the canonical Mirrlees model of optimal income taxation (Mirrlees, 1971) by introducing preference heterogeneity among consumers. While the traditional Mirrlees framework assumes homogeneous preferences across individuals, our research relaxes this assumption. We incorporate a dual source of heterogeneity: the well-established variation in wages/abilities and a novel dimension of diverse consumer preferences. Consequently, economies are characterized by a joint distribution of wages/abilities and preferences. In determining the distributional effects of income tax schedules, we analyze the interaction of tax policies with both wage differentials and preference variations.

The introduction of heterogeneous preferences potentially amplifies the distributional distortions of taxation on gross incomes inherent in the endogenous income framework. The effect of taxation on gross income distribution, now more complex due to two sources of variation in individual attributes — wages and preferences — across the population, may more intensely counteract the direct distributional effect of a tax on net incomes. This heightened complexity necessitates the identification of specific conditions on preference profiles that effectively resolve these potential trade-offs, ensuring a net reduction in both inequality and bipolarization. Our analysis aims to delineate these conditions, providing a more nuanced understanding of how diverse preferences interact with tax policies to shape income distribution outcomes.

We formulate a single crossing condition on model primitives that extends the standard agent monotonicity condition (Mirrlees, 1971; Seade, 1982) to the case of heterogeneous preferences. Under this condition, an extension of the results in Carbonell-Nicolau and Llavador (2018, 2021a) can be proven for families of utility vectors that are "sufficiently rich". Specifically, if $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a distribution of utility functions describing the preferences of n individuals, the distribution $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$ is called a *simple transformation* of $\boldsymbol{u}$ if $\boldsymbol{u}'$ takes the form $(u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$ for some i, i.e., if the first (respectively, last) i (respectively, $n-i$) individuals are endowed with the utility function u_i (respectively, u_{i+1}). A set of preference profiles is *closed under simple transformations* if it contains the simple transformations of its elements.

For families of preference profiles that are closed under simple transformations, the present study yields several insights. First, we demonstrate that inequality-reducing tax schedules — those producing a more equitable post-tax income distribution across all possible economies — are inherently progressive. We then establish the equivalence between inequality-reducing and bipolarization-reducing tax schedules in economies with heterogeneous preferences. Moreover, we precisely delineate the necessary and sufficient conditions on preference vectors that enable a progressive tax schedule to simultaneously reduce inequality and bipolarization.

The main results are illustrated by means of a simple example where the individuals' preferences are represented by a family of quasilinear utility functions.

2. Characterizing income tax progressivity

We extend the model in Carbonell-Nicolau and Llavador (2018, 2021a,b) by allowing for heterogeneity of preferences across individuals.

Individual preferences are represented by utility functions, which uniformly satisfy the following set of properties. First, they are assumed to be real-valued functions defined on consumption-labor pairs (x, l) in the product set $\mathbb{R}_+ \times [0, L]$, where $0 < L < +\infty$. For an individual endowed with a utility function u, $u(x, l)$ represents the individual's utility from x units of consumption and l units of labor. Throughout the sequel, all utility functions u are assumed to satisfy the following conditions:

(i) u is continuous on $\mathbb{R}_+ \times [0, L]$.

(ii) $u(\cdot, l)$ is strictly increasing in x for each $l \in [0, L)$ and $u(x, \cdot)$ is strictly decreasing in l for each $x > 0$.

(iii) u is strictly quasiconcave on $\mathbb{R}_{++} \times [0, L)$ and twice continuously differentiable on $\mathbb{R}_{++} \times (0, L)$.

(iv) For each $x > 0$,

$$\liminf_{l \uparrow L} MRS(x, l) = +\infty \quad \text{and} \quad \limsup_{l \downarrow 0} MRS(x, l) < +\infty, \tag{1}$$

where, for $(x, l) \in \mathbb{R}_{++} \times (0, L)$, $MRS(x, l)$ represents the marginal rate of substitution of labor for consumption, i.e.,

$$MRS(x, l) = -\frac{\partial u(x, l)}{\partial l} \Bigg/ \frac{\partial u(x, l)}{\partial x}.$$

(v) For each $a > 0$, there exists $l > 0$ such that $u(al, l) > u(0, 0)$.

The first condition in (iv) requires the marginal rate of substitution of x for l to diverge, for fixed $x > 0$, as leisure vanishes. The second condition in (iv) is a technical condition ensuring that indifference curves do not become arbitrarily steep as $l \downarrow 0$. The last condition, (v), implies that, in the absence of taxes, an individual whose wage rate is $a > 0$ always consumes a positive amount.

The set of all utility functions satisfying the conditions (i)–(v) is denoted by $\mathscr{U}$.

A *tax schedule* is a continuous and nondecreasing map $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

- $T(y) \le y$ for each $y \in \mathbb{R}_+$.
- The map $y \mapsto y - T(y)$ is nondecreasing (i.e., T is order-preserving).

For every pre-tax income level $y \in \mathbb{R}_+$, $T(y)$ represents the associated tax liability ($T(y)$ being a subsidy if $T(y) < 0$).

A tax schedule T is *piecewise linear* if $\mathbb{R}_+$ can be partitioned into finitely many intervals $I_1, \ldots, I_K$ satisfying the following: for each k, there exist $\beta \in \mathbb{R}$ and $t \in [0, 1)$ such that $T(y) = \beta + ty$ for all $y \in I_k$.[2]

This implies that there exists $M \in \{1, 2, \ldots\}$ such that

$$0 = e_0 < e_1 < \cdots < e_M = +\infty$$

and

$$T(y) = \begin{cases} -\beta_1 + t_1 y & \text{if } 0 = e_0 \le y \le e_1, \\ -\beta_1 + t_1 e_1 + t_2(y - e_1) & \text{if } e_1 < y \le e_2, \\ \vdots & \\ -\beta_1 + t_1 e_1 + t_2(e_2 - e_1) + \cdots + t_M(y - e_{M-1}) & \text{if } e_{M-1} < y < e_M = +\infty, \end{cases}$$

where $\beta_1 \ge 0$ and $t_1, \ldots, t_M \in [0, 1)$.

The set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

Consider an individual with ability $a > 0$ and utility function $u \in \mathscr{U}$. When this individual chooses to supply $l \in [0, L]$ units of labor under a tax schedule $T \in \mathscr{T}$, their consumption is equal to $al - T(al)$ units. Consequently, the individual's utility is given by $u(al - T(al), l)$.

The individual's optimization problem can thus be formulated as

$$\max_{l \in [0, L]} u(al - T(al), l). \tag{2}$$

A solution to (2) is denoted by $l^u(a, T)$.[3] It expresses the utility maximizing units of labor as a function of the "wage rate" a and the tax schedule T. Corresponding *pre-tax* and *post-tax income functions* are denoted by

$$y^u(a, T) = al^u(a, T) \quad \text{and} \quad x^u(a, T) = al^u(a, T) - T(al^u(a, T)),$$

respectively.[4,5] When the tax schedule T is identically zero, representing a scenario with no taxation, we denote the resulting income and consumption functions as $y^u(a, 0)$ and $x^u(a, 0)$, respectively. Note that

[2] Note that 100% marginal tax rates are ruled out by assumption.

[3] Note that (2) has a solution because the objective function is continuous and the feasible set is compact.

[4] A solution to (2) exists, but need not be unique, and so pre-tax and post-tax solution functions are not uniquely defined.

[5] Since marginal tax rates are less than unity for the tax schedules in $\mathscr{T}$, condition (v) ensures that income levels $x^u(a, T)$ are positive.

$x^u(a,0) = y^u(a,0)$.

In the special case when T is a fixed subsidy, i.e., $T(y) = -b$ for all y and some $b \geq 0$, (2) has a unique solution (by the strict quasiconcavity of u on $\mathbb{R}_{++} \times [0,L)$), denoted by $l^u(a,b)$, with associated pre-tax and post-tax incomes $y^u(a,b)$ and $x^u(a,b)$, respectively.

We now introduce the class of utility vectors that serves as the foundation for our principal characterizations of progressive income tax systems.

We consider groups of individuals of size n and describe their preferences by means of utility functions in $\mathscr{U}$. Thus, a vector $(u_1,\ldots,u_n) \in \mathscr{U}^n$ of utility functions lists the individual preferences for each member of the group, where u_i represents individual i's utility function ($i \in \{1,\ldots,n\}$).

An *wage rate distribution*, also referred to as an *ability distribution*, is a vector $(a_1,\ldots,a_n) \in \mathbb{R}^n_{++}$, with its coordinates arranged in increasing order, i.e., $a_1 \leq \cdots \leq a_n$.

Note that any a-individual's utility function $u(x,l)$ defined on income-labor pairs can be reformulated in terms of net income-gross income pairs, (x,y), via the equation $y = al$, which relates before-tax income, y, to the number of hours worked, l: $u(x,y/a)$. The marginal rate of substitution of x for y,

$$\eta^a_u(x,y) = -(1/a) \cdot \frac{\partial u(x,y/a)}{\partial l} \Bigg/ \frac{\partial u(x,y/a)}{\partial x}, \tag{3}$$

expresses the individual's required compensation, in terms of net income, for a one-unit marginal increase in the quantity of gross income.

Let $\mathbb{U}$ be the set of all utility vectors $\boldsymbol{u} = (u_1,\ldots,u_n) \in \mathscr{U}^n$ satisfying the following conditions:

(a) For each i,
$$\eta^a_{u_i}(x,y) \geq \eta^{a'}_{u_i}(x,y), \quad \text{for all } (x,y) \in \mathbb{R}_{++} \times (0,aL) \text{ and } a' \geq a > y/L.$$

(b) For each $i < n$ and $a > 0$,
$$\eta^a_{u_i}(x,y) \geq \eta^a_{u_{i+1}}(x,y), \quad \text{for all } (x,y) \in \mathbb{R}_{++} \times (0,aL).$$

The first condition, (a), specifies that the compensation individuals require, in terms of net income, for earning an additional dollar of gross income decreases as the wage rate increases.

The second condition, (b), compares consumption bundles across utility functions. It stipulates that the required compensation, in terms of net income, for an additional dollar of gross income decreases as we move to higher-order coordinates in the preference vector $(u_1,\ldots,u_n)$.

The above conditions generalize the standard agent monotonicity condition introduced by Mirrlees (1971) and further elaborated by Seade (1982) to accommodate vectors of heterogeneous preferences. In the special case of a common utility function across individuals, conditions (a) and (b) simplify to condition (a), which is equivalent to the original single-crossing condition proposed by Mirrlees (1971). It is worth noting that condition (b) also represents a single-crossing property analogous to the Mirrlees condition, but applied to preference heterogeneity rather than wage heterogeneity.

The conditions (a) and (b) impose constraints on the ordering of incomes across individuals. For a given ability distribution $(a_1,\ldots,a_n)$ with $a_1 \leq \cdots \leq a_n$, u_i denotes the utility function of individual i, where higher-order coordinates in the vector $\boldsymbol{u} = (u_1,\ldots,u_n)$ correspond to the preferences of higher-ability individuals. In this framework, a higher-order coordinate in $\boldsymbol{u}$, combined with higher ability, implies greater consumption.

Lemma 1. *Suppose that $T \in \mathscr{T}$. For each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$ and every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}$, the ordering*

$$x^{u_1}(a_1,T) \leq \cdots \leq x^{u_n}(a_n,T) \tag{4}$$

must hold.[6]

The proof of Lemma 1 is relegated to Appendix B.1.

It is instructive to examine why the monotonicity condition in (4) necessitates $(u_1,\ldots,u_n) \in \mathbb{U}$. For simplicity, consider the case of two individuals. We have

$$x^{u_1}(a_1,T) \leq x^{u_1}(a_2,T)$$

(by (a)) and

$$x^{u_1}(a_2,T) \leq x^{u_2}(a_2,T) \tag{5}$$

(by (b)). These inequalities together imply

$$x^{u_1}(a_1,T) \leq x^{u_2}(a_2,T). \tag{6}$$

Crucially, Eq. (5) is not guaranteed if $(u_1,u_2) \notin \mathbb{U}$. A violation of (5) may consequently invalidate the inequality in (6), hence the importance of the condition $(u_1,u_2) \in \mathbb{U}$ for maintaining the monotonicity property.

Given a utility vector $(u_1,\ldots,u_n) \in \mathscr{U}^n$, an ability distribution $0 < a_1 \leq \cdots \leq a_n$, and post-tax income functions $x^{u_1},\ldots,x^{u_n}$, a tax schedule T in $\mathscr{T}$ generates a post-tax income distribution

$$(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)).$$

Correspondingly, the income distribution in the absence of taxes is represented by

$$(x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0)).$$

Given two distributions $\boldsymbol{x} = (x_1,\ldots,x_n)$ and $\boldsymbol{y} = (y_1,\ldots,y_n)$ with positive total income, we say that $\boldsymbol{x}$ ***Lorenz dominates*** $\boldsymbol{y}$, a dominance relation denoted by "$\boldsymbol{x} \succcurlyeq_L \boldsymbol{y}$", if

$$\frac{\sum_{i=1}^{l} x_{[i]}}{\sum_{i=1}^{n} x_{[i]}} \geq \frac{\sum_{i=1}^{l} y_{[i]}}{\sum_{i=1}^{n} y_{[i]}}, \quad \text{for all } l \in \{1,\ldots,n\},$$

where $(x_{[1]},\ldots,x_{[n]})$ (respectively, $(y_{[1]},\ldots,y_{[n]})$) is a rearrangement of the coordinates in $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$) in increasing order: $x_{[1]} \leq \cdots \leq x_{[n]}$ and $y_{[1]} \leq \cdots \leq y_{[n]}$.

A tax schedule $T \in \mathscr{T}$ is said to be *inequality-reducing with respect to* $\mathbb{U}' \subseteq \mathbb{U}$, or $\mathbb{U}'$-ir, if

$$(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)) \succcurlyeq_L (x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0))$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1},\ldots,x^{u_n})$, and every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}'$.

The subset of all $\mathbb{U}'$-ir tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}'\text{-ir}}$.

We now define the families of utility vectors that will be used in the formulation of our main results. To this end, we first define the wage

[6] Recall that a solution to (2) exists, but need not be unique, and so the solutions functions $x^u(\cdot)$ are not uniquely defined. This fact introduces a technical subtlety in cases when, for some $i < n$, both the problems

$$\max_{l \in [0,L]} u_i(al - T(al), l) \quad \text{and} \quad \max_{l \in [0,L]} u_{i+1}(al - T(al), l)$$

happen to have multiple solutions for some a. In fact, in these particular cases, solutions $x^{u_i}(a,T)$ and $x^{u_{i+1}}(a,T)$ can be selected that violate the order in (4). To avoid these "pathologies", we shall impose a certain consistency in the choice of selections from the solution correspondence: for those points a for which said correspondence is multi-valued, the inequality "$x^{u_i}(a,T) \leq x^{u_{i+1}}(a,T)$" should be read to mean that for every solution function x^{u_i}, there exists a solution function $x^{u_{i+1}}$ such that

$$x^{u_i}(a,T) \leq x^{u_{i+1}}(a,T).$$

It is important to emphasize that the presence of multiple solutions does not pose a problem for tax schedules in $\mathscr{T}$ that exhibit marginal-rate progressivity — that is, those that are convex. For such progressive tax schedules T, the solution functions $x^{u_i}(a,T)$ are uniquely determined. Moreover, as demonstrated in *Theorem* 1, inequality-reducing tax schedules are exclusively found within the subset of marginal-rate progressive tax schedules.

elasticity of income for a utility function at wage rate a and non-wage income b.

For $u \in \mathscr{U}$ and $(a,b) \in \mathbb{R}_{++} \times \mathbb{R}_{+}$,

$$\zeta^u(a,b) := \frac{\partial(al^u(a,b)+b)}{\partial a} \cdot \frac{a}{al^u(a,b)+b}$$

defines the *wage elasticity of income* for u at (a,b).[7,8]

Recall that the set of all piecewise linear tax schedules is denoted by $\mathscr{T}$.

A *marginal-rate progressive tax schedule* is a convex tax schedule in $\mathscr{T}$, which exhibits marginal tax rates that increase with income. The set of marginal-rate progressive tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\text{m-prog}}$.

For every (piecewise linear) income tax schedule T in $\mathscr{T}_{\text{m-prog}}$, there exist

$$0 = e_0 < e_1 < \cdots < e_M = +\infty$$

and intervals

$$I_1 = [e_0, e_1], \ldots, I_M = [e_{M-1}, e_M),$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0,1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$, and

$$b_1 < \cdots < b_M \quad \text{and} \quad t_1 < \cdots < t_M,$$

where the inequalities follow from the convexity of T.

Note that the extension of

$$-b_m + t_m y$$

to the entire domain $\mathbb{R}_+$, which is denoted by $T_m(y)$, is itself an income tax schedule in $\mathscr{T}_{\text{m-prog}}$. Thus, there are M many such linear extensions in $\mathscr{T}_{\text{m-prog}}$.

More generally, the set of all the linear extensions obtained from $T \in \mathscr{T}_{\text{m-prog}}$ in this manner is contained in $\mathscr{T}_{\text{m-prog}}$, and the cardinality of this set is equal to the number of tax brackets in T.

Using the above terminology, and given $T \in \mathscr{T}_{\text{m-prog}}$, define the class $\mathbb{U}_T$ of all utility vectors $(u_1, \ldots, u_n)$ in $\mathbb{U}$ satisfying the following conditions:

(I) For each $i < n$,
$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}(a,T)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } a > 0.$$

(II) For each i,
$$\zeta^{u_i}((1-t_m)a, b_m) \leq \zeta^{u_i}(a,0)$$
whenever $a > 0$ and $m \in \{1, \ldots, M\}$ satisfy
$$y^{u_i}((1-t_m)a, b_m) \in [e_{m-1}, e_m]$$
(i.e., individual i's gross income under the linear tax T_m lies in the m-th tax bracket for T).

Condition (I) states that the net income $x^u(a,T)$ as a fraction of $x^u(a,0)$ decreases as the order rank for the vector of utility functions $(u_1, \ldots, u_n)$ increases.

Condition (II) states that, for a fixed utility function u and ability level a, the combined effect of the tax subsidy b and the proportional tax t decreases the wage elasticity of income.

Conditions (I) and (II) offer insights into the redistributive properties of the tax system T. Condition (I) is equivalent to the statement that T reduces inequality in a hypothetical economy characterized by uniform wage rates, where individual heterogeneity emerges solely through diverse preferences. By contrast, condition (II) characterizes the tax system's inequality-mitigating property in an economy featuring uniform preferences but a non-degenerate wage rate distribution.

Given the two fundamental sources of economic heterogeneity — simultaneous variation in both preferences and wage rates — both conditions are needed to fully characterize the tax system's inequality-reducing potential.

It is important to note that condition (II) is expressed in terms of linear tax schedules, while our main results characterize inequality-reducing, nonlinear income tax schedules. This distinction arises because, in the absence of heterogeneous preferences, behavioral responses to linear taxation are sufficient to fully characterize inequality-reducing, nonlinear income tax schedules.

However, this simplification does not hold when preferences are heterogeneous. In such cases, condition (I) becomes necessary for a complete characterization. Despite this added complexity, the piecewise linear nature of the tax schedule T allows for a relatively straightforward representation of the net income functions $x^{u_i}(a,T)$ and $x^{u_{i+1}}(a,T)$ in condition (I), as demonstrated by the following proposition.

Proposition 1. *Given $u \in \mathscr{U}$, $T \in \mathscr{T}_{\text{m-prog}}$, and $a > 0$, there exists $m \in \{1, \ldots, M\}$ such that one and only one of the following two conditions holds.*

1. $x^u(a,T) = x^u((1-t_m)a, b_m)$.
2. $x^u(a,T) = (1-t_m)e_m + b_m$ *and*
$$x^u((1-t_{m+1})a, b_{m+1}) < (1-t_m)e_m + b_m < x^u((1-t_m)a, b_m).$$

The proof of Proposition 1 is given in Appendix B.2.

The main results of the paper are stated for classes of preferences profiles that are "sufficiently" rich in the following sense.

Given $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}$, $\boldsymbol{u}' = (u'_1, \ldots, u'_n)$ is called a *simple transformation of* $\boldsymbol{u}$ if there exists $\imath \in \{0, 1, \ldots, n\}$ such that

$$u'_j = u_\imath, \qquad \text{for each } j \leq \imath,$$
$$u'_j = u_{\imath+1}, \qquad \text{for each } j \geq \imath + 1.$$

A subset $\mathbb{U}' \subseteq \mathbb{U}$ is *closed under simple transformations* if $\boldsymbol{u} \in \mathbb{U}'$ implies that $\boldsymbol{u}' \in \mathbb{U}'$ for every simple transformation $\boldsymbol{u}'$ of $\boldsymbol{u}$. In words, $\mathbb{U}'$ is closed under simple transformations if it contains the simple transformations of all of its members.

Given $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and given $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{U}'$ and $i \in \{1, \ldots, n\}$, $(u_i, \ldots, u_i)$ is also an element of $\mathbb{U}'$. To see this, note first that

$$(u_1, \ldots, u_1) \quad \text{and} \quad (u_n, \ldots, u_n)$$

are simple transformations of $\boldsymbol{u}$ (take $\imath = 0$ and $\imath = n$, respectively, in the definition of a simple transformation). Next, note that, for $i < n$,

$$\boldsymbol{u}' = (u_i, \ldots, u_i, u_{i+1}, \ldots, u_{i+1})$$

is a simple transformation of $\boldsymbol{u}$, and so $\boldsymbol{u}' \in \mathbb{U}'$. Now observe that

$$\boldsymbol{u}'' = (u_i, \ldots, u_i)$$

is a simple transformation of $\boldsymbol{u}'$, implying that $\boldsymbol{u}'' \in \mathbb{U}'$.

To clarify the concept of closure under simple transformations, let us examine a simplified scenario involving two individuals with utility functions u and v. In this case, the minimal set of preference profiles that is closed under simple transformations and contains the initial profile (u,v) is represented by

$$\{(u,v), (u,u), (v,v)\}. \tag{7}$$

This set encompasses all possible combinations resulting from simple transformations applied to the original profile.

When evaluating the capacity of tax schedules to consistently mitigate inequality and bipolarization, we require this property to hold for all initial wage distributions and across all utility vectors within a set closed under simple transformations. In the two-person example above, the set in (7) represents the minimal set of preference profiles for which

[7] The condition (v) guarantees that $al^u(a,b) + b$ is positive.

[8] For each $b \geq 0$, the derivative of the map $a \mapsto l^u(a,b)$ exists for all but at most one $a > 0$. See Carbonell-Nicolau and Llavador (2018, footnote 15).

a tax schedule would consistently achieve its intended redistributive objectives. Conceptually, the profiles (u, u) and (v, v) isolate income distribution variations stemming solely from wage differentials. In contrast, the profile (u, v) introduces an additional source of distributional variation arising from heterogeneous preferences.[9]

The following is the first main result of this paper. Its proof is relegated to Appendix B.3.

Theorem 1. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \text{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

This result characterizes income tax schedules that are inequality-reducing with respect to a universe of preference vectors $\mathbb{U}' \subseteq \mathbb{U}$: a tax schedule $T \in \mathscr{T}$ is inequality-reducing with respect to $\mathbb{U}'$ if and only if T is marginal-rate progressive (i.e., convex) and $\mathbb{U}'$ is contained in $\mathbb{U}_T$.

Theorem 1 can be extended to a second characterization of progressivity in terms of inequality *and* bipolarization-reducing tax schedules.

The Foster–Wolfson bipolarization order (Foster and Wolfson, 2010; Wang and Tsui, 2000; Chakravarty, 2009, 2015) is a measure of the degree of income polarization between two income groups, taking median income as the demarcation point.

For two income distributions $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ with the same median income, m, we write $\boldsymbol{y} \succcurlyeq_{FW} \boldsymbol{x}$ to indicate that $\boldsymbol{y}$ is no less bipolarized than $\boldsymbol{x}$, if

$$\sum_{k \le i < \frac{n+1}{2}} (m - x_i) \le \sum_{k \le i < \frac{n+1}{2}} (m - y_i), \quad \forall k : 1 \le k < \frac{n+1}{2},$$

$$\sum_{\frac{n+1}{2} < i \le k} (x_i - m) \le \sum_{\frac{n+1}{2} < i \le k} (y_i - m), \quad \forall k : \frac{n+1}{2} < k \le n.$$

The Foster–Wolfson bipolarization order compares income distributions on the basis of an aggregate measure of the deviation of income levels from median income, with lower aggregate deviations corresponding to less bipolarized distributions.

Assuming that proportional changes in income do not alter the degree of bipolarization, $\succcurlyeq_{FW}$ can be extended to pairs of income distributions with different median incomes as follows.

Let $m(\boldsymbol{x})$ (respectively, $m(\boldsymbol{y})$) denote the median income of $\boldsymbol{x}$ (respectively, $\boldsymbol{y}$), and suppose that $m(\boldsymbol{x}) > 0$ and $m(\boldsymbol{y}) > 0$. Then the transformation

$$\boldsymbol{y}' = \frac{m(\boldsymbol{x})}{m(\boldsymbol{y})}(y_1, \ldots, y_n)$$

of $\boldsymbol{y}$ has the same median as $\boldsymbol{x}$ and we write

$$\boldsymbol{y} \succcurlyeq_{FW} \boldsymbol{x} \Leftrightarrow \boldsymbol{y}' \succcurlyeq_{FW} \boldsymbol{x}.$$

A tax schedule $T \in \mathscr{T}$ is said to be *bipolarization-reducing with respect to* $\mathbb{U}' \subseteq \mathbb{U}$, or $\mathbb{U}'$-bpr, if

$$(x^{u_1}(a_1, 0), \ldots, x^{u_n}(a_n, 0)) \succcurlyeq_{FW} (x^{u_1}(a_1, T), \ldots, x^{u_n}(a_n, T))$$

for each ability distribution $0 < a_1 \le \cdots \le a_n$, each vector of income functions $(x^{u_1}, \ldots, x^{u_n})$, and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}'$.

The subset of all $\mathbb{U}'$-bpr tax schedules in $\mathscr{T}$ is denoted by $\mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

The equivalence between the inequality-reducing property and its counterpart formulation in terms of the Foster–Wolfson dominance relation was first established in Carbonell-Nicolau and Llavador (2021b, Theorem 4) for economies with homogeneous preferences. The following result extends the equivalence to economies with heterogeneous preferences. The proof is given in Appendix B.4.

Theorem 2. *If $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then $\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.*

Theorem 2 states that if a domain of utility vectors $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then a tax schedule is inequality-reducing with respect to $\mathbb{U}'$ if and only if it is bipolarization-reducing with respect to $\mathbb{U}'$.

Theorem 1 and Theorem 2 combined immediately give the following result.

Corollary 1. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \text{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

This result states that a tax schedule $T \in \mathscr{T}$ is inequality and bipolarization-reducing with respect to $\mathbb{U}'$ if and only if T is marginal-rate progressive (i.e., convex) and $\mathbb{U}'$ is contained in $\mathbb{U}_T$.

3. An example

Consider the quasilinear utility function

$$u_{(\alpha,\beta)}(x, l) = x - \alpha l^{\beta},$$

where $\alpha > 0$ and $\beta > 1$.

It is easy to verify that $u_{(\alpha,\beta)} \in \mathscr{U}$ (i.e., that the conditions (i)–(v) hold for the utility function $u_{(\alpha,\beta)}$).

To begin, we assume that α is common across individuals, while allowing β to vary. In this context, we can identify domains of utility vectors for which no tax schedule is inequality or bipolarization-reducing.

Since

$$\eta^a_{u_{(\alpha_i,\beta_i)}}(x, y) = \frac{\alpha_i \beta_i}{a} \left(\frac{y}{a}\right)^{\beta_i - 1},$$

we see that $\eta^a_{u_{(\alpha_i,\beta_i)}}(x, y)$ is nonincreasing in a for every (x, y). Moreover,

$$\eta^a_{u_{(\alpha_i,\beta_i)}}(x, y) \ge \eta^a_{u_{(\alpha_{i+1},\beta_{i+1})}}(x, y)$$
$$\Leftrightarrow \ln \alpha_i + \ln \beta_i + (\beta_i - 1)\ln(y/a) \ge \ln \alpha_{i+1} + \ln \beta_{i+1} + (\beta_{i+1} - 1)\ln(y/a),$$

Consequently, for fixed α, a vector of utilities

$$(u_{(\alpha,\beta_1)}, \ldots, u_{(\alpha,\beta_n)})$$

belongs to $\mathbb{U}$ if and only if

$$\beta_1 \ge \cdots \ge \beta_n. \tag{8}$$

Now fix any $T \in \mathscr{T}_{\text{m-prog}}$, and let $\mathbb{U}'$ be the set of all preference profiles of the form

$$(u_{(\alpha,\beta_1)}, \ldots, u_{(\alpha,\beta_n)})$$

satisfying (8). It is readily verified that $\mathbb{U}'$ is closed under simple transformations. Hence, by Corollary 1, T is inequality or bipolarization-reducing with respect to $\mathbb{U}'$ if and only if $\mathbb{U}' \subseteq \mathbb{U}_T$.

In Appendix A, we demonstrate that $\mathbb{U}' \nsubseteq \mathbb{U}_T$, which implies that the taxation scheme T fails to be inequality-reducing or bipolarization-reducing. This means that there exists at least one wage distribution and preference profile in $\mathbb{U}'$ where the post-tax income distribution does not Lorenz dominate the tax-free income distribution.

This example reveals a violation of condition (I) in the definition of $\mathbb{U}_T$. Within the context of this specific case, we can interpret the condition more intuitively: it requires that, across all wage rates, the elasticity of post-tax income with respect to the preference parameter β exceeds the elasticity of income in a tax-free scenario. Since higher-order individuals are assigned a lower β (as per Eq. (8)), and since individuals with a lower β have higher incomes, the tax system T introduces a distinctive income redistribution mechanism: a one-percent

[9] Ideally, one might prefer to limit redistribution to address income disparities stemming solely from differences in ability. However, in practice, the implementation of such targeted redistributive policies is infeasible, as both sources of variation — ability and preferences — are unobservable.

decrease in β induces a more compressed income progression under T compared to the no-tax scenario. This relationship effectively results in diminishing relative increases in post-tax income as one climbs the income ladder. This compression mechanism is precisely what drives the reduction in after-tax inequality as measured by the relative Lorenz criterion.

The preferences illustrated in the above example have the property that, for sufficiently low wage rates, the compression mechanism breaks down. In these instances, lower values of β, which typically correspond to higher individual incomes, amplify the relative increase of post-tax incomes across the income spectrum, regardless of the income tax. This phenomenon implies that for any marginal-rate progressive tax policy T, there exist homogeneous wage rate distributions (concentrated at sufficiently low wage rates and for which the entire pre-tax income distribution falls within the first tax bracket) such that the post-tax income distribution is Lorenz dominated by the tax-free income distribution, contradicting the expected redistributive effects of the tax system.

Next, we fix β and allow α to vary, and identify domains of utility vectors for which T is inequality or bipolarization-reducing.

First, note that for fixed β, a vector of utilities

$$(u_{(\alpha_1,\beta)}, \dots, u_{(\alpha_n,\beta)})$$

belongs to $\mathbb{U}$ if and only if

$$\alpha_1 \geq \cdots \geq \alpha_n. \tag{9}$$

Let $\mathbb{U}'$ be the set of all preference profiles of the form

$$(u_{(\alpha_1,\beta)}, \dots, u_{(\alpha_n,\beta)})$$

satisfying (9). Since $\mathbb{U}'$ is closed under simple transformations, and since $\mathbb{U}' \subseteq \mathbb{U}_T$, Corollary 1 implies that T is inequality and bipolarization-reducing.

We defer the proof that $\mathbb{U}' \subseteq \mathbb{U}_T$ to Appendix A and instead concentrate on a heuristic interpretation of the result that, under (9), any marginal-rate progressive tax schedule T is both inequality-reducing and bipolarization-reducing.

In the context of this example, the inclusion $\mathbb{U}' \subseteq \mathbb{U}_T$ implies two key relationships: first, the wage elasticity of post-tax income does not exceed that of tax-free income; second, the elasticity of post-tax income with respect to the preference parameter α is greater than the same elasticity in the tax-free scenario. These relationships indicate that as wages increase or α decreases (both corresponding to higher income levels), the relative increase in income is consistently lower for the post-tax income distribution.

The tax schedule T induces a monotonic compression effect, systematically reducing percentage income increments as one moves up the economic hierarchy. As established in Corollary 1, this mechanism ensures that the post-tax income distribution comprehensively dominates the tax-free distribution across both the relative Lorenz and Foster–Wolfson inequality measures. This dominance is robust—persisting uniformly across varying initial wage rate distributions and preference profiles within $\mathbb{U}'$.

4. Concluding remarks

We have studied inequality and bipolarization-reducing income tax schedules in economies with endogenous income and heterogeneous preferences. We have introduced a single crossing condition on vectors of utilities — a condition akin to the standard agent monotonicity condition of Mirrlees (1971) — ensuring that income increases with the wage rate. This property allows us to provide a full characterization of inequality and bipolarization-reducing income tax schedules in terms of taxpayer preference profiles and the structure of the tax code.

A recurring objection to our research approach highlights that using income as a proxy for "welfare" overlooks the welfare effects derived from leisure time utilization. There are, however, compelling reasons to eschew the classical welfare metric of utility and instead examine the distributional effects of income taxes on vectors of "felicity" indices. Indeed, most measures of inequality and polarization — being cardinal in nature — are fundamentally incompatible with a meaningful evaluation of utility distributions. These measures are not, in general, invariant to order-preserving utility transformations that do not alter consumer behavior in the neoclassical framework.

Alternatively, one might consider broader measures of "consumption", including the "value" of leisure, which requires, however, "comparable" metrics across individuals.[10] For example, one might consider using the "opportunity cost" of leisure. But this is, in general, a "censored" variable, since there is no observable wage rate for those individuals who do not work.

In conclusion, we highlight a significant connection between our model's dual source of heterogeneity — in wages and preferences — and the literature on voting over income tax functions.[11] This link offers potential avenues for future research, particularly in the realm of political economy and public choice theory.

Gans and Smart (1996) demonstrated that, under the standard Mirrlees single-crossing condition, majority voting equilibria exist for rich families of nonlinear income tax functions that can be completely ordered by increasing progressivity. Moreno-Ternero (2011) obtained a similar result in a related context, building on Young's taxation framework (Young, 1988). Moreno-Ternero showed that a Condorcet winner exists among a range of piecewise-linear functions, termed generalized talmudic tax methods. These methods incorporate a fairness principle of distributive justice, whereby each taxpayer faces a burden "similar" to that borne by the entire society. Crucially, both Gans and Smart (1996) and Moreno-Ternero (2011) rely on a single-crossing condition akin to the Mirrlees agent monotonicity property in their existence proofs.

Given these results, and considering that our analysis incorporates heterogeneity in both wages and preferences through a novel single-crossing condition on vectors of heterogeneous preferences — working in conjunction with the Mirrlees agent monotonicity condition — a natural question arises: Can the existence of majority equilibria be established within our more general framework? This question presents an intriguing direction for future research.

Declaration of competing interest

The author has nothing to declare.

Appendix A. Supplement to Section 3

This section expands upon the analysis presented in Section 3, which examines the family of quasilinear utility functions defined as

$$u_{(\alpha,\beta)}(x,l) = x - \alpha l^{\beta},$$

where $\alpha > 0$ and $\beta > 1$.

Recall from Section 3 that, for fixed α, a vector of utilities

$$(u_{(\alpha,\beta_1)}, \dots, u_{(\alpha,\beta_n)})$$

belongs to $\mathbb{U}$ if and only if

$$\beta_1 \geq \cdots \geq \beta_n. \tag{10}$$

Now fix any $T \in \mathscr{T}_{\text{m-prog}}$, and let $\mathbb{U}'$ be the set of all preference profiles of the form

$$(u_{(\alpha,\beta_1)}, \dots, u_{(\alpha,\beta_n)})$$

satisfying (10).

[10] In the special case of quasilinear preferences (see, e.g., Section 3), the consumers' "value" of leisure can be measured using the same monetary units as long as individual preferences are known.

[11] We are grateful to an anonymous referee for suggesting this connection.

We show that $\mathbb{U}' \not\subseteq \mathbb{U}_T$. To see this, choose

$$(u_{(\alpha,\beta_1)},\dots,u_{(\alpha,\beta_n)}) \in \mathbb{U}'$$

with $\beta_1 > \beta_2$ and fix $a > 0$ such that $y^{u_{(\alpha,\beta_1)}}(a,T) \in (0,e_1)$.[12] Because

$$0 < y^{u_{(\alpha,\beta_1)}}(a',T) \le y^{u_{(\alpha,\beta_1)}}(a,T), \quad \text{for all } 0 < a' < a$$

(where the second inequality follows from Lemma 1), a can be chosen close enough to 0 to ensure that $a/\alpha < 1$. In addition, because the map

$$\beta \mapsto y^{u_{(\alpha,\beta)}}(a,T)$$

is continuous, β_2 can be taken close enough to β_1 to ensure that $y^{u_{(\alpha,\beta_2)}}(a,T) \in (0,e_1)$. Consequently,

$$\frac{x^{u_{(\alpha,\beta_1)}}(a,T)}{x^{u_{(\alpha,\beta_1)}}(a,0)} = \frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta_1-1)} + b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta_1-1)}} < \frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta_2-1)} + b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta_2-1)}} = \frac{x^{u_{(\alpha,\beta_2)}}(a,T)}{x^{u_{(\alpha,\beta_2)}}(a,0)},$$

where the inequality follows from the fact that

$$\frac{\partial\left(\frac{(1-t_1)a\left(\frac{(1-t_1)a}{\alpha}\right)^{1/(\beta-1)}+b_1}{a\left(\frac{a}{\alpha}\right)^{1/(\beta-1)}}\right)}{\partial\beta} = \frac{b(\alpha a)^{1/(\beta-1)}\ln(a/\alpha)}{a^{(\beta+1)/(\beta-1)}(\beta-1)^2} < 0,$$

which holds by virtue of the inequality $a/\alpha < 1$.

Hence, $\mathbb{U}' \not\subseteq \mathbb{U}_T$.

Next, we fix β and allow α to vary. For fixed β, a vector of utilities

$$(u_{(\alpha_1,\beta)},\dots,u_{(\alpha_n,\beta)})$$

belongs to $\mathbb{U}$ if and only if

$$\alpha_1 \ge \cdots \ge \alpha_n. \tag{11}$$

Let $\mathbb{U}'$ be the set of all preference profiles

$$(u_{(\alpha_1,\beta)},\dots,u_{(\alpha_n,\beta)})$$

satisfying (11). The set $\mathbb{U}'$ is closed under simple transformations and we have $\mathbb{U}' \subseteq \mathbb{U}_T$.

To see that $\mathbb{U}' \subseteq \mathbb{U}_T$, note first that

$$\zeta^{u_{(\alpha,\beta)}}(a,b) = \frac{a^{\frac{\beta}{\beta-1}}\beta}{(\beta-1)\left(b(\alpha\beta)^{\frac{1}{\beta-1}} + a^{\frac{\beta}{\beta-1}}\right)},$$

and so $\zeta^{u_{(\alpha,\beta)}}(a,b)$ is decreasing in b and nondecreasing in a.[13] Consequently,

$$\zeta^{u_{(\alpha,\beta)}}((1-t_m)a,b_m) \le \zeta^{u_{(\alpha,\beta)}}(a,0)$$

whenever $a > 0$ and $m \in \{1,\dots,M\}$ satisfy

$$y^{u_{(\alpha,\beta)}}((1-t_m)a,b_m) \in [e_{m-1},e_m].$$

It remains to show that, for each $i < n$,

$$\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \ge \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}, \quad \text{for all } a > 0.$$

Fix $i < n$ and $a > 0$. For each $m \in \{1,\dots,M-1\}$, define $\overline{\alpha}(m)$ and $\underline{\alpha}(m)$ by

$$x^{u_{(\overline{\alpha}(m),\beta)}}((1-t_m)a,b_m) = (1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m = (1-t_m)e_m + b_m$$

and

$$x^{u_{(\underline{\alpha}(m),\beta)}}((1-t_{m+1})a,b_{m+1}) = (1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1} = (1-t_{m+1})e_m + b_{m+1}.$$

Note that $\overline{\alpha}(m) > \underline{\alpha}(m)$ for each $m \le M-1$ and $\underline{\alpha}(m) > \overline{\alpha}(m+1)$ for $m < M-1$. Indeed, because

$$\eta^a_{u_{(\overline{\alpha}(m),\beta)}}((1-t_m)e_m + b_m, e_m) = \frac{\overline{\alpha}(m)\beta}{a}\left(\frac{e_m}{a}\right)^{\beta-1} = 1-t_m,$$

$$\eta^a_{u_{(\underline{\alpha}(m),\beta)}}((1-t_{m+1})e_m + b_{m+1}, e_m) = \frac{\underline{\alpha}(m)\beta}{a}\left(\frac{e_m}{a}\right)^{\beta-1} = 1-t_{m+1},$$

and

$$\eta^a_{u_{(\overline{\alpha}(m+1),\beta)}}((1-t_{m+1})e_{m+1} + b_{m+1}, e_{m+1}) = \frac{\overline{\alpha}(m+1)\beta}{a}\left(\frac{e_{m+1}}{a}\right)^{\beta-1} = 1-t_{m+1},$$

and since $e_1 < \cdots < e_M$ and (by the convexity of T) $t_1 < \cdots < t_M$, we have $\overline{\alpha}(m) > \underline{\alpha}(m)$ and $\underline{\alpha}(m) > \overline{\alpha}(m+1)$.

Note also that

$$x^{u_{(\alpha,\beta)}}(a,T_m) = x^{u_{(\alpha,\beta)}}((1-t_m)a,b_m), \quad \text{for each } m \in \{1,\dots,M\},$$

where, recall, $T_m(y) = -b_m + t_m y$ for every y. This follows from the fact that both

$$l^{u_{(\alpha,\beta)}}(a,T_m) \quad \text{and} \quad l^{u_{(\alpha,\beta)}}((1-t_m)a,b_m)$$

solve the problem

$$\max_{l\in[0,L]} u_{(\alpha,\beta)}((1-t_m)al + b_m, l).$$

By Proposition 1 and Lemma 1, there are four cases to consider.

1. $m' \ge m$, $x^{u_{(\alpha_i,\beta)}}(a,T) = x^{u_{(\alpha_i,\beta)}}((1-t_m)a,b_m)$, and $x^{u_{(\alpha_{i+1},\beta)}}(a,T) = x^{u_{(\alpha_{i+1},\beta)}}((1-t_{m'})a,b_{m'})$.
2. $m' \ge m$, $x^{u_{(\alpha_i,\beta)}}(a,T) = x^{u_{(\alpha_i,\beta)}}((1-t_m)a,b_m)$, and $x^{u_{(\alpha_{i+1},\beta)}}(a,T) = (1-t_{m'})e_{m'} + b_{m'}$.
3. $m' \ge m$, $x^{u_{(\alpha_i,\beta)}}(a,T) = (1-t_m)e_m + b_m$, and $x^{u_{(\alpha_{i+1},\beta)}}(a,T) = x^{u_{(\alpha_{i+1},\beta)}}((1-t_{m'})a,b_{m'})$.
4. $m' \ge m$, $x^{u_{(\alpha_i,\beta)}}(a,T) = (1-t_m)e_m + b_m$, and $x^{u_{(\alpha_{i+1},\beta)}}(a,T) = (1-t_{m'})e_{m'} + b_{m'}$.

We consider only the first case, since the other three cases can be handled similarly. In the first case, we have, by (9) and Lemma 1,

$$\alpha_i \ge \overline{\alpha}(m) > \underline{\alpha}(m) \ge \alpha_{i+1} \qquad \text{if } m' = m+1,$$
$$\alpha_i \ge \overline{\alpha}(m) > \underline{\alpha}(m) > \cdots > \overline{\alpha}(m'-1) > \underline{\alpha}(m'-1) \ge \alpha_{i+1} \qquad \text{if } m' > m+1.$$

If $m = m'$, then

$$\begin{aligned}\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a,b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\ &= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}} \\ &\ge \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\ &= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)},\end{aligned}$$

where the inequality holds because

$$\frac{\partial\left(\frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha}\right)^{1/(\beta-1)}+b_m}{a\left(\frac{a}{\alpha}\right)^{1/(\beta-1)}}\right)}{\partial\alpha} \ge 0.$$

The sign of this partial derivative can be established by explicit differentiation.

[12] Such an a exists by Lemma 3 in Appendix B.3.

[13] To see that $\zeta^{u_{(\alpha,\beta)}}(a,b)$ is nondecreasing in a, note that

$$\frac{\partial\zeta^{u_{(\alpha,\beta)}}(a,b)}{\partial a} = \frac{ba^{\frac{1}{\beta-1}}\beta^{\frac{2\beta-1}{\beta-1}}\alpha^{\frac{1}{\beta-1}}}{(\beta-1)^2\left(b(\alpha\beta)^{\frac{1}{\beta-1}} + a^{\frac{\beta}{\beta-1}}\right)^2} \ge 0.$$

If $m' = m+1$, then

$$\begin{aligned}
\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\
&= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_m)e_m + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})e_m + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\
&= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}.
\end{aligned}$$

If $m' > m+1$, then

$$\begin{aligned}
\frac{x^{u_{(\alpha_i,\beta)}}(a,T)}{x^{u_{(\alpha_i,\beta)}}(a,0)} &= \frac{x^{u_{(\alpha_i,\beta)}}((1-t_m)a, b_m)}{x^{u_{(\alpha_i,\beta)}}(a,0)} \\
&= \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\alpha_i}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\alpha_i}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_m)a\left(\frac{(1-t_m)a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)} + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_m)e_m + b_m}{a\left(\frac{a}{\overline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})e_m + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\underline{\alpha}(m)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m+1})a\left(\frac{(1-t_{m+1})a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)} + b_{m+1}}{a\left(\frac{a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)}} \\
&= \frac{(1-t_{m+1})e_{m+1} + b_{m+1}}{a\left(\frac{a}{\overline{\alpha}(m+1)}\right)^{1/(\beta-1)}} \\
&\vdots \\
&\geq \frac{(1-t_{m'})a\left(\frac{(1-t_{m'})a}{\underline{\alpha}(m'-1)}\right)^{1/(\beta-1)} + b_{m'}}{a\left(\frac{a}{\underline{\alpha}(m'-1)}\right)^{1/(\beta-1)}} \\
&\geq \frac{(1-t_{m'})a\left(\frac{(1-t_{m'})a}{\alpha_{i+1}}\right)^{1/(\beta-1)} + b_{m'}}{a\left(\frac{a}{\alpha_{i+1}}\right)^{1/(\beta-1)}} \\
&= \frac{x^{u_{(\alpha_{i+1},\beta)}}(a,T)}{x^{u_{(\alpha_{i+1},\beta)}}(a,0)}.
\end{aligned}$$

Appendix B. Proofs

B.1. Proof of Lemma 1

Lemma 1. *Suppose that $T \in \mathscr{T}$. For each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$ and every vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}$, the ordering*

$$x^{u_1}(a_1, T) \leq \cdots \leq x^{u_n}(a_n, T) \tag{4}$$

must hold.[14]

Proof. Given a tax schedule $T \in \mathscr{T}$, a wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, and a vector of utility functions $(u_1, \ldots, u_n) \in \mathbb{U}$, the inequality

$$x^{u_i}(a_i, T) \leq x^{u_{i+1}}(a_{i+1}, T), \quad \text{for } i < n,$$

follows from the two inequalities

$$x^{u_i}(a_i, T) \leq x^{u_i}(a_{i+1}, T) \quad \text{and} \quad x^{u_i}(a_{i+1}, T) \leq x^{u_{i+1}}(a_{i+1}, T).$$

The first inequality is a consequence of the Mirrlees single crossing condition, (a) (see Mirrlees, 1971, Theorem 1).

Similarly, condition (b) implies the second inequality. This is a consequence of the fact that the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_{i+1}(x, y/a_{i+1}) \tag{12}$$

passing through the bundle $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$ must lie (weakly) above the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_i(x, y/a_{i+1}) \tag{13}$$

passing though the same bundle whenever $y \leq y^{u_i}(a_{i+1}, T)$, implying that the problem

$$\max_{(x,y) \in \mathbb{R}_+ \times [0, a_{i+1}L]} u_{i+1}(x, y/a_{i+1})$$
$$\text{s.t. } x \leq y - T(y)$$

has a solution (x^*, y^*) such that $y^* \geq y^{u_i}(a_{i+1}, T)$ and $x^* \geq x^{u_i}(a_{i+1}, T)$.[15]

Indeed, suppose that, on the contrary, there exists a point $y' < y^{u_i}(a_{i+1}, T)$ such that

$$u_i(x', y'/a_{i+1}) = \overline{u}_i, \quad u_{i+1}(x'', y'/a_{i+1}) = \overline{u}_{i+1}, \quad \text{and } x' > x'',$$

where $u_i(x, y/a_{i+1}) = \overline{u}_i$ (respectively, $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$) represents the indifference curve for the map in (13) (respectively, (12)) passing through the bundle $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$.

Note that, because

$$\eta^{a_{i+1}}_{u_i}(x, y) \geq \eta^{a_{i+1}}_{u_{i+1}}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL),$$

the indifference curve $u_i(x, y/a_{i+1}) = \overline{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \overline{u}_{i+1}$ for all $y \geq y'$. Moreover, both curves intersect at $y^{u_i}(a_{i+1}, T)$.

Now let $u_i(x, y/a_{i+1}) = \hat{u}_i$ represent the equation of the indifference curve for the map

$$(x, y) \in \mathbb{R}^2_{++} \mapsto u_i(x, y/a_{i+1})$$

passing through the point (x'', y'). Since $x'' < x'$, we have $\hat{u}_i < \overline{u}_i$.

Thus, we have three indifference curves,

$$u_i(x, y/a_{i+1}) = \overline{u}_i, \quad u_i(x, y/a_{i+1}) = \hat{u}_i, \quad \text{and } u_{i+1}(x, y/a_{i+1}) = \overline{u}_i,$$

satisfying the following conditions:

[14] When multiple solution functions x^{u_i} exist, "$x^{u_i}(a_i, T) \leq x^{u_{i+1}}(a_{i+1}, T)$" means that for every solution function x^{u_i}, there exists a solution function $x^{u_{i+1}}$ such that $x^{u_i}(a, T) \leq x^{u_{i+1}}(a, T)$. See footnote 6.

[15] If $y^* < y^{u_i}(a_{i+1}, T)$ for all solutions (x^*, y^*), then there exists a feasible bundle (x°, y°) with $y^\circ < y^{u_i}(a_{i+1}, T)$ strictly above the first indifference curve, and hence strictly above the second indifference curve, implying that an individual whose utility function is u_i and whose wage rate is a_{i+1} prefers (x°, y°) over $(x^{u_i}(a_{i+1}, T), y^{u_i}(a_{i+1}, T))$, a contradiction.

- $\bar{u}_i > \hat{u}_i$;
- the indifference curve $u_i(x, y/a_{i+1}) = \bar{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \bar{u}_{i+1}$ for all $y \geq y'$;
- the indifference curves $u_i(x, y/a_{i+1}) = \bar{u}_i$ and $u_{i+1}(x, y/a_{i+1}) = \bar{u}_i$ intersect at $y^{u_i}(a_{i+1}, T)$;
- the indifference curves $u_i(x, y/a_{i+1}) = \hat{u}_i$ and $u_{i+1}(x, y/a_{i+1}) = \bar{u}_i$ intersect at y'; and
- because

  $$\eta_{u_i}^{a_{i+1}}(x, y) \geq \eta_{u_{i+1}}^{a_{i+1}}(x, y), \quad \text{for all } (x, y) \in \mathbb{R}_{++} \times (0, aL),$$

  the indifference curve $u_i(x, y/a_{i+1}) = \hat{u}_i$ must lie (weakly) above the curve $u_{i+1}(x, y/a_{i+1}) = \bar{u}_{i+1}$ for all $y \geq y'$.

Consequently, the indifference curves $u_i(x, y/a_{i+1}) = \hat{u}_i$ and $u_i(x, y/a_{i+1}) = \bar{u}_i$ must intersect, which contradicts the inequality $\bar{u}_i > \hat{u}_i$. ■

B.2. Proof of Proposition 1

Proposition 1. *Given $u \in \mathscr{U}$, $T \in \mathscr{T}_{\text{m-prog}}$, and $a > 0$, there exists $m \in \{1, \dots, M\}$ such that one and only one of the following two conditions holds.*

1. $x^u(a, T) = x^u((1 - t_m)a, b_m)$.
2. $x^u(a, T) = (1 - t_m)e_m + b_m$ *and*

   $$x^u((1 - t_{m+1})a, b_{m+1}) < (1 - t_m)e_m + b_m < x^u((1 - t_m)a, b_m).$$

Proof. Pick $u \in \mathscr{U}$, $T \in \mathscr{T}_{\text{m-prog}}$, and $a > 0$.

If $y^u(a, T) = 0$, then

$$\eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) \geq 1 - t_1,$$

and, since $t_1 < \cdots < t_M$ (by the convexity of T), $y^u(a, T)$ solves both the problems

$$\max_{y \in [0, aL]} u(y - T(y), y/a) \quad \text{and} \quad \max_{y \in [0, aL]} u(y - T_1(y), y/a),$$

implying

$$y^u(a, T) = y^u(a, T_1) \quad \text{and} \quad x^u(a, T) = x^u(a, T_1) = x^u((1 - t_1)a, b_1).$$

Next, suppose that

$$y^u(a, T) > 0 = e_0.$$

If

$$e_{m-1} < y^u(a, T) < e_m, \quad \text{some } m \in \{1, \dots, M\},$$

then

$$\eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) = 1 - t_m, \tag{14}$$

and so

$$y^u(a, T) = y^u(a, T_m)$$

and

$$x^u(a, T) = x^u(a, T_m) = x^u((1 - t_m)a, b_m). \tag{15}$$

If

$$y^u(a, T) = e_m, \quad \text{some } m \in \{1, \dots, M\}, \tag{16}$$

then

$$1 - t_{m+1} \leq \eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) \leq 1 - t_m$$

must hold.[16]

If (14) and (16) hold, then (15) also holds.

If (16) holds and

$$\eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) = 1 - t_{m+1},$$

then

$$x^u(a, T) = x^u(a, T_{m+1}) = x^u((1 - t_{m+1})a, b_{m+1}).$$

If (16) holds and

$$1 - t_{m+1} < \eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) < 1 - t_m,$$

then

$$y^u(a, T_{m+1}) < e_m < y^u(a, T_m),$$

whence

$$x^u((1 - t_{m+1})a, b_{m+1}) = x^u(a, T_{m+1}) < (1 - t_m)e_m + b_m < x^u(a, T_m) = x^u((1 - t_m)a, b_m). \quad ■$$

B.3. Proof of Theorem 1

To begin, we state and prove a series of intermediate results.

Lemma 2. *Given $u \in \mathscr{U}$, $(x, y) \in \mathbb{R}^2_{++}$, and $\gamma \in (0, +\infty)$, there exists $a > 0$ satisfying*

$$a > y/L \quad \text{and} \quad \eta_u^a(x, y) = \gamma.$$

Proof. The statement follows from the Intermediate Value Theorem, since the map

$$a \mapsto \eta_u^a(x, y)$$

is continuous and, by (1),

$$\liminf_{a \downarrow y/L} \eta_u^a(x, y) = \liminf_{a \downarrow y/L} \frac{1}{a} MRS(x, y/a) = +\infty,$$

and

$$\limsup_{a \to +\infty} \eta_u^a(x, y) = \limsup_{a \to +\infty} \frac{1}{a} MRS(x, y/a) = 0. \quad ■$$

Lemma 3. *Given $u \in \mathscr{U}$, a linear $T \in \mathscr{T}$, and $e > 0$, there exists $a > 0$ such that $y^u(a, T) = e$.*

Proof. Choose $u \in \mathscr{U}$, a linear $T \in \mathscr{T}$, and $e > 0$. Let $t \in [0, 1)$ represent the marginal tax rate for the linear tax T. Note that, for $a > 0$ with $a > e/L$, the condition

$$\eta_u^a(e - T(e), e/a) = 1 - t$$

is sufficient for e to solve the problem

$$\max_{y \in [0, aL]} u(y - T(y), y/a).$$

By Lemma 2, there exists $a > 0$ such that

$$a > e/L \quad \text{and} \quad \eta_u^a(e - T(e), e/a) = 1 - t,$$

implying that $y^u(a, T) = e$. ■

Lemma 4. *Given $u \in \mathscr{U}$ and a linear $T \in \mathscr{T}$, the map $a \mapsto y^u(a, T)$ is continuous.*

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[16] Indeed, given that $t_1 < \cdots < t_M$,

$$1 - t_m < \eta_a^u(y^u(a, T) - T(y^u(a, T)), y^a(a, T))$$

implies $y^u(a, T) < e_m$ and

$$\eta_a^u(y^u(a, T) - T(y^u(a, T)), y^u(a, T)) < 1 - t_{m+1}$$

implies $y^u(a, T) > e_m$.

Proof. Fix $u \in \mathscr{U}$ and a linear $T \in \mathscr{T}$. Note that $y^u(a,T)$ is the unique solution to the problem

$$\max_{y\in[0,aL]} u(y - T(y), y/a),$$

and so the map $a \mapsto y^u(a,T)$ is continuous by the Maximum Theorem. ■

Lemma 5. *Suppose that $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations. Then a tax schedule $T \in \mathscr{T}$ is $\mathbb{U}'$-ir if and only if*

$$\frac{x^{u_1}(a_1,T)}{x^{u_1}(a_1,0)} \geq \cdots \geq \frac{x^{u_n}(a_n,T)}{x^{u_n}(a_n,0)} \tag{17}$$

for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1},\ldots,x^{u_n})$.

Proof. Suppose that (17) holds for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1},\ldots,x^{u_n})$. We must show that T is $\mathbb{U}'$-ir, i.e., that

$$(x^{u_1}(a_1,T),\ldots,x^{u_n}(a_n,T)) \succcurlyeq_L (x^{u_1}(a_1,0),\ldots,x^{u_n}(a_n,0))$$

for each wage rate distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of income functions $(x^{u_1},\ldots,x^{u_n})$, and every vector of utility functions $(u_1,\ldots,u_n) \in \mathbb{U}'$. But this follows from Marshall et al. (1967, Theorem 2.4), since $x^{u_1}(a_1,0) > 0$ (by condition (v)), and

$$x^{u_1}(a_1,T) \leq \cdots \leq x^{u_n}(a_n,T) \quad \text{and} \quad x^{u_1}(a_1,0) \leq \cdots \leq x^{u_n}(a_n,0)$$

(by Lemma 1).

We now prove the contrapositive of the converse assertion. Suppose that, for some $i < n$,

$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} < \frac{x^{u_{i+1}}(a',T)}{x^{u_{i+1}}(a',0)}, \quad \text{for some } a' \geq a > 0,$$

and some vectors $(x^{u_1},\ldots,x^{u_n})$ and $(u_1,\ldots,u_n) \in \mathbb{U}'$. We must show that T is not $\mathbb{U}'$-ir.

For the wage distribution

$$(a_1^*,\ldots,a_n^*), \quad \text{where } a_j^* = a \text{ for } j \leq i \text{ and } a_j^* = a' \text{ for } j > i,$$

and the preference profile

$$(u_1^*,\ldots,u_i^*,u_{i+1}^*,\ldots,u_n^*) = (u_i,\ldots,u_i,u_{i+1},\ldots,u_{i+1})$$

(which, being a simple transformation of $(u_1,\ldots,u_n)$, is an element of $\mathbb{U}'$), we have

$$\frac{x^{u_1^*}(a_1^*,0)}{x^{u_1^*}(a_1^*,T)} = \cdots = \frac{x^{u_i^*}(a_i^*,0)}{x^{u_i^*}(a_i^*,T)}$$

$$= \frac{x^{u_i}(a,0)}{x^{u_i}(a,T)} > \frac{x^{u_{i+1}}(a',0)}{x^{u_{i+1}}(a',T)} = \frac{x^{u_{i+1}^*}(a_{i+1}^*,0)}{x^{u_{i+1}^*}(a_{i+1}^*,T)} \cdots = \frac{x^{u_n^*}(a_n^*,0)}{x^{u_n^*}(a_n^*,T)}. \tag{18}$$

Applying Theorem 2.4 in Marshall et al. (1967), one obtains

$$(x^{u_1^*}(a_1^*,0),\ldots,x^{u_n^*}(a_n^*,0)) \succcurlyeq_L (x^{u_1^*}(a_1^*,T),\ldots,x^{u_n^*}(a_n^*,T)). \tag{19}$$

If

$$(x^{u_1^*}(a_1^*,0),\ldots,x^{u_n^*}(a_n^*,0)) \succ_L (x^{u_1^*}(a_1^*,T),\ldots,x^{u_n^*}(a_n^*,T)), \tag{20}$$

then T is not $\mathbb{U}'$-ir and the proof is complete.

To see that (20) holds, consider the following two (exhaustive) cases:

Case 1.

$$\frac{x^{u_1^*}(a_1^*,0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,0)} > \frac{x^{u_1^*}(a_1^*,T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,T)}.$$

In this case, (19) implies (20).

Case 2.

$$\frac{x^{u_1^*}(a_1^*,0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,0)} = \frac{x^{u_1^*}(a_1^*,T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,T)}.$$

In this case, since

$$x^{u_1^*}(a_1^*,0) = \cdots = x^{u_i^*}(a_i^*,0) \quad \text{and} \quad x^{u_1^*}(a_1^*,T) = \cdots = x^{u_i^*}(a_i^*,T),$$

it follows that

$$\frac{\sum_{\iota=1}^{i} x^{u_\iota^*}(a_\iota^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} = \frac{i x^{u_1^*}(a_1^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} = \frac{i x^{u_1^*}(a_1^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} = \frac{\sum_{\iota=1}^{i} x^{u_\iota^*}(a_\iota^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} \tag{21}$$

and

$$\frac{x^{u_i^*}(a_i^*,0)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,0)} = \frac{x^{u_i^*}(a_i^*,T)}{\sum_\iota x^{u_\iota^*}(a_\iota^*,T)}.$$

Note that the last equality, together with the inequality in (18), implies

$$\frac{x^{u_{i+1}^*}(a_{i+1}^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)} > \frac{x^{u_{i+1}^*}(a_{i+1}^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)}.$$

This inequality, together with (21), implies

$$\frac{\sum_{\iota=1}^{i+1} x^{u_\iota^*}(a_\iota^*,0)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,0)} < \frac{\sum_{\iota=1}^{i+1} x^{u_\iota^*}(a_\iota^*,T)}{\sum_{\iota=1}^{n} x^{u_\iota^*}(a_\iota^*,T)},$$

which contradicts (19).

Hence, (20) holds and the proof is complete. ■

Lemma 6. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Rightarrow T \in \mathscr{T}_{\text{m-prog}}.$$

Proof. For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$, suppose that $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$. Because $\mathbb{U}'$ is closed under simple transformations, $(u_1,\ldots,u_n) \in \mathbb{U}'$ implies $(u_1,\ldots,u_1) \in \mathbb{U}'$, and so T is inequality-reducing with respect to $\{(u_1,\ldots,u_1)\}$ whenever $(u_1,\ldots,u_n) \in \mathbb{U}'$. Applying Theorem 1 in Carbonell-Nicolau and Llavador (2018) gives $T \in \mathscr{T}_{\text{m-prog}}$. ■

Lemma 7. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Rightarrow \mathbb{U}' \subseteq \mathbb{U}_T.$$

Proof. Suppose that $\mathbb{U}' \not\subseteq \mathbb{U}_T$. Then there exists $(u_1,\ldots,u_n)$ in $\mathbb{U}' \setminus \mathbb{U}_T$. It will be shown that $T \notin \mathscr{T}_{\mathbb{U}'\text{-ir}}$. Note that, by Lemma 5, it suffices to show that there exist a wage distribution

$$0 < \alpha_1 \leq \cdots \leq \alpha_n$$

and a preference vector

$$(v_1,\ldots,v_n) \in \mathbb{U}'$$

such that

$$\frac{x^{v_i}(\alpha_i,T)}{x^{v_i}(\alpha_i,0)} < \frac{x^{v_{i+1}}(\alpha_{i+1},T)}{x^{v_{i+1}}(\alpha_{i+1},0)}, \quad \text{some } i < n.$$

Since $(u_1,\ldots,u_n) \in \mathbb{U}' \setminus \mathbb{U}_T$, either

$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} < \frac{x^{u_{i+1}}(a,T)}{x^{u_{i+1}}(a,0)}, \quad \text{some } i \text{ and } a, \tag{22}$$

or

$$\zeta^{u_i}((1-t_m)a, b_m) > \zeta^{u_i}(a,0),$$
$$\text{some } i,\ a,\ m \text{ such that } y^{u_i}((1-t_m)a, b_m) \in [e_{m-1}, e_m]. \tag{23}$$

Suppose first that (22) holds. Define the wage distribution

$$(\alpha_1, \dots, \alpha_n) = (a, \dots, a)$$

and the vector of utility functions

$$\boldsymbol{v} = (v_1, \dots, v_n) = (u_i, \dots, u_i, u_{i+1}, \dots, u_{i+1}).$$

Note that, because $(u_1, \dots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $\boldsymbol{v} \in \mathbb{U}'$.

The above definitions, together with (22), yield

$$\frac{x^{v_i}(\alpha_i, T)}{x^{v_i}(\alpha_i, 0)} = \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_{i+1}}(a, T)}{x^{u_{i+1}}(a, 0)} = \frac{x^{v_{i+1}}(\alpha_{i+1}, T)}{x^{v_{i+1}}(\alpha_{i+1}, 0)},$$

as we sought.

Next, suppose that (23) holds. Then the map

$$\beta \mapsto \frac{x^{u_i}(\beta, T_m)}{x^{u_i}(\beta, 0)} \tag{24}$$

is strictly increasing at a.[17]

Suppose first that

$$y^{u_i}((1 - t_m)a, b_m) \in [e_{m-1}, e_m).$$

Since the map in (24) is strictly increasing at a, for any $a' > a$ close enough to a, we have

$$\frac{x^{u_i}(a, T_m)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T_m)}{x^{u_i}(a', 0)}. \tag{25}$$

By Lemma 1,

$$y^{u_i}(a', T_m) \geq y^{u_i}(a, T_m),$$

and so, by continuity of the map $\beta \mapsto y^{u_i}(\beta, T_m)$ (Lemma 4), we have, for $a' > a$ close enough to a,

$$e_{m-1} \leq y^{u_i}(a, T_m) \leq y^{u_i}(a', T_m) < e_m, \tag{26}$$

where the first and last inequalities follow from (23). Note that (26) implies that

$$y^{u_i}(a, T_m) = y^{u_i}(a, T) \quad \text{and} \quad y^{u_i}(a', T_m) = y^{u_i}(a', T),$$

whence

$$x^{u_i}(a, T_m) = x^{u_i}(a, T) \quad \text{and} \quad x^{u_i}(a', T_m) = x^{u_i}(a', T).$$

Hence, (25) yields

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)}. \tag{27}$$

Define the wage distribution

$$(\alpha_1, \dots, \alpha_n) = (a, a', \dots, a')$$

and the vector of utility functions

$$\boldsymbol{v} = (v_1, \dots, v_n) = (u_i, \dots, u_i).$$

Note that, because $(u_1, \dots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $\boldsymbol{v} \in \mathbb{U}'$.

The above definitions, together with (27), yield

$$\frac{x^{v_1}(\alpha_1, T)}{x^{v_1}(\alpha_1, 0)} = \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} < \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)} = \frac{x^{v_2}(\alpha_2, T)}{x^{v_2}(\alpha_2, 0)},$$

as we sought.

It remains to consider the case when

$$y^{u_i}((1 - t_m)a, b_m) = e_m.$$

Since the map in (24) is strictly increasing at a, for any $a' < a$ close enough to a, we have

$$\frac{x^{u_i}(a, T_m)}{x^{u_i}(a, 0)} > \frac{x^{u_i}(a', T_m)}{x^{u_i}(a', 0)}. \tag{28}$$

By Lemma 1,

$$y^{u_i}(a', T_m) \leq y^{u_i}(a, T_m),$$

and so, by continuity of the map $\beta \mapsto y^{u_i}(\beta, T_m)$ (Lemma 4), we have, for $a' < a$ close enough to a,

$$e_m = y^{u_i}(a, T_m) \geq y^{u_i}(a', T_m) > e_{m-1}. \tag{29}$$

Note that (29) implies that

$$y^{u_i}(a, T_m) = y^{u_i}(a, T) \quad \text{and} \quad y^{u_i}(a', T_m) = y^{u_i}(a', T),$$

whence

$$x^{u_i}(a, T_m) = x^{u_i}(a, T) \quad \text{and} \quad x^{u_i}(a', T_m) = x^{u_i}(a', T).$$

Hence, (28) yields

$$\frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} > \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)}. \tag{30}$$

Define the wage distribution

$$(\alpha_1, \dots, \alpha_n) = (a', a, \dots, a)$$

and the vector of utility functions

$$\boldsymbol{v} = (v_1, \dots, v_n) = (u_i, \dots, u_i).$$

Note that, because $(u_1, \dots, u_n) \in \mathbb{U}'$, and since $\mathbb{U}'$ is closed under simple transformations, we have $\boldsymbol{v} \in \mathbb{U}'$. Moreover,

$$\frac{x^{v_1}(\alpha_1, T)}{x^{v_1}(\alpha_1, 0)} = \frac{x^{u_i}(a', T)}{x^{u_i}(a', 0)} < \frac{x^{u_i}(a, T)}{x^{u_i}(a, 0)} = \frac{x^{v_2}(\alpha_2, T)}{x^{v_2}(\alpha_2, 0)},$$

where the inequality follows from (30). ∎

Lemma 8. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \text{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

to show that

Proof. Suppose that $T \in \mathscr{T}_{\text{m-prog}}$ and $\mathbb{U}' \subseteq \mathbb{U}_T$. By Lemma 5, it suffices

$$\frac{x^{u_1}(a_1, T)}{x^{u_1}(a_1, 0)} \geq \cdots \geq \frac{x^{u_n}(a_n, T)}{x^{u_n}(a_n, 0)} \tag{31}$$

for each ability distribution $0 < a_1 \leq \cdots \leq a_n$, every vector of utility functions $(u_1, \dots, u_n) \in \mathbb{U}'$, and every vector of income functions $(x^{u_1}, \dots, x^{u_n})$.

Choose $0 < a_1 \leq \cdots \leq a_n$, $(u_1, \dots, u_n) \in \mathbb{U}'$, and $(x^{u_1}, \dots, x^{u_n})$.

The proof proceeds by induction on the number of brackets for T.

Suppose first that T is linear. By condition (II), we have, for each i,

[17] Indeed, $\zeta^{u_i}((1 - t_m)a, b_m) \leq \zeta^{u_i}(a, 0)$ if and only if the map $\beta \mapsto \frac{x^{u_i}(\beta, T_m)}{x^{u_i}(\beta, 0)}$ is nonincreasing at point a.

$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} \geq \frac{x^{u_i}(a',T)}{x^{u_i}(a',0)}, \quad \text{whenever } a' \geq a.$$

By condition (I), we have for each $i < n$,

$$\frac{x^{u_i}(a,T)}{x^{u_i}(a,0)} \geq \frac{x^{u_{i+1}}(a,T)}{x^{u_{i+1}}(a,0)}, \quad \text{for all } a > 0. \tag{32}$$

Hence, for each $i < n$,

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_i}(a_{i+1},T)}{x^{u_i}(a_{i+1},0)} \geq \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)},$$

implying (31), as we sought.

Now suppose that the lemma has been proven for any m-bracket tax schedule, where $m \in \{1,\ldots,M-1\}$, for some $M > 1$. It will be shown that the lemma is also true for an M-bracket tax schedule.

Suppose that T is an M-bracket tax schedule. Because T is piecewise linear in $\mathscr{T}_{\text{m-prog}}$, there exist

$$0 = e_0 < e_1 < \cdots < e_M = +\infty$$

and intervals

$$I_1 = [e_0, e_1], \ldots, I_M = [e_{M-1}, e_M),$$

satisfying the following: for each m, there exist $b_m \geq 0$ and $t_m \in [0,1)$ such that $T(y) = -b_m + t_m y$ for all $y \in I_m$, and

$$b_1 < \cdots < b_M \quad \text{and} \quad t_1 < \cdots < t_M.$$

For $m \in \{1,\ldots,M\}$, let

$$T_m(y) = -b_m + t_m y.$$

Because $(u_1,\ldots,u_n) \in \mathbb{U}$, we have

$$\begin{aligned} x^{u_1}(a_1,T_1) &= b_1 + (1-t_1)y^{u_1}(a_1,T_1) \leq \cdots \leq x^{u_n}(a_n,T_1) \\ &= b_1 + (1-t_1)y^{u_n}(a_n,T_1). \end{aligned}$$

Let i_1 be the largest i for which

$$x^{u_i}(a_i,T_1) \leq b_1 + (1-t_1)e_1.$$

Then

$$y^{u_1}(a_1,T_1) \leq \cdots \leq y^{u_{i_1}}(a_{i_1},T_1) \leq e_1,$$

and so

$$x^{u_i}(a_i,T_1) = x^{u_i}(a_i,T), \quad \text{for each } i \in \{1,\ldots,i_1\},$$

since T and T_1 coincide on $[e_0,e_1]$. Because T_1 is linear, the induction hypothesis implies that $T_1 \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_1}(a_1,T)}{x^{u_1}(a_1,0)} = \frac{x^{u_1}(a_1,T_1)}{x^{u_1}(a_1,0)} \geq \cdots \geq \frac{x^{u_{i_1}}(a_{i_1},T_1)}{x^{u_{i_1}}(a_{i_1},0)} = \frac{x^{u_{i_1}}(a_{i_1},T)}{x^{u_{i_1}}(a_{i_1},0)}. \tag{33}$$

Next, let T^* be defined as follows:

$$T^*(y) = \begin{cases} T(y) & \text{if } y \geq e_1, \\ T_2(y) & \text{if } y < e_1. \end{cases}$$

It is easy to see that T is an $M-1$-bracket tax schedule in $\mathscr{T}_{\text{m-prog}}$. Consequently, the induction hypothesis gives $T^* \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_1}(a_1,T^*)}{x^{u_1}(a_1,0)} \geq \cdots \geq \frac{x^{u_n}(a_n,T^*)}{x^{u_n}(a_n,0)}. \tag{34}$$

Now let i_2 be the smallest $i \geq i_1$ for which

$$x^{u_i}(a_i,T) \geq b_2 + (1-t_2)e_1.$$

Then Lemma 1 implies that

$$e_1 \leq y^{u_{i_2}}(a_{i_2},T) \leq \cdots \leq y^{u_n}(a_n,T).$$

and so

$$x^{u_i}(a_i,T^*) = x^{u_i}(a_i,T), \quad \text{for each } i \in \{i_2,\ldots,n\},$$

since T^* and T coincide on $[e_1,+\infty)$. Consequently, (34) gives

$$\frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)} \geq \cdots \geq \frac{x^{u_n}(a_n,T)}{x^{u_n}(a_n,0)}. \tag{35}$$

Note that the definition of i_1 and i_2 entails

$$i_1 \leq i_2 \leq i_1 + 1,$$

and so, in light of (33) and (35), the proof will be complete if we show that

$$\frac{x^{u_{i_1}}(a_{i_1},T)}{x^{u_{i_1}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)}. \tag{36}$$

Using (32), we see that

$$\frac{x^{u_{i_1}}(a_{i_1},T)}{x^{u_{i_1}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)}. \tag{37}$$

Consequently, it suffices to show that

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)}, \tag{38}$$

since this inequality, combined with (37), gives (36).

Since $a_{i_1} \leq a_{i_2}$, we have

$$y^{u_{i_2}}(a_{i_1},T) \leq y^{u_{i_2}}(a_{i_2},T)$$

(see Lemma 1).

Let m be the bracket for the gross income $y^{u_{i_2}}(a_{i_2},T)$, i.e.,

$$y^{u_{i_2}}(a_{i_2},T) \in [e_{m-1},e_m].$$

Suppose first that there is no m' such that

$$y^{u_{i_2}}(a_{i_1},T) \leq e_{m'} \leq y^{u_{i_2}}(a_{i_2},T).$$

In this case,

$$e_{m-1} < y^{u_{i_2}}(a_{i_1},T) = y^{u_{i_2}}(a_{i_1},T_m) \leq y^{u_{i_2}}(a_{i_2},T_m) = y^{u_{i_2}}(a_{i_2},T) < e_m.$$

Since T_m is linear, we know that $T_m \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies (38).[18]

Now suppose that there are exactly k thresholds

$$y^{u_{i_2}}(a_{i_1},T) \leq e_{m_1} < \cdots < e_{m_k} \leq y^{u_{i_2}}(a_{i_2},T) \tag{39}$$

between $y^{u_{i_2}}(a_{i_1},T)$ and $y^{u_{i_2}}(a_{i_2},T)$, for some $k \in \{1,\ldots,M-1\}$. Suppose further that (38) has been proven when the number of thresholds between $y^{u_{i_2}}(a_{i_1},T)$ and $y^{u_{i_2}}(a_{i_2},T)$ is less than k. It will be shown that (38) holds.

First, we show that there exist $\alpha \leq \alpha'$ such that

$$y^{u_{i_2}}(\alpha,T_{m_1}) = e_{m_1} = y^{u_{i_2}}(\alpha',T_{m_1+1}). \tag{40}$$

The existence of α and α' satisfying (40) follows from Lemma 3. To see that $\alpha \leq \alpha'$, it suffices to observe that

$$\begin{aligned} \eta^{\alpha}_{u_{i_2}}(x^{u_{i_2}}(\alpha,T_{m_1}), y^{u_{i_2}}(\alpha,T_{m_1})) &= 1 - t_{m_1} > 1 - t_{m_1+1} \\ &= \eta^{\alpha'}_{u_{i_2}}(x^{u_{i_2}}(\alpha',T_{m_1+1}), y^{u_{i_2}}(\alpha',T_{m_1+1})), \end{aligned}$$

implying that

$$y^{u_{i_2}}(\alpha,T_{m_1}) < y^{u_{i_2}}(\alpha',T_{m_1}),$$

whence $\alpha \leq \alpha'$ (by Lemma 1).

Next, observe that

$$y^{u_{i_2}}(\beta,T) = e_{m_1}, \quad \text{for all } \beta \in [\alpha,\alpha']. \tag{41}$$

Indeed, (40) implies that

$$y^{u_{i_2}}(\alpha,T) = e_{m_1} = y^{u_{i_2}}(\alpha',T),$$

[18] This can be seen by applying Lemma 5 to the wage rate distribution $(a_{i_1}, a_{i_2}, \ldots, a_{i_2})$ and the preference vector $(u_{i_2}, \ldots, u_{i_2})$.

and so Lemma 1 implies (41).

Note that, since $x^{u_{i_2}}(\alpha,0) \leq x^{u_{i_2}}(\alpha',0)$ (by Lemma 1), (41) implies that

$$\frac{x^{u_{i_2}}(\alpha,T)}{x^{u_{i_2}}(\alpha,0)} = \frac{(1-t_{m_1})e_{m_1}+b_{m_1}}{x^{u_{i_2}}(\alpha,0)} \geq \frac{(1-t_{m_1})e_{m_1}+b_{m_1}}{x^{u_{i_2}}(\alpha',0)} = \frac{x^{u_{i_2}}(\alpha',T)}{x^{u_{i_2}}(\alpha',0)}. \quad (42)$$

We are now ready to prove (38). First, consider the case when

$$a_{i_1} \leq \alpha \leq \alpha' \leq a_{i_2}. \quad (43)$$

Since $a_{i_1} \leq \alpha$, we have

$$e_{m_1-1} < y^{u_{i_2}}(a_{i_1},T) \leq y^{u_{i_2}}(\alpha,T) = e_{m_1},$$

implying that

$$y^{u_{i_2}}(a_{i_1},T) = y^{u_{i_2}}(a_{i_1},T_{m_1}) \quad \text{and} \quad y^{u_{i_2}}(\alpha,T) = y^{u_{i_2}}(\alpha,T_{m_1}).$$

Since T_{m_1} is linear, we know that $T_{m_1} \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, and so Lemma 5 implies that

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\alpha,T)}{x^{u_{i_2}}(\alpha,0)}. \quad (44)$$

Combining this inequality with (42) gives

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\alpha',T)}{x^{u_{i_2}}(\alpha',0)}. \quad (45)$$

If $\alpha' = a_{i_2}$, we see that (38) holds.

It remains to consider the case when $\alpha' < a_{i_2}$. If

$$x^{u_{i_2}}(\alpha',T) = x^{u_{i_2}}(a_{i_2},T),$$

then

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\alpha',T)}{x^{u_{i_2}}(\alpha',0)} = \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(\alpha',0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)},$$

where the first inequality uses (45) and the last inequality follows from the inequality

$$x^{u_{i_2}}(\alpha',0) \leq x^{u_{i_2}}(a_{i_2},0),$$

which is implied by Lemma 1. Thus, (38) holds.

Now suppose that

$$x^{u_{i_2}}(\alpha',T) < x^{u_{i_2}}(a_{i_2},T).$$

By Lemma 3, there exists $\beta \in (\alpha', a_{i_2})$ close enough to α' such that

$$y^{u_{i_2}}(\alpha',T) = e_{m_1} < y^{u_{i_2}}(\beta,T) < e_{m_1+1}. \quad (46)$$

Note that

$$y^{u_{i_2}}(\alpha',T) = y^{u_{i_2}}(\alpha',T_{m_1+1}) \quad \text{and} \quad y^{u_{i_2}}(\beta,T) = y^{u_{i_2}}(\beta,T_{m_1+1}).$$

Since T_{m_1+1} is linear, we obtain

$$\frac{x^{u_{i_2}}(\alpha',T)}{x^{u_{i_2}}(\alpha',0)} \geq \frac{x^{u_{i_2}}(\beta,T)}{x^{u_{i_2}}(\beta,0)}.$$

Combining this inequality with (45) gives

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\beta,T)}{x^{u_{i_2}}(\beta,0)}. \quad (47)$$

Next, note that (39) and (46) imply

$$e_{m_1} < y^{u_{i_2}}(\beta,T) \leq e_{m_1+1} < \cdots < e_{m_k} \leq y^{u_{i_2}}(a_{i_2},T),$$

i.e., the number of thresholds between $y^{u_{i_2}}(\beta,T)$ and $y^{u_{i_2}}(a_{i_2},T)$ is less than k. Consequently,

$$\frac{x^{u_{i_2}}(\beta,T)}{x^{u_{i_2}}(\beta,0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)}.$$

This inequality, together with (47), gives (38), as we sought.

Next, consider the case when

$$a_{i_1} \leq a_{i_2} \leq \alpha.$$

Then

$$e_{m_1-1} < y^{u_{i_2}}(a_{i_1},T) \leq y^{u_{i_2}}(a_{i_2},T) \leq y^{u_{i_2}}(\alpha,T) = e_{m_1},$$

implying that

$$y^{u_{i_2}}(a_{i_1},T) = y^{u_{i_2}}(a_{i_1},T_{m_1}) \quad \text{and} \quad y^{u_{i_2}}(a_{i_2},T) = y^{u_{i_2}}(a_{i_2},T_{m_1}).$$

Since T_{m_1} is linear, we obtain (38).

It remains to consider the case when

$$a_{i_1} \leq \alpha \leq a_{i_2} \leq \alpha'.$$

This implies (44), as in the first case (43). To see that (38) holds, note that

$$\frac{x^{u_{i_2}}(a_{i_1},T)}{x^{u_{i_2}}(a_{i_1},0)} \geq \frac{x^{u_{i_2}}(\alpha,T)}{x^{u_{i_2}}(\alpha,0)} = \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(\alpha,0)} \geq \frac{x^{u_{i_2}}(a_{i_2},T)}{x^{u_{i_2}}(a_{i_2},0)},$$

where the first inequality uses (44), the equality follows from (41), and the last inequality is a consequence of the inequality $x^{u_{i_2}}(\alpha,0) \leq x^{u_{i_2}}(a_{i_2},0)$ (which follows from Lemma 1). ■

We are now ready to prove Theorem 1.

Theorem 1. *For $\mathbb{U}' \subseteq \mathbb{U}$, where $\mathbb{U}'$ is closed under simple transformations, and $T \in \mathscr{T}$,*

$$T \in \mathscr{T}_{\mathbb{U}'\text{-ir}} \Leftrightarrow \left[T \in \mathscr{T}_{\text{m-prog}} \quad \textit{and} \quad \mathbb{U}' \subseteq \mathbb{U}_T\right].$$

Proof. The equivalence is an immediate consequence of Lemma 6, Lemma 7, and Lemma 8. ■

B.4. Proof of Theorem 2

Theorem 2. *If $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations, then $\mathscr{T}_{\mathbb{U}'\text{-ir}} = \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.*

Proof. Suppose that $\mathbb{U}' \subseteq \mathbb{U}$ is closed under simple transformations. First, we prove the containment $\mathscr{T}_{\mathbb{U}'\text{-ir}} \subseteq \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Pick $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$, $0 < a_1 \leq \cdots \leq a_n$, $(u_1,\ldots,u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1},\ldots,x^{u_n})$.

First, suppose that n is odd. Let a_m denote the median ability level. For $i < m$, we have

$$\begin{aligned}
&\frac{1}{x^{u_m}(a_m,T)}(x^{u_m}(a_m,T) - x^{u_i}(a_i,T)) \\
&= \frac{1}{x^{u_m}(a_m,0)}\left[x^{u_m}(a_m,0) - \left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \cdot \frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0)\right] \\
&\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)),
\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_i \leq a_m$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Similarly, for $i > m$, we have

$$\begin{aligned}
&\frac{1}{x^{u_m}(a_m,T)}(x^{u_i}(a_i,T) - x^{u_m}(a_m,T)) \\
&= \frac{1}{x^{u_m}(a_m,0)}\left[\left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \cdot \frac{x^{u_m}(a_m,0)}{x^{u_m}(a_m,T)}\right)x^{u_i}(a_i,0) - x^{u_m}(a_m,0)\right] \\
&\leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0) - x^{u_m}(a_m,0)),
\end{aligned}$$

where the last inequality follows from the inequality

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \leq \frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)},$$

which holds because $a_m \leq a_i$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Because

$$\frac{1}{x^{u_m}(a_m,T)}(x^{u_m}(a_m,T) - x^{u_i}(a_i,T)) \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_m}(a_m,0) - x^{u_i}(a_i,0)), \text{ for } i < n,$$

$$\frac{1}{x^{u_m}(a_m,T)}(x^{u_i}(a_i,T)-x^{u_m}(a_m,T)) \leq \frac{1}{x^{u_m}(a_m,0)}(x^{u_i}(a_i,0)-x^{u_m}(a_m,0)),\ \text{for } i>n,$$

we see that

$$\frac{x^{u_m}(a_m,T)}{x^{u_m}(a_m,0)}(x^{u_1}(a_1,0),\dots,x^{u_n}(a_n,0)) \succcurlyeq_{FW} (x^{u_1}(a_1,T),\dots,x^{u_n}(a_n,T)),$$

whence

$$(x^{u_1}(a_1,0),\dots,x^{u_n}(a_n,0)) \succcurlyeq_{FW} (x^{u_1}(a_1,T),\dots,x^{u_n}(a_n,T)).$$

Consequently, $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Next, suppose that n is even. Let

$$m = m(x^{u_1}(a_1,0),\dots,x^{u_n}(a_n,0)) = \frac{x^{u_{n/2}}(a_{n/2},0)+x^{u_{(n/2)+1}}(a_{(n/2)+1},0)}{2}$$

and

$$m' = m(x^{u_1}(a_1,T),\dots,x^{u_n}(a_n,T)) = \frac{x^{u_{n/2}}(a_{n/2},T)+x^{u_{(n/2)+1}}(a_{(n/2)+1},T)}{2}.$$

For $i \leq n/2$, we have

$$\frac{1}{m'}(m'-x^{u_i}(a_i,T)) = \frac{1}{m}\left[m-\left(\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)}\frac{m}{m'}\right)x^{u_i}(a_i,0)\right] \leq \frac{1}{m}(m-x^{u_i}(a_i,0)),$$

where the last inequality follows from the inequalities

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{n/2}}(a_{n/2},T)}{x^{u_{n/2}}(a_{n/2},0)} \geq \frac{m'}{m};$$

the first inequality holds because $a_i \leq a_{n/2}$ and T is $\mathbb{U}'$-ir (see Lemma 5); the second inequality is expressible as

$$\frac{x^{u_{n/2}}(a_{n/2},T)}{x^{u_{n/2}}(a_{n/2},0)} \geq \frac{x^{u_{(n/2)+1}}(a_{(n/2)+1},T)}{x^{u_{(n/2)+1}}(a_{(n/2)+1},0)},$$

which holds because $a_{n/2} \leq a_{(n/2)+1}$ and T is $\mathbb{U}'$-ir (see Lemma 5).

Because

$$\frac{1}{m'}(m'-x^{u_i}(a_i,T)) \leq \frac{1}{m}(m-x^{u_i}(a_i,0)),\quad \text{for } i \leq n/2,$$
$$\frac{1}{m'}(x^{u_i}(a_i,T)-m') \leq \frac{1}{m}(x^{u_i}(a_i,0)-m),\quad \text{for } i \geq (n/2)+1,$$

we have

$$(x^{u_1}(a_1,0),\dots,x^{u_n}(a_n,0)) \succcurlyeq_{FW} (x^{u_1}(a_1,T),\dots,x^{u_n}(a_n,T)).$$

Consequently, $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

It remains to prove the containment $\mathscr{T}_{\mathbb{U}'\text{-ir}} \supseteq \mathscr{T}_{\mathbb{U}'\text{-bpr}}$.

Choose $T \in \mathscr{T}_{\mathbb{U}'\text{-bpr}}$, $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathbb{U}'$, and a vector of income functions $(x^{u_1},\dots,x^{u_n})$.

First, suppose that n is odd. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \leq \cdots \leq a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \leq i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \geq i+1.$$

Note that

$$a'_{m-1} = a_i \leq a'_m \leq a_{i+1} = a'_{m+1},$$

where a'_m represents the median ability level. Moreover, either $a'_m = a_i$ or $a'_m = a_{i+1}$. Suppose that $a'_m = a_{i+1}$ (the other case can be handled similarly).

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1,\dots,u'_n)$, where

$$u'_j = u_i, \qquad \text{for each } j \leq i,$$
$$u'_j = u_{i+1}, \qquad \text{for each } j \geq i+1,$$

is a member of $\mathbb{U}'$.

Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{x^{u'_m}(a'_m,T)}\sum_{j=i}^{m}(x^{u'_m}(a'_m,T)-x^{u'_j}(a'_j,T)) \leq \frac{1}{x^{u'_m}(a'_m,0)}\sum_{j=i}^{m}(x^{u'_m}(a'_m,0)-x^{u'_j}(a'_j,0)). \tag{48}$$

Since $a'_m = a_{i+1}$, $a'_j = a_{i+1}$, and $u'_j = u_{i+1}$ for $j \geq i+1$, we have

$$\sum_{j=i+1}^{m}(x^{u'_m}(a'_m,T)-x^{u'_j}(a'_j,T)) = 0 \quad \text{and} \quad \sum_{j=i+1}^{m}(x^{u'_m}(a'_m,0)-x^{u'_j}(a'_j,0)) = 0.$$

Consequently, (48) can be expressed as

$$\frac{1}{x^{u'_m}(a'_m,T)}(x^{u'_m}(a'_m,T)-x^{u'_i}(a'_i,T)) \leq \frac{1}{x^{u'_m}(a'_m,0)}(x^{u'_m}(a'_m,0)-x^{u'_i}(a'_i,0)),$$

whence

$$\frac{x^{u'_i}(a'_i,T)}{x^{u'_i}(a'_i,0)} \geq \frac{x^{u'_m}(a'_m,T)}{x^{u'_m}(a'_m,0)}.$$

Now since $u'_i = u_i$, $u'_m = u_{i+1}$, $a'_i = a_i$, and $a'_m = a_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)}.$$

Since $i < n$ was arbitrary, we see that

$$\frac{x^{u_1}(a_1,T)}{x^{u_1}(a_1,0)} \geq \cdots \geq \frac{x^{u_n}(a_n,T)}{x^{u_n}(a_n,0)}.$$

Since $0 < a_1 \leq \cdots \leq a_n$, $\boldsymbol{u} = (u_1,\dots,u_n) \in \mathbb{U}'$, and $(x^{u_1},\dots,x^{u_n})$ were arbitrary, Lemma 5 implies that $T \in \mathscr{T}_{\mathbb{U}'\text{-ir}}$.

Next, suppose that n is even. Pick $i < n$ and a_i, and define an ability distribution $0 < a'_1 \leq \cdots \leq a'_n$ satisfying

$$a'_j = a_i \text{ for each } j \leq i \quad \text{and} \quad a'_j = a_{i+1} \text{ for each } j \geq i+1.$$

Because $\mathbb{U}'$ is closed under simple transformations, the utility vector $\boldsymbol{u}' = (u'_1,\dots,u'_n)$, where

$$u'_j = u_i, \qquad \text{for each } j \leq i,$$
$$u'_j = u_{i+1}, \qquad \text{for each } j \geq i+1,$$

is a member of $\mathbb{U}'$.

Note that the income distributions

$$(x^{u'_1}(a'_1,T),\dots,x^{u'_n}(a'_n,T)) \quad \text{and} \quad (x^{u'_1}(a'_1,0),\dots,x^{u'_n}(a'_n,0))$$

satisfy

$$x^{u'_1}(a'_1,T) = \cdots = x^{u'_i}(a'_i,T) \leq m' \leq x^{u'_{i+1}}(a'_{i+1},T) = \cdots = x^{u'_n}(a'_n,T), \tag{49}$$

where m' represents the median income for $(x^{u'_1}(a'_1,T),\dots,x^{u'_n}(a'_n,T))$, and

$$x^{u'_1}(a'_1,0) = \cdots = x^{u'_i}(a'_i,0) \leq m \leq x^{u'_{i+1}}(a'_{i+1},0) = \cdots = x^{u'_n}(a'_n,0), \tag{50}$$

where m represents the median income for $(x^{u'_1}(a'_1,0),\dots,x^{u'_n}(a'_n,0))$.

As in the previous case, it suffices to show that

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)}.$$

If

$$x^{u'_1}(a'_1,T) = \cdots = x^{u'_i}(a'_i,T) = m' = x^{u'_{i+1}}(a'_{i+1},T) = \cdots = x^{u'_n}(a'_n,T), \quad (51)$$

then

$$x^{u'_1}(a'_1,0) = \cdots = x^{u'_i}(a'_i,0) = m = x^{u'_{i+1}}(a'_{i+1},0) = \cdots = x^{u'_n}(a'_n,0). \quad (52)$$

Indeed, $x^{u'_i}(a'_i,0) < x^{u'_{i+1}}(a'_{i+1},0)$ implies that $x^{u'_i}(a'_i,T) < x^{u'_{i+1}}(a'_{i+1},T)$, since marginal tax rates are less than unity. Under (51)–(52), we have

$$\frac{x^{u'_i}(a'_i,T)}{x^{u'_i}(a'_i,0)} = \frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} = \frac{m'}{m} = \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)} = \frac{x^{u'_{i+1}}(a'_{i+1},T)}{x^{u'_{i+1}}(a'_{i+1},0)}.$$

If $x^{u'_i}(a'_i,T) < x^{u'_{i+1}}(a'_{i+1},T)$ then $x^{u'_i}(a'_i,0) < x^{u'_{i+1}}(a'_{i+1},0)$. Thus, if $m' = x^{u'_i}(a'_i,T)$ (respectively, $m' = x^{u'_{i+1}}(a'_{i+1},T)$), then $m = x^{u'_i}(a'_i,0)$ (respectively, $m = x^{u'_{i+1}}(a'_{i+1},0)$). We consider the case when $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$ and omit the other case, which can be handled similarly.

Suppose that $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$. Because T is $\mathbb{U}'$-bpr,

$$\frac{1}{m'} \sum_{\frac{n+1}{2} < j \leq i+1} (x^{u'_j}(a'_j,T) - m') \leq \frac{1}{m} \sum_{\frac{n+1}{2} < j \leq i+1} (x^{u'_j}(a'_j,0) - m). \quad (53)$$

Given (49)–(50), and since $m' = x^{u'_i}(a'_i,T)$ and $m = x^{u'_i}(a'_i,0)$, we see that (53) can be expressed as

$$\frac{1}{m'}(x^{u'_{i+1}}(a'_{i+1},T) - m') \leq \frac{1}{m}(x^{u'_{i+1}}(a'_{i+1},0) - m),$$

whence

$$\frac{x^{u'_i}(a'_i,T)}{x^{u'_i}(a'_i,0)} = \frac{m'}{m} \geq \frac{x^{u'_{i+1}}(a'_{i+1},T)}{x^{u'_{i+1}}(a'_{i+1},0)}.$$

Now since $u'_i = u_i$ and $u'_{i+1} = u_{i+1}$, it follows that

$$\frac{x^{u_i}(a_i,T)}{x^{u_i}(a_i,0)} \geq \frac{x^{u_{i+1}}(a_{i+1},T)}{x^{u_{i+1}}(a_{i+1},0)},$$

as we sought. ■

Data availability

No data was used for the research described in the article.

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