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Perfect and limit admissible perfect equilibria in discontinuous games☆

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ABSTRACT

We compare the properties of several notions of trembling-hand perfection within classes of compact, metric, and possibly discontinuous games, and show that in the presence of payoff discontinuities, standard notions of trembling-hand perfection fail a weakening of admissibility termed limit admissibility. We also provide conditions ensuring the existence of a limit admissible perfect equilibrium.

1. Introduction

The notion of trembling-hand perfect equilibrium was introduced by Selten (1975) as a refinement of the Nash equilibrium concept. Nash equilibria survive trembling-hand perfection if they are "good" predictors of equilibrium behavior in some perturbed games in which the players make slight mistakes in the execution of their strategies.

It is well-known that for finite games, trembling-hand perfect equilibria exist in the set of Nash equilibria and put no mass on weakly dominated strategies. In other words, trembling-hand perfect equilibria satisfy existence, (E), and admissibility, (Ad). Moreover, a trembling-hand perfect equilibrium is the limit of a sequence of exact equilibria in perturbed games in which the players are constrained to play completely mixed strategies, (P).

Ideally, one would like an infinite-game extension of the standard solution concept that satisfies (E), (Ad), (P), and reduces to the set of trembling-hand perfect equilibria in finite games, (R). For infinite games, however, properties (E) and (Ad) are not generally compatible. In fact, there are (compact, continuous) games that have a *unique* Nash equilibrium in weakly dominated strategies (e.g., Simon and Stinchcombe, 1995, Example 2.1). However, condition (E) is compatible with a weaker property, termed *limit admissibility* (LA) in Simon and Stinchcombe (1995). This property requires that equilibria put mass only on the limits of weakly undominated strategies. For compact, continuous games, Simon and Stinchcombe (1995) present three infinite-game generalizations of Selten's (1975) trembling-hand perfect equilibrium: *strong perfection* (satisfying the four criteria, (E), (LA), (P), and (R)), *weak perfection* (satisfying all but (P)), and *limit-of-finite perfection* (which fails (LA) but satisfies the remaining three properties). Of the three approaches, it is argued in Simon and Stinchcombe (1995) that strong perfection most closely respects the strategic aspects of infinite games.

Our treatment allows for potential discontinuities in the payoffs of a game. In the presence of discontinuities, condition (E) is not guaranteed by standard arguments. This, in turn, influences (P) (through the potential nonexistence of exact equilibria in perturbations of the original game). Moreover, discontinuities may affect (LA) via their effect on the topological properties of the set of weakly dominated strategies.

Carbonell-Nicolau (2011b) considers an infinite-game extension of the standard notion of perfection and identifies a rich class of possibly discontinuous games for which the set of trembling-hand perfect equilibria is nonempty.[1] The extension considered in Carbonell-Nicolau (2011b) is termed *trembling-hand perfection*. Building on the existence results of Carbonell-Nicolau (2011b), in this paper we compare several notions of trembling-hand perfection in terms of their properties. We first present the equilibrium concepts of Simon and Stinchcombe (1995) and

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[1] The existence of pure-strategy trembling-hand perfect equilibria is addressed in Carbonell-Nicolau (2011c).

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state an analogue of the standard three-way characterization of trembling-hand perfection (e.g. van Damme, 2002, p. 28) for discontinuous games (Theorem 2). We then study a class of possibly discontinuous games (subsuming the set of all compact, metric, continuous games) for which the behavior of the existing concepts of perfection is standard (i.e., as in Simon and Stinchcombe, 1995). The properties of the various solution concepts within this class are summarized in Theorem 4. This result considers two notions of weak domination. *Weak* domination* confines attention to domination by pure strategies, while *weak domination* allows for domination by both pure and mixed strategies. *Limit admissibility** (LA*) is based on weak* domination, while *limit admissibility* (LA) is consistent with weak domination.

In Section 4, we show, by means of an example, that trembling-hand perfection and strong (or weak) perfection fail limit admissibility* (and hence limit admissibility) in rather well-behaved discontinuous games. Thus, as the universe of games is expanded to include larger classes of discontinuous games, the strong approach, which is equivalent to trembling-hand perfection, loses some of its appeal: it ceases to satisfy (LA*) and (LA). It is therefore natural to ask whether there are alternative approaches to trembling-hand perfection (or stronger refinements) that do not suffer from this drawback.

We argue that for the classes of discontinuous games considered in this paper, natural strengthenings of trembling-hand perfection for infinite games are likely to pose existence problems, in the sense that, for the said classes, conditions leading to existence (e.g., better-reply security in perturbed games) fail. By contrast, the set of trembling-hand perfect equilibria minus the interior of the set of weakly dominated (resp. weakly* dominated) strategy profiles, a solution concept that we call *limit admissible perfection* (resp. *limit admissible perfection**), need not be subject to this trade-off: there are rich collections of games for which the set of limit admissible perfect (resp. limit admissible perfect*) equilibria is nonempty. For these collections, limit admissible perfection (resp. limit admissible perfection*) selects a subset of the set of trembling-hand perfect equilibria and, unlike the existing formulations of perfection, meets (LA) (resp. (LA*)) and (E), as well as the other desiderata. These properties, along with those of the other solution concepts, are summarized in Theorems 6 and 7.

2. Preliminaries

A ***metric game*** is a collection

$$G = (X_i, u_i)_{i=1}^N,$$

where N is a finite number of players, each X_i is a nonempty metric space, and each $u_i : X \to \mathbb{R}$ is bounded and Borel measurable, with domain $X := \times_{i=1}^N X_i$. A metric game $G = (X_i, u_i)_{i=1}^N$ such that each X_i is compact is called a ***compact metric game***.

In the sequel, by X_{-i} we mean the set $\times_{j\neq i} X_j$, and, given i, $x_i \in X_i$, and

$$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_{-i},$$

we slightly abuse notation and denote the point $(x_1, \ldots, x_N)$ by (x_i, x_{-i}).

The ***mixed extension*** of G is the game

$$\overline{G} = (M_i, U_i)_{i=1}^N,$$

where each M_i represents the set of Borel probability measures on X_i, endowed with the weak* topology, and $U_i : M \to \mathbb{R}$ is defined by

$$U_i(\mu) := \int_X u_i \mathrm{d}\mu,$$

where $M := \times_{i=1}^N M_i$.

Henceforth, the set $\times_{j\neq i} M_j$ is denoted by M_{-i}, and given i, $\mu_i \in M_i$, and

$$\mu_{-i} = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N) \in M_{-i},$$

we sometimes represent the point $(\mu_1, \ldots, \mu_N)$ as (μ_i, μ_{-i}).

Given $x_i \in X_i$, let δ_{x_i} be the Dirac measure on X_i with support $\{x_i\}$. We sometimes write, by a slight abuse of notation, x_i in place of δ_{x_i}. For $\delta \in [0, 1]$ and $(\mu_i, \nu_i) \in M_i^2$,

$$(1-\delta)\nu_i + \delta\mu_i$$

denotes the member σ_i of M_i for which $\sigma_i(B) = (1-\delta)\nu_i(B) + \delta\mu_i(B)$ for every Borel set $B \subseteq X_i$. When $\nu_i = \delta_{x_i}$ for some $x_i \in X_i$, we sometimes write $(1-\delta)x_i + \delta\mu_i$ for $(1-\delta)\nu_i + \delta\mu_i$. Similarly, given $(\nu, \mu) \in M^2$,

$$(1-\delta)\nu + \delta\mu$$

denotes the point

$$((1-\delta)\nu_1 + \delta\mu_1, \ldots, (1-\delta)\nu_N + \delta\mu_N),$$

where $\nu = (\nu_1, \ldots, \nu_N)$ and $\mu = (\mu_1, \ldots, \mu_N)$.

The following notation shall be used in Section 3 to formally define various notions of trembling-hand perfection.

A Borel probability measure μ_i on X_i is said to be *strictly positive* if $\mu_i(O) > 0$ for every nonempty open set $O \subseteq X_i$.

For each i, let $\widehat{M}_i$ represent the set of all strictly positive members of M_i. Set $\widehat{M} := \times_{i=1}^N \widehat{M}_i$. For $\nu_i \in \widehat{M}_i$ and

$$\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N,$$

define

$$M_i(\delta_i\nu_i) := \{\mu_i \in M_i : \mu_i \geq \delta_i\nu_i\}$$

and $M(\delta\nu) := \times_{i=1}^N M_i(\delta_i\nu_i)$. Given $\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N$ and $\nu = (\nu_1, \ldots, \nu_N) \in \widehat{M}$, the game

$$\overline{G}_{\delta\nu} = (M_i(\delta_i\nu_i), U_i|_{M(\delta\nu)})_{i=1}^N$$

is called a ***Selten perturbation*** of G. We often work with perturbations $\overline{G}_{\delta\nu}$ satisfying $\delta_1 = \cdots = \delta_N$. When referring to these objects, we simply write $\overline{G}_{\delta\nu}$ with $\delta = \delta_1 = \cdots = \delta_N$.

3. Comparison of perfect equilibria

In this section, we introduce several formulations of perfection and compare their properties in families of possibly discontinuous games. Let $\mathfrak{g}_c$ designate the set of compact, metric, and continuous games. We show that the strengths of the various solution concepts within $\mathfrak{g}_c$, as highlighted by Simon and Stinchcombe (1995), need not extend to larger classes of games. In this section, we consider a strict superset of $\mathfrak{g}_c$ for which the existing solution concepts behave as in $\mathfrak{g}_c$. In a subsequent section (Section 4), we point to the following fact: in the presence of discontinuities, the existing concepts fail limit admissibility. We then study refinements of trembling-hand perfection that do not suffer from this limitation.

Definition 1. A strategy profile $x = (x_1, \ldots, x_N) \in X$ is a ***Nash equilibrium*** of G if for each i, $u_i(x) \geq u_i(y_i, x_{-i})$ for every $y_i \in X_i$.

A Nash equilibrium of the mixed extension $\overline{G}$ is called a ***mixed-strategy Nash equilibrium*** of G. By a slight abuse of terminology, we sometimes refer to a mixed-strategy Nash equilibrium of G simply as a Nash equilibrium of G.

We first define trembling-hand perfection in terms of Selten perturbations.

Definition 2. A strategy profile $\mu \in M$ is a ***trembling-hand perfect (thp) equilibrium*** of G if there are sequences (δ^n), (ν^n), and (μ^n) such that $(0, 1)^N \ni \delta^n \to 0$, $\nu^n \in \widehat{M}$, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of the perturbed game $\overline{G}_{\delta^n\nu^n}$.

In words, μ is a *thp* equilibrium of G if it is the limit of some sequence of equilibria of neighboring Selten perturbations of G. See Carbonell-Nicolau (2011b) for an intuitive interpretation of Definition 2.

Remark 1. Note that, in Definition 2, we do not require that μ be a Nash equilibrium of G. It is well-known that, for continuous games, the fact that a strategy profile μ is the limit of some sequence of equilibria of Selten perturbations of G guarantees that μ is a Nash equilibrium of G. While we do not impose continuity of payoff functions, our conditions also ensure that the limit point is an equilibrium (cf. Carbonell-Nicolau, 2011b).

The following definitions adapt the perfection concepts in Simon and Stinchcombe (1995) to potential discontinuities in the payoff functions of a game.

For $\mu \in M$, let $\text{Br}_i(\mu)$ denote player i's set of best responses in M_i to the vector of strategies μ:

$$\text{Br}_i(\mu) := \left\{\sigma_i \in M_i : U_i(\sigma_i, \mu_{-i}) \geq \sup_{\varrho_i \in M_i} U_i(\varrho_i, \mu_{-i})\right\}.$$

Consider the following distance functions between members of M_i:

$$\rho_i^s(\mu_i, \nu_i) := \sup_B |\mu_i(B) - \nu_i(B)|,$$

and

$$\rho_i^w(\mu_i, \nu_i) := \inf\{\epsilon > 0 : \forall B, \mu_i(B) \leq \nu_i(N_\epsilon(B)) + \epsilon \text{ and } \nu_i(B) \leq \mu_i(N_\epsilon(B)) + \epsilon\},$$

where $N_\epsilon(B)$ denotes the ϵ-neighborhood of the (measurable) set B (i.e., $N_\epsilon(B) := \bigcup_{x \in B} N_\epsilon(x)$).

Definition 3 (*Simon and Stinchcombe, 1995*)**.** Given $\epsilon > 0$, a ***strong ϵ-perfect equilibrium*** of G is a vector $\mu^\epsilon \in \widehat{M}$ such that for each i,

$$\rho_i^s(\mu_i^\epsilon, \text{Br}_i(\mu^\epsilon)) < \epsilon,$$

and a ***weak ϵ-perfect equilibrium*** of G satisfies

$$\rho_i^w(\mu_i^\epsilon, \text{Br}_i(\mu^\epsilon)) < \epsilon.$$

A strategy profile in G is a ***strong*** (resp. ***weak***) ***perfect equilibrium*** of G if it is the weak* limit as $\epsilon^n \to 0$ of strong (resp. weak) ϵ^n-perfect equilibria.

Strong closedness implies weak closedness, and hence all strong perfect equilibria are weak perfect.[2]

It is well-known that for finite games, *thp* equilibria exist in the set of Nash equilibria and put no mass on weakly dominated strategies. In other words, *thp* equilibria satisfy existence, (E), and admissibility, (Ad). Moreover, a *thp* equilibrium is the limit of a sequence of exact equilibria in perturbed games in which players are constrained to play completely mixed strategies, (P).

As demonstrated by Simon and Stinchcombe (1995, Example 2.1), (E) and (Ad) are not generally compatible, for there are infinite games with a *unique* Nash equilibrium in weakly dominated strategies. Simon and Stinchcombe propose a weakening of (Ad), termed *limit admissibility* (LA), which states that a Nash equilibrium is limit admissible if it puts mass only on the limits of undominated strategies, or, equivalently, if no player assigns positive mass to the interior of the set of weakly dominated strategies.

For compact, continuous games, the set of *thp* equilibria (according to Definition 2) satisfies (E), (LA), (P), and reduces to the set of trembling-hand perfect equilibria for finite games, (R) (Theorem 3). Our purpose is to compare the properties of the solution concepts in Definitions 2 and 3 in the presence of payoff discontinuities.[3]

We first state the existence result of Carbonell-Nicolau (2011b).

Condition (A). There exists $(\mu_1, \ldots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for all $z_{-i} \in O_{y_{-i}}$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.[4]

Theorem 1 (*Carbonell-Nicolau, 2011b, Theorem 2*)**.** *Suppose that a compact, metric game G satisfies Condition (A). Suppose further that $\sum_{i=1}^N u_i$ is upper semicontinuous. Then G has a trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

The following result establishes the relationship between trembling-hand perfection and strong perfection. The equivalence of (1)–(3) is analogous to the standard characterization of trembling-hand perfect equilibria in finite games (e.g. van Damme, 2002, p. 28).

Theorem 2. *For a metric game $G = (X_i, u_i)_{i=1}^N$, the following three conditions are equivalent:*

(1) *μ is a trembling-hand perfect equilibrium of G.*
(2) *μ is a strong perfect equilibrium of G.*
(3) *μ is the limit of a sequence (μ^n) in $\widehat{M}$ with the property that for each i and every $\epsilon > 0$,*

$$\mu_i^n\left(\left\{x_i \in X_i : U_i(x_i, \mu_{-i}^n) \geq \sup_{y_i \in X_i} U_i(y_i, \mu_{-i}^n)\right\}\right) \geq 1 - \epsilon,$$

for any sufficiently large n.

Proof. Let $G = (X_i, u_i)_{i=1}^N$ be a metric game. Suppose that μ is a *thp* equilibrium of G. Then there are sequences (δ^n), (ν^n), and (μ^n) such that $(0,1)^N \ni \delta^n \to 0$, $\nu^n \in \widehat{M}$, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of the perturbed game $\overline{G}_{\delta^n \nu^n}$. Since $\mu^n \in M(\delta^n \nu^n)$ for each n, it is easily seen that for each i and n there exists $\rho_i^n \in M_i$ such that

$$\mu_i^n = (1 - \delta_i^n)\rho_i^n + \delta_i^n \nu_i^n.$$

Because each μ^n is a Nash equilibrium of $\overline{G}_{\delta^n \nu^n}$, $\rho_i^n \in \text{Br}_i(\mu^n)$ for each i and n. In fact, $\rho_i^n \notin \text{Br}_i(\mu^n)$ implies that

$$U_i(p_i^n, \mu_{-i}^n) > U_i(\rho_i^n, \mu_{-i}^n)$$

for some $p_i^n \in M_i$, so

$$U_i((1 - \delta_i^n)p_i^n + \delta_i^n \nu_i^n, \mu_{-i}^n) > U_i((1 - \delta_i^n)\rho_i^n + \delta_i^n \nu_i^n, \mu_{-i}^n) = U_i(\mu_i^n, \mu_{-i}^n),$$

thereby contradicting that μ^n is a Nash equilibrium of $\overline{G}_{\delta^n \nu^n}$. Consequently, because $\mu_i^n = (1 - \delta_i^n)\rho_i^n + \delta_i^n \nu_i^n$ and $\rho_i^n \in \text{Br}_i(\mu^n)$ for each i and n, and since $\delta_i^n \to 0$ for each i, we have

$$\rho_i^s(\mu_i^n, \text{Br}_i(\mu^n)) \to 0$$

[2] The converse is false (see Simon and Stinchcombe, 1995, Example 2.3).

[3] Simon and Stinchcombe (1995) introduced a third equilibrium refinement, termed *limit-of-finite (lof) perfection*. Roughly speaking, μ is a *limit-of-finite perfect* equilibrium in G if it is the limit of ϵ-perfect equilibria for successively larger finite approximations of G. As pointed out in Simon and Stinchcombe (1995), the *lof* approach is ill-suited as a general solution concept even in continuous games. We shall not reiterate the limitations of the *lof* approach. These limitations extend to families of possibly discontinuous games. See Carbonell-Nicolau and McLean (2011) for a limit-of-finite formulation that does not suffer from the drawbacks of Simon and Stinchcombe's (1995) approach.

[4] Condition (A) can be slightly weakened. See footnote 11 in Carbonell-Nicolau (2011b).

for each i, so that μ is a strong perfect equilibrium. Hence, (1) implies (2).

If μ is a strong perfect equilibrium of G, then there is a sequence (μ^n) in $\widehat{M}$ such that for each i and every $\epsilon > 0$, $\rho_i^s(\mu_i^n, \mathrm{Br}_i(\mu^n)) < \epsilon$ for any sufficiently large n, and this implies that

$$\mu_i^n\Big(\Big\{x_i \in X_i : U_i(x_i, \mu_{-i}^n) \geq \sup_{y_i \in X_i} U_i(y_i, \mu_{-i}^n)\Big\}\Big) \geq 1 - \epsilon$$

for any sufficiently large n. Hence, (2) implies (3).

Now assume (3). Let

$$X_i^n := \Big\{x_i \in X_i : U_i(x_i, \mu_{-i}^n) < \sup_{y_i \in X_i} U_i(y_i, \mu_{-i}^n)\Big\}.$$

Let $\sigma_i^n \in M_i$ be defined by

$$\sigma_i^n(B) := \frac{\mu_i^n(B \cap X_i^n) + \frac{1}{n}\mu_i^n(B \cap (X_i \setminus X_i^n))}{\mu_i^n(X_i^n) + \frac{1}{n}\mu_i^n(X_i \setminus X_i^n)}.$$

For large enough n, $\mu_i^n(X_i^n) + \frac{1}{n}\mu_i^n(X_i \setminus X_i^n) < 1$, and μ^n is an equilibrium of $\overline{G}_{\delta^n\sigma^n}$, where

$$\delta^n = (\delta_1^n, \ldots, \delta_N^n) := \Big(\mu_1^n(X_1^n) + \tfrac{1}{n}\mu_1^n(X_1 \setminus X_1^n), \ldots, \mu_N^n(X_N^n) + \tfrac{1}{n}\mu_N^n(X_N \setminus X_N^n)\Big).$$

In fact, for large enough n,

$$\mu_i^n(B) \geq \mu_i^n(B \cap X_i^n) + \tfrac{1}{n}\mu_i^n(B \cap (X_i \setminus X_i^n)) = \delta_i^n\sigma_i^n(B)$$

for every B, and so $\mu_i^n \in M_i(\delta_i^n\sigma_i^n)$ for each i. Moreover, for each i and every $p_i \in M_i(\delta_i^n\sigma_i^n)$, since $p_i(X_i^n) \geq \delta_i^n\sigma_i^n(X_i^n) = \mu_i^n(X_i^n)$, it follows that $U_i(p_i, \mu_{-i}^n) \leq U_i(\mu^n)$. Consequently, μ_i^n is a best response to μ_{-i}^n.

Because μ^n is an equilibrium of $\overline{G}_{\delta^n\sigma^n}$ (for large enough n) and we have $\delta^n \to 0$ and $\mu^n \to \mu$, μ is a *thp* equilibrium of G. Hence, (3) implies (1). □

Theorems 1 and 2 give the following:

Corollary 1. *Suppose that a compact, metric game G satisfies Condition (A). Suppose further that $\sum_{i=1}^N u_i$ is upper semicontinuous. Then G possesses a trembling-hand perfect equilibrium, which is also strong perfect and weak perfect, and all trembling-hand perfect equilibria of G are Nash.*

In this section we consider the family of games satisfying the conditions of Theorem 1, and impose an additional constraint on the dominance relations between strategies (to be formally defined below). We drop the extra requirement in Section 4.

Recall that $\mathfrak{g}_c$ denotes the set of compact, metric, and continuous games. Let $\mathfrak{g}_{dd}$ represent the set of compact, metric games $G = (X_i, u_i)_{i=1}^N$ satisfying Condition (A) and upper semicontinuity of the sum of payoffs $\sum_{i=1}^N u_i$.

Remark 2. Not all games satisfy Condition (A) (or some of its weakenings (such as Reny's (1999) *payoff security*) for that matter) or upper semicontinuity of $\sum_{i=1}^N u_i$. For example, the game $G = ([0,1],[0,1],u_1,u_2)$, where $u_1(0,0) := 1$, $u_1 := 0$ elsewhere, and u_2 is identically zero fails Condition (A) (even payoff security). In fact, we have $u_1(0,0) = 1$ and for each $x_1 \in [0,1]$ and every neighborhood $[0,\epsilon)$ of 0, $u_1(x_1, \frac{1}{2}\epsilon) = 0 < u_1(0,0) = 1$. Nevertheless, $\mathfrak{g}_{dd}$ is a rich class in that it contains several economic games of interest (cf. Carbonell-Nicolau (2011b, Section 3) and Carbonell-Nicolau (2011c, Section 4)).

On the other hand, for fixed action spaces $X_1, \ldots, X_N$ (with $X := \times_i X_i$), it can be shown that the class $\mathfrak{g}_{dd}(X)$ of games $G = (X_i, u_i)_{i=1}^N$ in $\mathfrak{g}_{dd}$ is closed when viewed as a metric subspace of the metric space $(B(X)^N, \rho_X)$, where $B(X)$ represents the set of bounded maps $f : X \to \mathbb{R}$ and $\rho_X : B(X)^N \times B(X)^N \to \mathbb{R}$ is defined by

$$\rho_X((f_1, \ldots, f_N), (g_1, \ldots, g_N)) := \sum_{i=1}^N \sup_{x \in X} |f_i(x) - g_i(x)|.$$

Consequently, this class is complete and metric (since $B(X)$ is complete and metric) and hence (by the Baire category theorem) a Baire space.

To see that $\mathfrak{g}_{dd}(X)$ is closed in $B(X)^N$, suppose that $(u^n) = (u_1^n, \ldots, u_N^n)$ is a sequence of games in $\mathfrak{g}_{dd}(X)$ with $u^n \to u = (u_1, \ldots, u_N)$. We need to show that u lies in $\mathfrak{g}_{dd}(X)$. The proof that $\sum_{i=1}^N u_i$ is upper semicontinuous is left to the reader. We show that $(X_i, u_i)_{i=1}^N$ satisfies Condition (A).

For each n there exists $(\mu_1^n, \ldots, \mu_N^n) \in \widehat{M}$ such that for each i and every $\alpha > 0$ there is a Borel measurable map $f_{(i,\alpha)}^n : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i^n(f_{(i,\alpha)}^n(x_i), z_{-i}) > u_i^n(x_i, y_{-i}) - \alpha$ for all $z_{-i} \in O_{y_{-i}}$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i^n(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i^n(f_{(i,\alpha)}^n(x_i), z_{-i}) < u_i^n(x_i, z_{-i}) + \alpha$ for all $z_{-i} \in V_{y_{-i}}$.

Fix i and $\varepsilon > 0$. In light of (a), and since $u^n \to u$, if n is large enough and $\alpha > 0$ is small enough, then, for every $(x_i, y_{-i}) \in X$, $u_i^n(x_i, y_{-i}) - \alpha$ is close enough to $u_i(x_i, y_{-i})$ to ensure that

$$u_i^n(f_{(i,\alpha)}^n(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \tfrac{\varepsilon}{2} \quad \text{for all } z_{-i} \in O_{y_{-i}},$$

and, since $u^n \to u$, we see that for any large enough n,

$$u_i(f_{(i,\alpha)}^n(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon \quad \text{for all } z_{-i} \in O_{y_{-i}}.$$

We conclude that given i and $\varepsilon > 0$, and for any sufficiently large n and any sufficiently small α, the map $f_{(i,\alpha)}^n$ satisfies the following: for each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f_{(i,\alpha)}^n(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for all $z_{-i} \in O_{y_{-i}}$. Hence, to show that $(X_i, u_i)_{i=1}^N$ satisfies Condition (A), it suffices to prove that for any large enough n and any small enough α, the map $f_{(i,\alpha)}^n$ satisfies the following: For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i^n(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_{(i,\alpha)}^n(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$. But this flows from the following observations. In light of (b), because $u^n \to u$, for large enough n and small enough α, and given any $y_{-i} \in X_{-i}$, given a subset Y_i satisfying condition (b), an $x_i \in Y_i$, and a $z_{-i} \in V_{y_{-i}}$ (where $V_{y_{-i}}$ is given by condition (b)), $u_i^n(x_i, z_{-i})$ is close enough to $u_i(x_i, z_{-i})$ to ensure that $u_i^n(f_{(i,\alpha)}^n(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \frac{\varepsilon}{2}$. Consequently, since $u^n \to u$, for any large enough n and any small enough α we have $u_i(f_{(i,\alpha)}^n(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$.

Two notions of weak domination have been considered in the literature.

Definition 4. A strategy $x_i \in X_i$ is ***weakly* dominated for i*** if there exists a strategy $y_i \in X_i$ such that $u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, with strict inequality for some x_{-i}.

Definition 5. A strategy $x_i \in X_i$ is ***weakly dominated for i*** if there exists a strategy $\mu_i \in M_i$ such that $U_i(\mu_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, with strict inequality for some x_{-i}.

The first definition is less restrictive and has been considered in Salonen (1996), while the second definition is used in Simon and Stinchcombe (1995) to formulate limit admissibility.

Recall that $\mathfrak{g}_{dd}$ denotes the set of compact, metric games $G = (X_i, u_i)_{i=1}^N$ satisfying Condition (A) and upper semicontinuity of the sum of payoffs $\sum_{i=1}^{N} u_i$. Define the set $\mathfrak{g}_d$ of all members $G = (X_i, u_i)_{i=1}^N$ of $\mathfrak{g}_{dd}$ with the following property: if x_i is weakly dominated in G for player i, then for some μ_i that weakly dominates x_i, there exists $y_{-i} \in X_{-i}$ with $U_i(\mu_i, z_{-i}) > u_i(x_i, z_{-i})$ for all $z_{-i} \in O_{y_{-i}}$ and for some neighborhood $O_{y_{-i}}$ of y_{-i}.

In words, $\mathfrak{g}_d$ is the set of members of $\mathfrak{g}_{dd}$ for which any strategy x_i that is weakly dominated for i has the property that there is some μ_i that weakly dominates x_i and is a better response than x_i to some neighborhood of action profiles of the other players. This condition is clearly satisfied by all continuous games, but need not be met in the set $\mathfrak{g}_{dd}$ (Example 1, Section 4). We have $\mathfrak{g}_c \subsetneqq \mathfrak{g}_d \subsetneqq \mathfrak{g}_{dd}$.

The next results are concerned with the properties of the various notions of perfection in the families of games $\mathfrak{g}_c$, $\mathfrak{g}_d$, and $\mathfrak{g}_{dd}$. For the axioms that are not affected by potential payoff discontinuities, the characterization of the solution concepts follows from the analysis in Simon and Stinchcombe (1995). We expand on the cases when Simon and Stinchcombe (1995) uses continuity to verify a property.

Definition 6. A strategy profile $\mu \in M$ is ***admissible**** if $\mu_i(D_i^*) = 0$ for all i, where D_i^* denotes the set of strategies weakly* dominated for i.

Definition 7. A strategy profile $\mu \in M$ is ***limit admissible**** if $\mu_i(O_i^*) = 0$ for all i, where O_i^* denotes the interior of the set of strategies weakly* dominated for player i.

Definition 8. A strategy profile $\mu \in M$ is ***admissible*** if $\mu_i(D_i) = 0$ for all i, where D_i denotes the set of strategies weakly dominated for i.

Definition 9. A strategy profile $\mu \in M$ is ***limit admissible*** if $\mu_i(O_i) = 0$ for all i, where O_i denotes the interior of the set of strategies weakly dominated for player i.

We say that a refinement specification satisfies property (N) (resp. (W), (S)) if it selects a set of strategy profiles that is closed in the set of Nash equilibria (resp. the set of weak perfect equilibria and the set of strong perfect equilibria); property (E) if it chooses a nonempty set of strategy profiles; and property (LA*) (resp. (LA)) if it selects a set of limit admissible* (resp. limit admissible) strategy profiles.

A notion of perfection satisfies property (R) if it reduces to the standard set of trembling-hand perfect equilibria in finite games; and property (P) if it chooses strategy profiles that can be viewed as limits of equilibria in perturbed games, so that the agents can be thought of as optimizing against the play of others, subject to the constraint that they and the others make mistakes.

We follow Simon and Stinchcombe (1995), where the interpretation of condition (P) varies across the various equilibrium refinements: in each case, the conception of a perturbed game arises "naturally" from the "spirit" of the corresponding approach to refinement.

Formally, a perturbation for the strong approach is a game $(M_i^s(\epsilon), U_i)$, where $M_i^s(\epsilon)$ denotes a convex, weak* compact subset of $\widehat{M}_i$ within Hausdorff distance ϵ of M_i, where the strong distance, ρ_i^s, is the associated metric. The definition of a perturbation for the weak approach is analogous, with the weak distance, ρ_i^w, replacing ρ_i^s.

When applied to strong perfection, condition (P) requires that any strong perfect equilibrium be the limit of a sequence (μ^n), where each μ^n is a Nash equilibrium of a perturbation $(M_i^s(\epsilon^n), U_i)$, for some $\epsilon^n \to 0$.[5] The requirement is analogous for the weak approach.

[5] Even when the choice of a perturbation for G has been made, one can furnish (at least) two definitions of (P). The alternative to the above definition (for strong perfection) imposes the equivalence of the set of strong perfect equilibria and the set of limits of sequences (μ^n) with the said properties. We adopt the weak criterion.

Simon and Stinchcombe (1995) study the properties of the strong and weak approaches within the class $\mathfrak{g}_c$.

Theorem 3 (*Simon and Stinchcombe, 1995*)**.** *For the family of games $\mathfrak{g}_c$, we have the following:*

- *Strong perfection and trembling-hand perfection satisfy* (E), (LA) *(hence* (LA*)*),* (P), (R), *and* (W).
- *Weak perfection satisfies* (E), (LA) *(hence* (LA*)*),* (R), *and* (N), *and fails* (P).

Theorem 3 holds with $\mathfrak{g}_d$ replacing $\mathfrak{g}_c$.

Theorem 4. *For the family of games $\mathfrak{g}_d$, we have the following:*

- *Strong perfection and trembling-hand perfection satisfy* (E), (LA) *(hence* (LA*)*),* (P), (R), *and* (W).
- *Weak perfection satisfies* (E), (LA) *(hence* (LA*)*),* (R), *and* (N), *and fails*(P).

Proof. Theorems 1 and 2 give (E) in $\mathfrak{g}_d$ for all the equilibrium concepts.

We now show that weak perfect equilibrium profiles must be limit admissible. Suppose that μ is a weak perfect equilibrium of $G \in \mathfrak{g}_d$. Let O_i denote the interior of the set of strategies weakly dominated for i. Because μ is a weak perfect equilibrium, there are sequences (μ^{ϵ^n}) and (ϵ^n) with $\widehat{M} \ni \mu^{\epsilon^n} \to \mu$ and $\epsilon^n \searrow 0$ such that for each i,

$$\rho_i^w(\mu_i^{\epsilon^n}, \mathrm{Br}_i(\mu^{\epsilon^n})) < \epsilon^n, \quad \text{for all } n. \tag{1}$$

Fix i. Suppose that $\nu_i(O_i) > 0$ and $\nu_i \in \mathrm{Br}_i(\mu^{\epsilon^n})$. Then there exists z_i in $\mathrm{Br}_i(\mu^{\epsilon^n}) \cap O_i$. But this is a contradiction, since $z_i \in O_i$ implies that there exists a strategy $\sigma_i \in M_i$ such that $U_i(\sigma_i, x_{-i}) \geq U_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and, for some y_{-i} and some neighborhood $O_{y_{-i}}$ of y_{-i}, $U_i(\sigma_i, z_{-i}) > U_i(z_i, z_{-i})$ for all $z_{-i} \in O_{y_{-i}}$ (recall that $G \in \mathfrak{g}_d$), so (because $\mu^{\epsilon^n} \in \widehat{M}$) one cannot have $z_i \in \mathrm{Br}_i(\mu^{\epsilon^n})$. Consequently, $\nu_i(O_i) = 0$ for every $\nu_i \in \mathrm{Br}_i(\mu^{\epsilon^n})$, and this implies that for every sequence (ν_i^n) in M_i such that $\nu_i^n \in \mathrm{Br}_i(\mu^{\epsilon^n})$ for each n, we have

$$\liminf_{n\to\infty} \nu_i^n(O_i) = 0. \tag{2}$$

Now, given (1), one can choose a sequence (ν_i^n) in M_i such that $\nu_i^n \in \mathrm{Br}_i(\mu^{\epsilon^n})$ for each n and

$$\rho_i^w(\mu_i^{\epsilon^n}, \nu_i^n) \to 0.$$

Consequently, because $\mu^{\epsilon^n} \to \mu$, we have $\nu_i^n \to \mu_i$. This yields

$$\liminf_{n\to\infty} \nu_i^n(O_i) \geq \mu_i(O_i),$$

since O_i is open. Therefore, using (2), we see that $\mu_i(O_i) = 0$.

Thus, $\mu_i(O_i) = 0$ for each i, and μ is limit admissible. Because weak perfection satisfies (LA) (and *thp* equilibria are weak perfect), so do strong perfection and trembling-hand perfection (Theorem 2).

Now consider condition (P). It is clear that trembling-hand perfection satisfies (P), and hence so does strong perfection (Theorem 2). Simon and Stinchcombe (1995) show, in their Example 2.6, that there are games in which the limit of some sequence (μ^n), where each μ^n is a Nash equilibrium of $(M_i^w(\epsilon^n), U_i)$, and $\epsilon^n \to 0$, fails to be a weak perfect equilibrium. Moreover, according to Simon and Stinchcombe (1995) (footnote 23) one can construct a (five-agent) game with a weak perfect equilibrium that is not the limit of any sequence of weakly perturbed games. Thus, weak perfection fails (P).

Because the metrics ρ^s and ρ^w are equivalent for mixtures on finite sets, for finite games weak and strong perfection coincide, and are identical to Selten's (1975) definition.

By Theorem 1, trembling-hand perfection selects Nash equilibria. Furthermore, the set of *thp* equilibria is closed. In fact, for $\alpha \in (0, 1)$, let $P(\alpha)$ be the set of profiles μ such that μ is a Nash equilibrium of $\overline{G}_{\delta\nu}$, for some $\delta \in (0, \alpha]^N$ and some $\nu \in \widehat{M}$. Since the set of *thp* equilibria can be expressed as $\bigcap_\alpha \text{cl}(P(\alpha))$, trembling-hand perfection and strong perfection satisfy (W).

To see that weak perfect equilibria are Nash, let μ be a weak perfect equilibrium of G. Then there is a sequence (ϵ^n) with $\epsilon^n \to 0$ such that $\mu^n \to \mu$, where each μ^n is a weak ϵ^n-perfect equilibrium. Hence, for each i and every n,

$$\mu_i^n(N_{\epsilon^n}(\widehat{\text{Br}}_i(\mu^n))) > 1 - \epsilon^n,$$

where

$$\widehat{\text{Br}}_i(\mu) := \left\{x_i \in X_i : U_i(x_i, \mu_{-i}) \geq \sup_{y_i \in X_i} U_i(y_i, \mu_{-i})\right\},$$

and, since $\mu^n \to \mu$, μ_i puts mass 1 on $\text{Ls}_n\widehat{\text{Br}}_i(\mu^n)$, the limes superior (in the sense of Kuratowski) of $\widehat{\text{Br}}_i(\mu^n)$. Thus it suffices to show that $\text{Ls}_n\widehat{\text{Br}}_i(\mu^n) \subseteq \widehat{\text{Br}}_i(\mu)$. To establish this containment, let $x_i^k \to x_i$, where $x_i^k \in \widehat{\text{Br}}_i(\mu^k)$, for some subsequence $(\widehat{\text{Br}}_i(\mu^k))$ of $(\widehat{\text{Br}}_i(\mu^n))$, and $x_i \notin \widehat{\text{Br}}_i(\mu)$. Because $x_i^k \to x_i$, $\mu^k \to \mu$, and each u_i is bounded, we have (passing to a subsequence if necessary)

$$\begin{aligned}&((x_i^k, \mu_{-i}^k), (U_1(x_i^k, \mu_{-i}^k), \ldots, U_N(x_i^k, \mu_{-i}^k)))\\ &\quad \to ((x_i, \mu_{-i}), (\alpha_1, \ldots, \alpha_N))\end{aligned} \tag{3}$$

for some $\alpha := (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. Therefore, $((x_i, \mu_{-i}), \alpha)$ belongs to the closure of the set

$$\{(\rho, a) \in M \times \mathbb{R}^N : U_i(\rho) = a_i, \text{ for all } i\}.$$

Consequently, since $G \in \mathfrak{g}_d$, and Condition (A), together with upper semicontinuity of $\sum_{i=1}^N u_i$, implies Reny's (1999) *better-reply security* of $\overline{G}$, the fact that $x_i \notin \widehat{\text{Br}}_i(\mu)$ (so that $\delta_{x_i} \notin \text{Br}_i(\mu)$) gives i and $\sigma_i \in M_i$ such that, for some neighborhood $O_{\mu_{-i}}$ of μ_{-i} and some $\gamma > 0$,

$$U_i(\sigma_i, \sigma_{-i}) \geq \alpha_i + \gamma, \quad \text{for all } \sigma_{-i} \in O_{\mu_{-i}}.$$

We therefore have, in view of (3),

$$U_i(\sigma_i, \mu_{-i}^k) > U_i(x_i^k, \mu_{-i}^k)$$

for any sufficiently large k, thereby contradicting that x_i^k is a best response to μ_{-i}^k (i.e., that $x_i^k \in \widehat{\text{Br}}_i(\mu^k)$). Therefore, $x_i \in \text{Ls}_n\widehat{\text{Br}}_i(\mu^n)$ implies $x_i \in \widehat{\text{Br}}_i(\mu)$, as desired. We omit the proof that the set of weak perfect equilibrium profiles is closed. □

4. Limit admissibility in discontinuous games

While trembling-hand perfection and strong perfection appear to better respect the structure of infinite games relative to weak perfection (cf. Simon and Stinchcombe, 1995), expanding the domain of games beyond $\mathfrak{g}_c$ or $\mathfrak{g}_d$ affects the properties of these solution concepts. In fact, there are members of $\mathfrak{g}_{dd}$ in which carefully chosen "trembles" may render a weakly dominated strategy no less appealing than a corresponding dominant strategy: if the opponent's "tremble" gives measure zero to the strategies against which the dominant strategy is superior, associated Selten perturbations may exhibit equilibria assigning large mass to the interior of the set of weakly* dominated strategies. As a result, trembling-hand perfection and strong perfection fail (LA*) and (LA). This is illustrated in the following example.

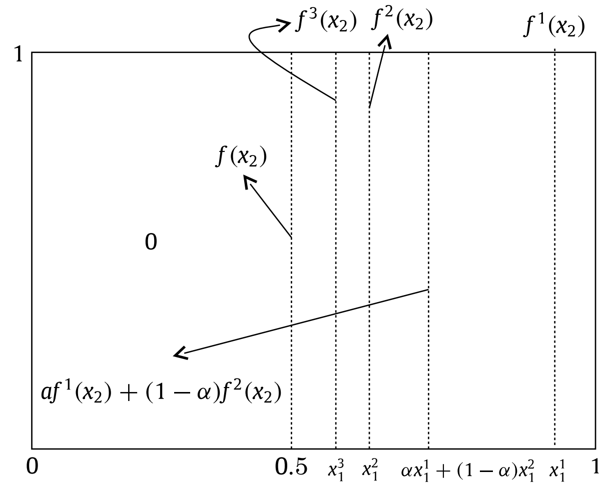

Fig. 1. Example 1: the payoff function for player 1.

Example 1. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Let (f^n) be a sequence of continuous, strictly decreasing functions $f^n : [0, 1] \to \mathbb{R}$ with the following properties:

(a) Each f^n intersects with f only once at $x_2^n \in \left(0, \frac{1}{2}\right)$, where $x_2^n \to 0$, and $f^n(1) = -\frac{1}{n} < 0 = f(1)$ for each n.
(b) f^n converges to f pointwise.

Let (x_1^n) be a sequence from $\left(\frac{1}{2}, 1\right)$ with $x_1^n \searrow \frac{1}{2}$.

Consider the two-player game $G = ([0, 1], [0, 1], u_1, u_2)$, where

$$u_1(x_1, x_2) := \begin{cases} 0 & \text{if } (x_1, x_2) \in \left[0, \frac{1}{2}\right) \times [0, 1], \\ f(x_2) & \text{if } x_1 = \frac{1}{2}, \\ f^n(x_2) & \text{if } x_1 = x_1^n, \\ \alpha f^n(x_2) + (1-\alpha)f^{n+1}(x_2) & \text{if } x_1 = \alpha x_1^n + (1-\alpha)x_1^{n+1}, \\ & \text{some } \alpha \in (0, 1), \end{cases}$$

and $u_2(x_1, x_2) := x_2$ (Fig. 1).

For this game, $\sum_{i=1}^N u_i$ is upper semicontinuous (in fact, each u_i is upper semicontinuous), and Condition (A) is fulfilled.[6]

The action profile $(0, 1)$ is clearly a Nash equilibrium in weakly dominated strategies. In fact, 0, and any sufficiently small neighborhood around 0, is weakly dominated for player 1 by $\frac{1}{2}$.

We claim that $(0, 1)$ is a *thp* equilibrium of G. In fact, consider a tremble sequence $(\nu_2^n) \in \widehat{M}_2$ for player 2 such that ν_2^n is the Lebesgue measure on $[0, 1]$. Note that for large enough n, choosing the action 0 (in fact, choosing any action in $\left[0, \frac{1}{2}\right]$) is a best response for player 1 to the perturbed strategy $\left(1 - \frac{1}{n}\right) 1 + \left(\frac{1}{n}\right) \nu_2^n$. To see this, note that for every $x_1 \in \left[0, \frac{1}{2}\right]$ we have

$$U_1\left(x_1, \left(1 - \tfrac{1}{n}\right)1 + \tfrac{1}{n}\nu_2^n\right) = 0,$$

and for every $x_1 \in \left(\frac{1}{2}, 1\right]$ we have

$$\begin{aligned} U_1\left(x_1, \left(1 - \tfrac{1}{n}\right)1 + \tfrac{1}{n}\nu_2^n\right) &= \left(1 - \tfrac{1}{n}\right)u_1(x_1, 1) + \tfrac{1}{n}U_1(x_1, \nu_2^n) \\ &\leq -\left(1 - \tfrac{1}{n}\right)\tfrac{1}{n} + \tfrac{1}{n}\tfrac{1}{2} \\ &= \tfrac{1}{2}\left(-\tfrac{1}{2} + \tfrac{1}{n}\right), \end{aligned}$$

[6] The game G satisfies even Carbonell-Nicolau's (2011b) *generic entire payoff security* and *generic local equi-upper semicontinuity*.

so, for large enough n,

$$U_1\left(0, \left(1-\tfrac{1}{n}\right)1+\tfrac{1}{n}\nu_2^n\right) \geq U_1\left(x_1, \left(1-\tfrac{1}{n}\right)1+\tfrac{1}{n}\nu_2^n\right)$$

for every $x_1 \in [0, 1]$. Consequently, since 0 is a best response for player 1 to the perturbed strategy $\left(1-\frac{1}{n}\right)1+\left(\frac{1}{n}\right)\nu_2^n$, and since the action 1 is strictly dominant for player 2, for large enough n,

$$\left(\left(1-\tfrac{1}{n}\right)0+\left(\tfrac{1}{n}\right)\nu_1^n, \left(1-\tfrac{1}{n}\right)1+\left(\tfrac{1}{n}\right)\nu_2^n\right)$$

is an equilibrium of the Selten perturbation $\overline{G}_{n^{-1}\nu^n}$ for any $\nu_1^n \in \widehat{M}_1$. Hence, (0, 1) is trembling-hand perfect, and therefore strongly (hence weakly) perfect (Theorem 2).

Since the action 0 lies in the interior of the set of strategies weakly dominated for 1, it follows that trembling-hand perfection and strong perfection fail limit admissibility* and limit admissibility.

The following result summarizes the properties of strong perfection, trembling-hand perfection, and weak perfection in $\mathfrak{g}_{dd}$. The statements that are given with no direct proof can be established as in the proof of Theorem 4.

Theorem 5. *For the family of games* $\mathfrak{g}_{dd}$, *we have the following:*

- *Strong perfection and trembling-hand perfection satisfy* (E), (P), (R), *and* (W), *and fail* (LA*) *and* (LA).
- *Weak perfection satisfies* (E), (R), *and* (N), *and fails* (P), (LA*), *and* (LA).

5. Limit admissible perfect equilibria

Because (limit) admissibility is regarded as a basic requirement for stability of equilibria (e.g. Kohlberg and Mertens, 1986, and references listed therein), the fact that the existing notions of trembling-hand perfection for infinite games fail (LA*) and (LA) raises questions on the existence of alternative refinement specifications that do not suffer from this drawback. We argue that, even if there were refinements of trembling-hand perfection, along the lines of Kohlberg and Mertens' (1986) *stability*, satisfying (LA*) or (LA), these refinements would pose existence problems for the classes of games considered in this paper. By contrast, there are rich families of games for which the concept of *limit admissible perfect (lap) equilibrium* (defined below) is immune to this problem, and meets other criteria.

We first define a *lap* equilibrium and compare the properties of the various solution concepts.

We consider two concepts for limit admissible perfection. The first one uses the less restrictive notions of weak domination or (limit) admissibility in Definitions 4 and 7. The second one is consistent with Definitions 5 and 9.

Definition 10. A strategy profile μ in G is a ***limit admissible perfect* (lap*) equilibrium*** if it is a limit admissible*, trembling-hand perfect equilibrium of G.

Definition 11. A strategy profile μ in G is a ***limit admissible perfect (lap) equilibrium*** if it is a limit admissible, trembling-hand perfect equilibrium of G.

Remark 3. A related solution concept eliminates the interior of the set of weakly dominated strategies and applies trembling-hand perfection to the resulting reduced form. Formally, given a game $G = (X_i, u_i)_{i=1}^N$, and letting $G_r = (Y_i, u_i)_{i=1}^N$ stand for the 'subgame' of G that results from eliminating, for each i, the interior of the set of all weakly dominated strategies in X_i, this variant picks the set of trembling-hand perfect equilibria in G_r.

Like limit admissible perfection, this specification satisfies (LA). In finite games, it chooses *thp* equilibria, but there are finite games with the property that adding dominated strategies enlarges the set of trembling-hand perfect equilibria (cf. Myerson, 1978). In infinite games, however, it need not select *thp* equilibria or *even* Nash equilibria, unlike limit admissible perfection. Thus, for infinite games, adding weakly dominated strategies may alter the set of *thp* equilibria in such a way that some strategy profiles *cease* to be *thp* or *even* Nash.

This is illustrated in the following example. Define the maps $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} -1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) := \begin{cases} 1 & \text{if } x \in \{0, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let (g^n) be a sequence of continuous maps $g^n : [0, 1] \to \mathbb{R}$ with the following properties:

- (g^n) converges pointwise to g; and
- $g^n(x) < 1$ for all x.

Let $G = ([0, 1], [0, 1], u_1, u_2)$ be a two-player game with

$$u_1(x_1, x_2) := \begin{cases} f(x_2) & \text{if } x_1 = 1, \\ 0 & \text{if } (x_1, x_2) = \left(0, \frac{1}{2}\right), \\ x_1 - 1 & \text{otherwise,} \end{cases}$$

and

$$u_2(x_1, x_2) := \begin{cases} g(x_1) & \text{if } x_2 = 0, \\ 1 & \text{if } (x_1, x_2) = \left(1, \frac{1}{2}\right), \\ g^n(x_1) & \text{if } x_2 = \frac{1}{2} + \frac{1}{2n} \text{ and } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The sum $u_1 + u_2$ is upper semicontinuous, and G satisfies Condition (A), so G is a member of $\mathfrak{g}_{dd}$. Moreover, it is easily seen that the action 0 is weakly dominated by 1 for player 1, and that any action $x_1 \in (0, 1)$ is dominated by any action in $(x_1, 1)$ for player 1. In addition, any action in $(0, \frac{1}{2}] \cup \left(\left(\frac{1}{2}, 1\right] \setminus \left(\bigcup_n \left\{\frac{1}{2} + \frac{1}{2n}\right\}\right)\right)$ is weakly dominated by 0 for player 2. Consequently, deleting the interior of the set of weakly dominated strategies yields the reduced form

$$G_r = \left(\{1\}, \left\{0, \tfrac{1}{2}\right\} \cup \left(\bigcup_n \left\{\tfrac{1}{2} + \tfrac{1}{2n}\right\}\right), u_1, u_2\right),$$

which has two Nash equilibria, $\left(1, \frac{1}{2}\right)$ and $(1, 0)$, and it is easy to see that $(1, 0)$ is *thp* in G_r (note that $\frac{1}{2}$ is weakly dominated by 0 for player 2). However, the strategy profile $(1, 0)$ is not even a Nash equilibrium in G, for

$$u_1\left(\tfrac{1}{2}, 0\right) = -\tfrac{1}{2} > -1 = u_1(1, 0).$$

Proving or disproving the existence of *lap* equilibria (resp. *lap** equilibria) in $\mathfrak{g}_{dd}$ seems highly nontrivial. We can furnish results for other rich collections of games. We first define these collections and then derive corresponding analogues of Theorems 3–5 for the refinement specifications considered in this section and in Section 4.

Given i, recall that O_i^* denotes the interior of the set of strategies weakly* dominated for player i. Given $G = (X_i, u_i)_{i=1}^N$, define, for each i, $X_i^* := X_i \setminus O_i^*$.

Condition $(\widehat{A})$. There exists $(\mu_1, \ldots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i^*$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for all $z_{-i} \in O_{y_{-i}}$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

Given i, recall that O_i denotes the interior of the set of strategies weakly dominated for player i. Let $\overline{O}_i$ be the set of $x_i \in O_i$ such that for every σ_i that weakly dominates x_i, if $U_i(\sigma_i, x_{-i}) > u_i(x_i, x_{-i})$, there is no neighborhood $V_{x_{-i}}$ of x_{-i} such that $U_i(\sigma_i, y_{-i}) > u_i(x_i, y_{-i})$ for all $y_{-i} \in V_{x_{-i}}$.

Define $\overline{M}$ as the set of $\mu \in \widehat{M}$ such that there exist $Q_1, \ldots, Q_N$, where each Q_i is a countable dense subset of X_i, satisfying the following: for each i and every $x_i \in Q_i \cap \overline{O}_i$, $\mu_{-i}(\{\phi_i(x_i)\}) > 0$, where $\phi_i : Q_i \cap \overline{O}_i \to X_{-i}$ is defined as follows: $\phi_i(x_i) := y_{-i}$, where y_{-i} is such that, for some $\rho_i \in M_i$ that weakly dominates x_i, $U_i(\rho_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $U_i(\rho_i, y_{-i}) > u_i(x_i, y_{-i})$.

Note that when each X_i is compact and metric (and hence separable) the set $\overline{M}$ is nonempty. In fact, if each X_i is separable there exists, for each i, a countable dense subset Q_i of X_i. Letting $\rho(i) = (\rho_1(i), \ldots, \rho_N(i))$ be a member of M such that $\rho_{-i}(i)(\{\phi_i(x_i)\}) > 0$ for every $x_i \in Q_i \cap \overline{O}_i$ and each i, and defining $\mu := \alpha\sigma + (1-\alpha)\sum_{i\in\{1,\ldots,N\}} \frac{1}{N}\rho(i)$, where $\alpha \in (0, 1)$ and $\sigma \in \widehat{M}$, we see that $\mu \in \overline{M}$.

Remark 4. If $\overline{O}_i = \emptyset$ for each i, then $\overline{M} = \widehat{M}$.

Given $G = (X_i, u_i)_{i=1}^N$, define, for each i, $X_i^\bullet := X_i \setminus O_i$.

Condition ($\overline{\text{A}}$)**.** There exists $(\mu_1, \ldots, \mu_N) \in \overline{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i^\bullet$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) > u_i(x_i, y_{-i}) - \varepsilon$ for all $z_{-i} \in O_{y_{-i}}$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) < u_i(x_i, z_{-i}) + \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

We define the following collections of games:

- $\widehat{\mathfrak{g}}_{dd}$: the set of compact, metric games $(X_i, u_i)_{i=1}^N$ with the following properties: (1) $X_i^* \neq \emptyset$ for each i; (2) $(X_i, u_i)_{i=1}^N$ satisfies Condition ($\widehat{\text{A}}$); (3) for each i, if $O_i^* \neq \emptyset$, then $u_i(\cdot, x_{-i})$ is upper semicontinuous; and (4) $\sum_{i=1}^N u_i$ is upper semicontinuous.
- $\overline{\mathfrak{g}}_{dd}$: the set of compact, metric games $(X_i, u_i)_{i=1}^N$ with the following properties: (1) $X_i^\bullet \neq \emptyset$ for each i; (2) $(X_i, u_i)_{i=1}^N$ satisfies Condition ($\overline{\text{A}}$); (3) for each i, if $O_i \neq \emptyset$, then $u_i(\cdot, x_{-i})$ is upper semicontinuous; and (4) $\sum_{i=1}^N u_i$ is upper semicontinuous.

Given $(\delta, \mu) \in [0, 1) \times M$, define $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ by

$$u_i^{(\delta,\mu)}(x_1, \ldots, x_N) := U_i((1-\delta)x_1 + \delta\mu_1, \ldots, (1-\delta)x_N + \delta\mu_N).$$

Given $(\delta, \mu) \in [0, 1) \times M$ and $G = (X_i, u_i)_{i=1}^N$ with $X_i^* \neq \emptyset$ (resp. $X_i^\bullet \neq \emptyset$) for each i, let $G^*_{(\delta,\mu)} := (X_i^*, u_i^{(\delta,\mu)})_{i=1}^N$ (resp. $G^\bullet_{(\delta,\mu)} := (X_i^\bullet, u_i^{(\delta,\mu)})_{i=1}^N$), where $u_i^{(\delta,\mu)}$ denotes the restriction of $u_i^{(\delta,\mu)}$ to $\times_{j=1}^N X_j^*$ (resp. $\times_{j=1}^N X_j^\bullet$). Let $\overline{G}^*_{(\delta,\mu)}$ (resp. $\overline{G}^\bullet_{(\delta,\mu)}$) be the mixed extension of $G^*_{(\delta,\mu)}$ (resp. $G^\bullet_{(\delta,\mu)}$).

The following definition is taken from Reny (1999).

Definition 12. The game $G = (X_i, u_i)_{i=1}^N$ is ***payoff secure*** if for each $\varepsilon > 0$, $x \in X$, and i, there exists $y_i \in X_i$ such that $u_i(y_i, y_{-i}) > u_i(x) - \varepsilon$ for every $y_{-i} \in O_{x_{-i}}$ and for some neighborhood $O_{x_{-i}}$ of x_{-i}.

The proofs of the following lemmas are similar to the proof of Lemma 1 in Carbonell-Nicolau (2011b). We omit the details and refer the reader to Carbonell-Nicolau (2011b).

Lemma 1. *Suppose that a compact, metric game $G = (X_i, u_i)_{i=1}^N$ with $X_i^* \neq \emptyset$ for each i satisfies Condition ($\widehat{\text{A}}$). Then there exists $\mu \in \widehat{M}$ such that $\overline{G}^*_{(\delta,\mu)}$ is payoff secure for every $\delta \in [0, 1)$.*

Lemma 2. *Suppose that a compact, metric game $G = (X_i, u_i)_{i=1}^N$ with $X_i^\bullet \neq \emptyset$ for each i satisfies Condition ($\overline{\text{A}}$). Then there exists $\mu \in \overline{M}$ such that $\overline{G}^\bullet_{(\delta,\mu)}$ is payoff secure for every $\delta \in [0, 1)$.*

The following lemmas establish the existence of *lap** equilibria (resp. *lap* equilibria) in $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$).

Lemma 3. *Every member of $\widehat{\mathfrak{g}}_{dd}$ has a limit admissible perfect* equilibrium.*

Proof. Fix $G = (X_i, u_i)_{i=1}^N$ in $\widehat{\mathfrak{g}}_{dd}$. Take a sequence (δ^n) with $(0, 1) \ni \delta^n \to 0$. By Lemma 1, there exists $\mu \in \widehat{M}$ such that each $\overline{G}^*_{(\delta^n,\mu)}$ is payoff secure. This, together with upper semicontinuity of $\sum_{i=1}^N u_i$, gives, for each n, a Nash equilibrium σ^n of $\overline{G}^*_{(\delta^n,\mu)}$. In fact, upper semicontinuity of $\sum_{i=1}^N u_i$ (which gives upper semicontinuity of $\sum_{i=1}^N U_i$ (e.g., Aliprantis and Border, 2006, Theorem 15.5)) and payoff security of $\overline{G}^*_{(\delta^n,\mu)}$ imply that $\overline{G}^*_{(\delta^n,\mu)}$ is better-reply secure (Reny, 1999, Proposition 3.2) and therefore, $\overline{G}^*_{(\delta^n,\mu)}$, being a compact, quasiconcave game, possesses a Nash equilibrium σ^n (Reny, 1999, Theorem 3.1). We claim that σ^n is also a Nash equilibrium of the mixed extension of the game $(X_i, u_i^{(\delta^n,\mu)})$. To see this, suppose that σ^n is not an equilibrium of the mixed extension of $(X_i, u_i^{(\delta^n,\mu)})$. Then, there exist i and $\varrho_i \in M_i$ such that $\int_X u_i^{(\delta^n,\mu)} d(\varrho_i, \sigma^n_{-i}) > \int_X u_i^{(\delta^n,\mu)} d\sigma^n$, so there is a $y_i \in X_i$ such that $\int_X u_i^{(\delta^n,\mu)} d(y_i, \sigma^n_{-i}) > \int_X u_i^{(\delta^n,\mu)} d\sigma^n$. Therefore, since for any $x_i \in O_i^*$ there exists an action $z_i \in X_i^*$ that weakly* dominates x_i (Salonen, 1996, Corollary 1), there exists $z_i \in X_i^*$ such that $\int_X u_i^{(\delta^n,\mu)} d(z_i, \sigma^n_{-i}) > \int_X u_i^{(\delta^n,\mu)} d\sigma^n$, thereby contradicting that σ^n is a Nash equilibrium of $\overline{G}^*_{(\delta^n,\mu)}$.

Now, since σ^n is a Nash equilibrium of the mixed extension of the game $(X_i, u_i^{(\delta^n,\mu)})$, $(1-\delta^n)\sigma^n + \delta^n\mu$ is a Nash equilibrium of $\overline{G}_{\delta^n\mu}$, and because M is sequentially compact, we may write (passing to a subsequence if necessary) $\sigma^n \to \sigma$ for some $\sigma \in M$. This, together with the fact that $\sigma_i^n(O_i^*) = 0$ for each i and all n, gives $\sigma_i(O_i^*) = 0$. Consequently, σ is a *lap** equilibrium of G. □

Lemma 4. *Every member of $\overline{\mathfrak{g}}_{dd}$ has a limit admissible perfect equilibrium.*

Proof. Fix $G = (X_i, u_i)_{i=1}^N$ in $\overline{\mathfrak{g}}_{dd}$. Take a sequence (δ^n) with $(0, 1) \ni \delta^n \to 0$. By Lemma 2, there exists $\mu \in \overline{M}$ such that each $\overline{G}^\bullet_{(\delta^n,\mu)}$ is payoff secure. This, together with upper semicontinuity of $\sum_{i=1}^N u_i$, gives, for each n, a Nash equilibrium σ^n of $\overline{G}^\bullet_{(\delta^n,\mu)}$. In fact, upper semicontinuity of $\sum_{i=1}^N u_i$ (which gives upper semicontinuity of $\sum_{i=1}^N U_i$ (e.g., Aliprantis and Border, 2006, Theorem 15.5)) and payoff security of $\overline{G}^\bullet_{(\delta^n,\mu)}$ imply that $\overline{G}^\bullet_{(\delta^n,\mu)}$ is better-reply secure (Reny, 1999, Proposition 3.2) and therefore, $\overline{G}^\bullet_{(\delta^n,\mu)}$, being a compact, quasiconcave game, possesses a Nash equilibrium σ^n (Reny, 1999, Theorem 3.1). We claim that σ^n is also a Nash equilibrium of the mixed extension of the game

$(X_i, u_i^{(\delta^n,\mu)})$. To see this, suppose that σ^n is not an equilibrium of the mixed extension of $(X_i, u_i^{(\delta^n,\mu)})$. Then, there exist i and $p_i \in M_i$ such that

$$\int_X u_i^{(\delta^n,\mu)} \mathrm{d}(p_i, \sigma_{-i}^n) > \int_X u_i^{(\delta^n,\mu)} \mathrm{d}\sigma^n. \quad (4)$$

If $O_i = \emptyset$ it is clear that (4) contradicts the fact that σ^n is a Nash equilibrium of $\overline{G}^{\bullet}_{(\delta^n,\mu)}$, so in the sequel we assume that $O_i \neq \emptyset$. If $O_i \neq \emptyset$, there is no loss of generality in assuming that no member of M_i weakly dominates p_i and that

$$p_i \in \arg\max_{\nu_i \in M_i} \int_X u_i^{(\delta^n,\mu)} \mathrm{d}(\nu_i, \sigma_{-i}^n).^{7} \quad (5)$$

To see that p_i may be taken weakly undominated, suppose that $\nu_i \in M_i$ weakly dominates p_i. Then, because $u_i(\cdot, x_{-i})$ is upper semicontinuous for every $x_{-i} \in X_{-i}$ (recall that $O_i \neq \emptyset$ and $G \in \overline{\mathfrak{g}}_{dd}$), the map $U_i(\cdot, p_{-i})$ is upper semicontinuous for every $p_{-i} \in M_{-i}$ (this is shown below). Consequently, by Corollary 1 of Salonen (1996), there exists a strategy $\varrho_i \in M_i$ that weakly dominates p_i and is not weakly dominated by any member of M_i. For this strategy, we have

$$\varrho_i \in \arg\max_{\nu_i \in M_i} \int_X u_i^{(\delta^n,\mu)} \mathrm{d}(\nu_i, \sigma_{-i}^n)$$

and $\int_X u_i^{(\delta^n,\mu)} \mathrm{d}(\varrho_i, \sigma_{-i}^n) > \int_X u_i^{(\delta^n,\mu)} \mathrm{d}\sigma^n$.

If $p_i(O_i) = 0$ it is clear that (4) contradicts the fact that σ^n is a Nash equilibrium of $\overline{G}^{\bullet}_{(\delta^n,\mu)}$. If p_i has atoms in O_i (i.e., if $p_i(\{x_i\}) > 0$ for some $x_i \in O_i$) it is easy to construct a $p_i^{\bullet} \in M_i$ that weakly dominates p_i, a contradiction. It remains to consider the case when p_i has no atoms in O_i and $p_i(O_i) > 0$.

Suppose that p_i has no atoms in O_i and $p_i(O_i) > 0$. Then, because $Q_i \cap O_i$ is dense in O_i, there exists $z_i \in Q_i \cap O_i \cap \mathrm{supp}(p_i)$. Since $z_i \in O_i$, there exists $\nu_i \in M_i$ such that $U_i(\nu_i, x_{-i}) \geq u_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $U_i(\nu_i, y_{-i}) > u_i(z_i, y_{-i})$ for some $y_{-i} \in X_{-i}$. Note that if $z_i \in O_i \setminus \overline{O}_i$, there is no loss of generality in assuming that there exists a neighborhood $V_{y_{-i}}$ of y_{-i} such that $U_i(\nu_i, z_{-i}) > u_i(z_i, z_{-i})$ for all $z_{-i} \in V_{y_{-i}}$. Let (ϵ^k) be a sequence in $(0, 1)$ with $\epsilon^k \searrow 0$. Define $p_i^k \in M_i$ by

$$p_i^k(B_i) := p_i(B_i \setminus N_{\epsilon^k}(z_i)) + \nu_i(B_i)p_i(N_{\epsilon^k}(z_i))$$

and observe that p_i satisfies (for every Borel set $B_i \subseteq X_i$)

$$p_i(B_i) = p_i(B_i \setminus N_{\epsilon^k}(z_i)) + \nu_i^k(B_i)p_i(N_{\epsilon^k}(z_i)),$$

where $\nu_i^k \in M_i$ is defined by $\nu_i^k(B_i) := \frac{p_i(B_i \cap N_{\epsilon^k}(z_i))}{p_i(N_{\epsilon^k}(z_i))}$.[8]

Recall that $U_i(\nu_i, x_{-i}) \geq u_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}$ and $U_i(\nu_i, y_{-i}) > u_i(z_i, y_{-i})$. Also recall that if $z_i \in O_i \setminus \overline{O}_i$, there exists a neighborhood $V_{y_{-i}}$ of y_{-i} such that $U_i(\nu_i, z_{-i}) > u_i(z_i, z_{-i})$ for all $z_{-i} \in V_{y_{-i}}$, and since $\mu \in \widehat{M}$, we have $\mu_{-i}(V_{y_{-i}}) > 0$. Finally, observe that if $z_i \in \overline{O}_i$, then $\mu_{-i}(\{y_{-i}\}) > 0$. Therefore, we have

$$U_i(\nu_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) > U_i(z_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}).$$

Now, since $U_i(\cdot, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i})$ is upper semicontinuous at z_i, for large enough k we have

$$U_i(\nu_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) > \eta > U_i(w_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i})$$

for some $\eta \in \mathbb{R}$ and all $w_i \in N_{\epsilon^k}(z_i)$, so

$$U_i(\nu_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) > U_i(\nu_i^k, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}).$$

Consequently, for large enough k we have

$$\begin{aligned} &U_i(p_i^k, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) \\ &= \int_{X_i \setminus N_{\epsilon^k}(z_i)} U_i(\cdot, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i})\mathrm{d}p_i \\ &\quad + p_i(N_{\epsilon^k}(z_i))U_i(\nu_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) \\ &> \int_{X_i \setminus N_{\epsilon^k}(z_i)} U_i(\cdot, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i})\mathrm{d}p_i \\ &\quad + p_i(N_{\epsilon^k}(z_i))U_i(\nu_i^k, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}) \\ &= U_i(p_i, (1-\delta^n)\sigma_{-i}^n + \delta^n\mu_{-i}), \end{aligned}$$

thereby contradicting (5).

Now, since σ^n is a Nash equilibrium of the mixed extension of the game $(X_i, u_i^{(\delta^n,\mu)})$, $(1-\delta^n)\sigma^n + \delta^n\mu$ is a Nash equilibrium of $\overline{G}_{\delta^n\mu}$. Because M is sequentially compact, we may write (passing to a subsequence if necessary) $\sigma^n \to \sigma$ for some $\sigma \in M$. This, together with the fact that $\sigma_i^n(O_i) = 0$ for each i and all n, gives $\sigma_i(O_i) = 0$. Consequently, σ is a *lap* equilibrium of G.

It remains to show that the map $U_i(\cdot, p_{-i})$ is upper semicontinuous for every $p_{-i} \in M_{-i}$ if $u_i(\cdot, x_{-i})$ is upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$. Because $u_i(\cdot, x_{-i})$ is upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is upper semicontinuous on M_i for every $x_{-i} \in X_{-i}$ (e.g., Aliprantis and Border, 2006, Theorem 15.5), and so, given $\nu_i^n \to \nu_i$,

$$\limsup_{n\to\infty} U_i(\nu_i^n, x_{-i}) \leq U_i(\nu_i, x_{-i}), \quad \text{for all } (\nu_i, x_{-i}) \in M_i \times X_{-i}. \quad (6)$$

Now take $\nu_i^n \to \nu_i$. For each n, define $f_i^n : X_{-i} \to \mathbb{R}$ by $f_i^n(x_{-i}) := U_i(\nu_i^n, x_{-i})$. Given $p_{-i} \in M_{-i}$, we have, by Fatou's lemma,

$$\limsup_{n\to\infty} \int_{X_{-i}} f_i^n \mathrm{d}p_{-i} \leq \int_{X_{-i}} \limsup_{n\to\infty} f_i^n \mathrm{d}p_{-i}.$$

This, combined with (6), gives

$$\limsup_{n\to\infty} U_i(\nu_i^n, p_{-i}) \leq U_i(\nu_i, p_{-i}),$$

so $U_i(\cdot, p_{-i})$ is upper semicontinuous on M_i for every $p_{-i} \in M_{-i}$. $\square$

Remark 5. The conditions for existence of *lap** equilibria (resp. *lap* equilibria) provided in Lemma 3 (resp. Lemma 4) apply in various economic games. For example, the games in Examples 4 and 6 of Carbonell-Nicolau (2011b) are members of $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$).

Limit admissible perfection (resp. limit admissible perfection*) satisfies (LA) (resp. (LA*)) and (P) (every *lap* equilibrium (resp. *lap** equilibrium) is *thp*, and every *thp* is the limit of Nash equilibria in Selten perturbations). Furthermore, since trembling-hand perfect equilibria are admissible (hence admissible*) in finite games, the set of *lap* equilibria (resp. *lap** equilibria) coincides with the set of trembling-hand perfect equilibria in finite games (i.e., limit admissible perfection (resp. limit admissible perfection*) meets (R)).[9]

On the other hand, trembling-hand perfection (and therefore strong perfection) fails (LA) (resp. (LA*)) within $\overline{\mathfrak{g}}_{dd}$ (resp. $\widehat{\mathfrak{g}}_{dd}$). In

[7] The set $\arg\max_{\nu_i \in M_i} \int_X u_i^{(\delta^n,\mu)} \mathrm{d}(\nu_i, \sigma_{-i}^n)$ is well-defined because if $O_i \neq \emptyset$ the map $U_i(\cdot, p_{-i})$ is upper semicontinuous for every $p_{-i} \in M_{-i}$ (this is shown below), and this implies that the map $\nu_i \mapsto \int_X u_i^{(\delta^n,\mu)} \mathrm{d}(\nu_i, \sigma_{-i}^n)$ defined on M_i is upper semicontinuous.

[8] Observe that because $z_i \in \mathrm{supp}(p_i)$ we have $p_i(N_{\epsilon^k}(z_i)) > 0$.

[9] Also, note that, in $\mathfrak{g}_d$, *thp* equilibria are *lap* (resp. *lap**), and conversely.

fact, the game in Example 1 belongs to $\overline{\mathfrak{g}}_{dd} \cap \widehat{\mathfrak{g}}_{dd}$,[10] and for this game $O_i = O_i^*$ for each i.

These observations, along with Lemmas 3 and 4, are summarized by the following statements.

Theorem 6. *For the family of games* $\widehat{\mathfrak{g}}_{dd}$*, we have the following:*

- *Limit admissible perfection* satisfies* (LA*), (P), (R), *and* (E).
- *Strong perfection and trembling-hand perfection satisfy* (E), (P), *and* (R), *and fail* (LA*).
- *Weak perfection satisfies* (E) *and* (R), *and fails* (P) *and* (LA*).

Theorem 7. *For the family of games* $\overline{\mathfrak{g}}_{dd}$*, we have the following:*

- *Limit admissible perfection satisfies* (LA), (P), (R), *and* (E).
- *Strong perfection and trembling-hand perfection satisfy* (E), (P), *and* (R), *and fail* (LA).
- *Weak perfection satisfies* (E) *and* (R), *and fails* (P) *and* (LA).

Remark 6. Observe that properties (S), (W), and (N) do not appear in the statement of Theorems 6 and 7. For the class of games $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$), strong perfection (or trembling-hand perfection) and weak perfection need not select Nash equilibria. This is a consequence of the fact that $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$) does not sufficiently constrain the players' payoff functions in the interior of the set of weakly* dominated (resp. weakly dominated) strategies. There is, however, an easy fix to this problem: impose Reny's (1999) better-reply security of $\overline{G}$ on the members G of $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$).[11]

We now indicate the difficulties that may arise when studying certain strengthenings of trembling-hand perfection. The game in Example 1 has a *thp* equilibrium μ that is not limit admissible because, at μ, even though player 1 chooses a strategy, 0, that is weakly dominated by $\frac{1}{2}$, the action $\frac{1}{2}$ outperforms 0 only when the opponent chooses 0, and there are 'trembles' (completely mixed strategies) of the opponent that assign measure zero to the action 0. Thus, if player 1 expects player 2 to mischoose according to one such tremble, he finds it optimal to choose the strategy 0, even though 0 is weakly dominated by $\frac{1}{2}$. But if player 1 entertained the possibility that player 2's imperfect choice might follow one of many possible trembles, some of which put positive mass on the action 0, then player 1 would never choose the weakly dominated action, 0. For example, requiring robustness of Nash equilibria to the players' choice of *any* slight tremble within a small neighborhood of completely mixed strategies would guarantee the selection of limit admissible strategy profiles.

While it is well-known that *strictly perfect equilibria* (Okada, 1981) may fail to exist even in finite games, the above discussion suggests that Kohlberg and Merten's (1986) set-valued notion of stability might select only limit admissible equilibrium points. Even if this were true, we point out that establishing the existence of strategically stable sets for the classes of games considered in this paper poses difficulties. Our argument is informal and confines attention to the collection $\mathfrak{g}_{dd}$.

The existence of strategically stable sets in a game G requires the existence of Nash equilibria in open sets of slight Selten perturbations of G. Given $\delta \in (0,1)^N$, some $\mu \in \widehat{M}$ must exist such that the games in $\{\overline{G}_{\delta\varrho} : \varrho \in O_\mu\}$ possess a Nash equilibrium, for some neighborhood O_μ of μ. However, we have found a game G in $\mathfrak{g}_{dd}$ satisfying the following: given any $\mu \in \widehat{M}$ and $\delta \in (0,1)^N$, one can find many ϱ arbitrarily close to μ such that $\overline{G}_{\delta\varrho}$ fails Reny's (1999) payoff security and better-reply security. Consequently, Reny's (1999) results cannot be used to establish the existence of Nash equilibria in open sets of Selten perturbations of G.[12]

6. Concluding remarks

We have studied trembling-hand perfect equilibria in possibly discontinuous, normal-form games, by comparing the properties of several infinite-game generalizations of Selten's (1975) notion of perfection. The behavior of the various formulations of trembling-hand perfection in the set $\mathfrak{g}_c$ of compact, metric, continuous games extends to certain supersets of $\mathfrak{g}_c$. For larger classes of games, however, the existing solution concepts may select equilibria in the interior of the set of weakly dominated strategies. While this problem might be remedied by using stronger refinement specifications, for the classes of games considered in this paper these specifications are likely to fail conditions that are known to give existence. However, for rich classes of discontinuous games, the notion of limit admissible perfect equilibrium is not subject to this problem, and meets other desiderata.

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10 For this game, $\overline{O}_2 = \emptyset$ (so μ_1 in Condition ($\overline{\text{A}}$) can be freely chosen from $\widehat{M}_1$), $\overline{O}_1 = \left[0, \frac{1}{2}\right)$, and, letting Q_1 be the set of rational numbers in [0, 1], $\phi_1(x_1) = 0$ for each $x_1 \in O_1^* \cap Q_1$. Hence, for any $\mu = (\mu_1, \mu_2) \in \widehat{M}$ such that $\mu_2(\{0\}) > 0$ we have $\mu \in \overline{M}$. Given that $X_1^\bullet = \left[\frac{1}{2}, 1\right]$ and $X_2^\bullet = \{1\}$, it is easy to verify items (a) and (b) of Condition ($\overline{\text{A}}$) for any such μ.

11 Better-reply security of $\overline{G}$ for the members G of $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$) is satisfied, for example, by the members of $\widehat{\mathfrak{g}}_{dd}$ (resp. $\overline{\mathfrak{g}}_{dd}$) that are *uniformly payoff secure* (cf. Monteiro and Page, 2007).

12 There are at least two solutions to this problem. First, one can further constrain the points of discontinuity of the players' payoff functions. This is done in Carbonell-Nicolau (2011a), where the existence of stable sets is established for a strict subset of $\mathfrak{g}_{dd}$. Second, one can generalize some aspect of the main theorem in Reny (1999) in such a way that open sets of neighboring Selten perturbations can be shown to satisfy an *appropriate* weakening of better-reply security.