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RESEARCH ARTICLE

Perfect equilibria in games of incomplete information

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Abstract
This paper extends Selten's (Int J Game Theory 4:25–55, 1975) notion of perfection to normal-form games of incomplete information and provides conditions on the primitives of a game that ensure the existence of a perfect Bayes–Nash equilibrium. The existence results, which allow for arbitrary (compact, metric) type and/or action spaces and payoff discontinuities, are illustrated in the context of all-pay auctions and Cournot games with incomplete information and cost discontinuities.

Keywords Infinite game of incomplete information · Perfect Bayes–Nash equilibrium · Payoff security

JEL Classification C72

1 Introduction

The notion of perfect equilibrium was introduced by Selten (1975). For normal-form games with complete information, Selten's (1975) perfect equilibrium refines the Nash equilibrium concept by requiring that equilibrium strategies be immune to slight trembles in the execution of the players' actions. The standard definition of perfect equilibrium for normal-form games with finite action spaces (see, *e.g.*, van Damme 2002) can be extended to normal-form games with infinitely many actions, and these extensions have been studied by several authors (see, *e.g.*, Al-Najjar 1995; Simon and Stinchcombe 1995; Carbonell-Nicolau 2011a, b, c, 2014b; Carbonell-Nicolau and McLean 2013, 2014, 2015; Scalzo 2014; Bajoori et al. 2013). For applications of the notion of perfection as an equilibrium selection criterion in complete-information games, see, *e.g.*, Bagnoli and Lipman (1989), Broecker (1990), Pitchik and Schotter (1988), and Allen (1988).

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This paper considers an extension of the notion of perfection to normal-form games of incomplete information, also called Bayesian games, that refines the standard Bayes–Nash equilibrium concept. Roughly, a Bayes–Nash equilibrium is perfect if there are nearby Bayes–Nash equilibria in slightly perturbed Bayesian games in which each type of each player makes slight mistakes in the execution of her strategies. Conditions on the primitives of a Bayesian game are furnished under which a Bayesian game with infinitely many types and/or actions (henceforth infinite Bayesian games), and possibly with payoff discontinuities in type and action profiles, possesses a perfect Bayes–Nash equilibrium.[1]

While there is a substantial literature on the existence of Bayes–Nash equilibria in infinite Bayesian games,[2] there is very little work dealing with refinements. Jackson et al. (2002) employ the notion of perfection to eliminate "undesirable" Nash equilibria in one of their applications to second-price auction games. A stronger notion of perfection than the one studied here has been considered in Bajoori et al. (2016), where the refinement is applied to a particular class of second-price auctions. Bajoori et al. (2016) also obtain an existence result for Bayesian games with countable type spaces and finite action spaces. The results developed here allow for arbitrary (compact, metric) type and/or action spaces and payoff discontinuities.

Methodologically, the analysis builds on the work in Carbonell-Nicolau and McLean (2018), which obtains conditions on the primitives of a Bayesian game ensuring that the corresponding "behavioral normal-form game" (*i.e.*, the normal form defined in terms of behavioral strategies) satisfies the Reny (1999) criteria for existence of a Nash equilibrium. While these conditions are sufficient to establish existence of Nash equilibria in Bayesian games, they are generally not strong enough to ensure that the Reny (1999) existence result can be applied to slight *Selten perturbations* (of a Bayesian game's behavioral normal-form) in which the players "tremble" by playing a completely mixed strategy with positive, yet low, probability.[3] Because the perfection refinement requires existence of Nash equilibria in Selten perturbations, stronger conditions than those in Carbonell-Nicolau and McLean (2018) are needed here to establish existence of perfect Bayes–Nash equilibria via Reny's (1999) conditions.

The main condition developed in this paper is termed *strong uniform payoff security*. This property, which strengthens the uniform payoff security condition in Carbonell-Nicolau and McLean (2018) and collapses to Condition (A) in Carbonell-Nicolau

[1] The refinement of Bayes–Nash equilibrium for normal-form Bayesian games considered in this paper should not be confused with the so-called *perfect Bayesian equilibrium* concept for dynamic games of incomplete information (with finitely many types and actions) (see, *e.g*., Fudenberg and Tirole (1991, Chapter 8)), which extends the notion of subgame perfection to extensive-form games with incomplete information.

[2] See, *e.g*., Milgrom and Weber (1985); Balder (1988); Carbonell-Nicolau and McLean (2018, 2019, 2020); He and Yannelis (2016); Yannelis and Rustichini (1991); Hellman and Levy (2017); Athey (2001); McAdams (2003); Reny (2011). Prokopovych and Yannelis (2019) and He and Sun (2019) study certain robustness properties of (pure-strategy) Bayes–Nash equilibria. These properties are similar in spirit—insofar as they involve continuity of equilibrium points with respect to slight perturbations of a Bayesian game—but different in nature from the ones considered in this paper.

[3] A proof of this assertion can be found in Carbonell-Nicolau (2011b, Example 3, p. 243), which features a complete-information game satisfying the conditions in Carbonell-Nicolau and McLean (2018), which guarantee that the game's "behavioral normal-form" (*i.e*., in this case, its mixed extension) satisfies the Reny (1999) conditions, while the game's Selten perturbations do *not* satisfy them.

(2011a, b) in the special case of complete-information games, implies that the Selten perturbations of a Bayesian game satisfy Reny's (1999) *payoff security*.

Strong uniform payoff security, together with the standard upper semicontinuity of the sum of the game's payoff functions in the players' pure strategies, yields existence of perfect Bayes–Nash equilibria (via the Reny 1999 existence criteria) (Theorem 1). Verifying the strong uniform payoff security condition in applications can be relatively straightforward, as illustrated in the context of all-pay auctions, in Sect. 4.1.

The strong uniform payoff security condition can be decomposed into two independent properties, which are easily verified in certain applications, such as Cournot games with incomplete information and cost discontinuities. A second existence result (Theorem 2) is presented in terms of these two conditions and illustrated in the context of Cournot competition.

2 Preliminaries

Throughout the paper, the following definitions will be adopted. If A is a topological space, then $\mathscr{B}(A)$ will denote the σ-algebra of the Borel subsets of A. If A is a σ-algebra of subsets of A, then $\Delta(A, \mathscr{A})$ will represent the set of probability measures on $(A, \mathscr{A})$, and $C^b(A)$ will denote the set of all bounded continuous real-valued functions on A.

Definition 1 Let A be a topological space and let $\mathscr{A}$ be a σ-alebra of subsets of A containing $\mathscr{B}(A)$. The ***w-topology*** on $\Delta(A, \mathscr{A})$ is defined as the coarsest topology for which all the functionals in

$$\left\{\mu \in \Delta(A, \mathscr{A}) \mapsto \int_A f(a)\mu(da) \in \mathbb{R} : f \in C^b(A)\right\}$$

are continuous.

We will refer to convergence of measures in $\Delta(A, \mathscr{A})$ with respect to the w-topology as ***weak convergence*** of measures and we will write $\mu^n \underset{w}{\rightarrow} \mu$ to indicate that the sequence of measures (μ^n) converges weakly to μ.

When $\mathscr{A} = \mathscr{B}(A)$, we write $\Delta(A)$ for $\Delta(A, \mathscr{B}(A))$. In this case, the members of $\Delta(A)$ are Borel probability measures, and the topology from Definition 1, defined on $\Delta(A)$, coincides with that studied in Varadarajan (1965).

If A is a complete, separable metric space, the w-topology on $\Delta(A)$ is metrizable, and the Prokhorov metric defines a compatible metric (Prokhorov 1956, Theorem 1.11).

2.1 Games and strategies

Definition 2 A ***normal-form game*** (or simply a ***game***) is a collection $G = (Z_i, g_i)_{i=1}^N$, where N is a finite number of players, Z_i is a nonempty set of actions for player i, and $g_i : Z \to \mathbb{R}$ represents player i's payoff function, defined on the set of action profiles $Z := \times_{i=1}^N Z_i$. The game G is called a ***metric game*** (resp. a ***compact game***) if each Z_i is a metric (resp. compact) space. A compact metric game $G = (Z_i, g_i)_{i=1}^N$ is called a ***Borel game*** if each g_i is bounded and $(\mathscr{B}(Z), \mathscr{B}(\mathbb{R}))$-measurable.

Throughout the sequel, given N sets $Z_1, \ldots, Z_N$, we adhere to the following standard notation: for $i \in \{1, \ldots, N\}$, $Z_{-i} := \times_{j \neq i} Z_j$; given i, the set $\times_{j=1}^N Z_j$ is sometimes denoted as $Z_i \times Z_{-i}$, and a member z of $\times_{j=1}^N Z_j$ is sometimes represented as $z = (z_i, z_{-i}) \in Z_i \times Z_{-i}$.

The following definition of a Bayesian game is standard in the literature.

Definition 3 A ***Bayesian game*** is a collection

$$\Gamma = (T_i, X_i, u_i, p)_{i=1}^N,$$

where

- $\{1, \ldots, N\}$ is a finite set of players;
- T_i is a nonempty, compact, metric space of types for player i;
- X_i is a nonempty, compact, metric space of actions for player i;
- u_i is a real-valued map on $T \times X$, where $T := \times_{i=1}^N T_i$ and $X := \times_{i=1}^N X_i$; it represents player i's payoff function, and it is assumed bounded and $(\mathscr{B}(T \times X), \mathcal{B}(\mathbb{R}))$-measurable; and
- p is a probability measure on $(T, \mathscr{B}(T))$ (a member of $\Delta(T)$) describing the players' common priors over type profiles.

For each $i \in \{1, \ldots, N\}$, let p_i be the marginal probability measure induced by p on T_i, *i.e.*, the probability measure in $\Delta(T_i)$ defined by

$$p_i(S) := p(S \times T_{-i}), \quad \text{for every } S \in \mathscr{B}(T_i). \tag{1}$$

Definition 4 Let $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ be a Bayesian game. A ***distributional strategy*** for player i in Γ is a probability measure σ_i in $\Delta(T_i \times X_i)$ such that

$$\sigma_i(A \times X_i) = p_i(A), \quad \text{for all } A \in \mathscr{B}(T_i).$$

Let $\mathscr{D}_i$ represent the set of distributional strategies for player i, and define $\mathscr{D} := \times_{i=1}^N \mathscr{D}_i$.

Given $\sigma_i \in \mathscr{D}_i$, the map $t_i \in T_i \mapsto \sigma_i(\cdot|t_i) \in \Delta(X_i)$ will denote a corresponding version of the regular conditional probability measure on X_i.

Definition 5 A distributional strategy $\sigma_i \in \mathscr{D}_i$ is *strictly positive* if for each $t_i \in T_i$, $\sigma_i(V|t_i) > 0$ for every nonempty open set V in X_i.

The set of all strictly positive distributional strategies in $\mathscr{D}_i$ is denoted by $\widehat{\mathscr{D}_i}$, and the Cartesian product $\times_{j=1}^N \widehat{\mathscr{D}_j}$ is denoted by $\widehat{\mathscr{D}}$. Each $\mathscr{D}_i$ will be endowed with the relative w-topology (Definition 1) on $\Delta(T_i \times X_i)$, and $\mathscr{D}$ will be endowed with the corresponding product topology.

Given a Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$, the *normal form* of Γ is defined as

$$\mathfrak{G}_\Gamma := (\mathscr{D}_i, U_i)_{i=1}^N, \tag{2}$$

where $U_i : \mathscr{D} \to \mathbb{R}$ is given by

$$U_i(\sigma_1, \ldots, \sigma_N) := \int_T \int_{X_N} \cdots \int_{X_1} u_i(t, x)\sigma_1(dx_1|t_1)\cdots\sigma_N(dx_N|t_N)p(dt).$$

As is standard in the literature (see Milgrom and Weber 1985; Balder 1988, and Carbonell-Nicolau and McLean 2018) a *Bayes–Nash equilibrium* of Γ (Definition 6 below) is defined as a Nash equilibrium of the normal-form game (Definition 2) $\mathfrak{G}_\Gamma$.

Also standard in the literature is the following assumption on the joint information of the players in Γ, described by the common prior p: p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$ (recall the definition of the marginals p_i in (1)). This condition, called *absolutely continuous information* in Milgrom and Weber (1985), allows one to express the payoffs U_i in the normal form $\mathfrak{G}_\Gamma$ of Γ as follows:

$$U_i(\sigma_1, \ldots, \sigma_N) = \int_{T_N \times X_N} \cdots \int_{T_1 \times X_1} [u_i(t, x)g(t)]\sigma_1(d(t_1, x_1))\cdots\sigma_N(d(t_N, x_N)), \tag{3}$$

where g is a density of p with respect to $p_1 \otimes \cdots \otimes p_N$.[4] This fact will be used repeatedly in this paper.

Next, we define a *Selten perturbation* of the normal form $\mathfrak{G}_\Gamma$, a variant of $\mathfrak{G}_\Gamma$ in which, with certain probability α_i, each player i "trembles" by playing a completely mixed strategy μ_i.

Given $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$ and $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$, define the normal-form game

$$\mathfrak{G}_\Gamma^{(\alpha,\mu)} := (\mathscr{D}_i, U_i^{(\alpha,\mu)})_{i=1}^N, \tag{4}$$

where $U_i^{(\alpha,\mu)} : \mathscr{D} \to \mathbb{R}$ is defined by

$$U_i^{(\alpha,\mu)}(\sigma_1, \ldots, \sigma_N) := U_i((1 - \alpha_1)\sigma_1 + \alpha_1\mu_1, \ldots, (1 - \alpha_N)\sigma_N + \alpha_N\mu_N).$$

Selten perturbations of the form $\mathfrak{G}_\Gamma^{(\alpha,\mu)}$ (see (4)) are instrumental in the definition of a *perfect Bayes–Nash equilibrium* (Definition 7 below).

2.2 Equilibrium

The following definition of a Bayes–Nash equilibrium of a Bayesian game is standard in the literature.

4 Without the absolutely continuous information condition, Bayes–Nash equilibria need not exist (see Simon 2003; Hellman 2014; Hellman and Levy 2017). There are, however, certain classes of Bayesian games for which this condition is not needed for existence (see Athey 2001; McAdams 2003; Reny 2011; Yannelis and Rustichini 1991; Hellman and Levy 2017; Carbonell-Nicolau and McLean 2020).

Definition 6 A ***Bayes–Nash equilibrium*** of a Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ is a Nash equilibrium of the game $\mathfrak{G}_\Gamma$ defined in (2), *i. e.*, a profile $(\sigma_1, \dots, \sigma_N) \in \mathscr{D}$ such that for each i,

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\nu_i, \sigma_{-i}), \quad \text{for all } \nu_i \in \mathscr{D}_i.$$

This paper introduces the following refinement of Definition 6.

Definition 7 A Bayes–Nash equilibrium σ of a Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ is ***perfect*** if there exist sequences (α^n), (μ^n), and (σ^n) such that the following holds:

- For each n, $\alpha^n \in (0, 1)^N$, $\mu^n \in \widehat{\mathscr{D}}$, and σ^n is a Nash equilibrium of the game $\mathfrak{G}_\Gamma^{(\alpha^n,\mu^n)}$ defined in (4).
- $\alpha^n \to 0$ and $\sigma^n \underset{w}{\to} \sigma$.

Thus, a Bayes–Nash equilibrium σ of a Bayesian game Γ is perfect if for nearby Selten perturbations of $\mathfrak{G}_\Gamma$ one can find Nash equilibria close to σ.

In the special case of complete information games (*i.e.*, when type spaces are singletons), this definition collapses to the notion of perfection considered in Al-Najjar (1995), Carbonell-Nicolau (2011a, b, c, 2014b), Carbonell-Nicolau and McLean (2013, 2014, 2015), and the strong notion of perfection defined in Simon and Stinchcombe (1995).

Bajoori et al. (2016) consider a stronger notion of perfection whereby, roughly speaking, the convergence condition '$\sigma^n \underset{w}{\to} \sigma$' in the second bullet point is replaced by pointwise convergence of a version of the regular conditional probability measures. They obtain an existence result for Bayesian games with countable type spaces and finite action spaces, and present an application to a class of second-price auctions. The results developed here allow for arbitrary (compact, metric) type and/or action spaces and payoff discontinuities.[5]

3 Existence of perfect equilibrium

This section contains the main existence results for the refinement in Definition 7. The aim of the paper is to obtain conditions on the objects in a Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ (Definition 3) that guarantee the existence of a perfect Bayes–Nash equilibrium.

The notion of *strong uniform payoff security* (Definition 9 below) plays a central role in the results of this paper. This condition has the flavor of the so-called *payoff security* condition, which was used in Reny (1999) to prove results on the existence of Nash equilibrium in discontinuous normal-form games.

Definition 8 (Reny 1999). A metric game $(Z_i, g_i)_{i=1}^N$ is ***payoff secure*** if for each $\varepsilon > 0$, $z \in \times_{i=1}^N Z_i$, and i, there exist a $y_i \in Z_i$ and a neighborhood $V_{z_{-i}}$ of z_{-i} such that $g_i(y_i, y_{-i}) > g_i(z) - \varepsilon$ for every $y_{-i} \in V_{z_{-i}}$.

[5] It should be pointed out that the existence results obtained here do not apply to the class of second-price auctions considered in Bajoori et al. (2016). Existence for this class is established, in Bajoori et al. (2016), by direct construction of a perfect Bayes–Nash equilibrium.

Payoff security of a game does not generally imply that the game's mixed extension is itself payoff secure. Monteiro and Page (2007) introduced a strengthening of payoff security, termed *uniform payoff security*, which does ensure that a normal-form game has a payoff secure mixed extension. In a similar fashion, Carbonell-Nicolau and McLean (2018) introduced a notion of uniform payoff security, defined on the primitives of a Bayesian game, which coincides with the Monteiro-Page condition in the special case of complete information games (*i.e.*, when type spaces are singletons), and which ensures that the normal form of a Bayesian game (recall (2)) is itself payoff secure. Carbonell-Nicolau and McLean's (2018) uniform payoff security condition, however, is generally too weak to guarantee the payoff security of the game's Selten perturbations, as defined in (4).

This paper proposes the following strengthening of Carbonell-Nicolau and McLean's (2018) condition, which is also an extension of Condition (A) in Carbonell-Nicolau (2011a, b) to the class of Bayesian games. This condition is strong enough to ensure that the Selten perturbations of a Bayesian game are payoff secure.

Definition 9 The Bayesian game $(T_i, X_i, u_i, p)_{i=1}^N$ satisfies ***strong uniform payoff security*** if there exists $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$ there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying the following:

(a) For each k and $(t, x) \in T \times X$, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) > u_i(t, x) - \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

(b) For each $(t, x_{-i}) \in T \times X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i|t_i) = 1$ satisfying the following: for each $x_i \in Y_i$, there exists K such that for all $k \geq K$, there is a neighborhood $V'_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) < u_i(t, (x_i, y_{-i})) + \varepsilon, \quad \text{for all } y_{-i} \in V'_{x_{-i}}.$$

Intuitively, the sequence of maps (f^k) in Definition 9 must satisfy the following two conditions. First, for each type t_i and each action x_i of player i, the action $f^k(t_i, x_i)$ from player i's action space X_i secures a payoff "virtually" as large as $u_i(t, x)$, for every $(t_{-i}, x_{-i}) \in T_{-i} \times X_{-i}$, even when the action $f^k(t_i, x_i)$ is played against a perturbed action profile y_{-i} for the rest of the players that is sufficiently close to x_{-i}. Second, for each type profile $t \in T$ and every action profile $x_{-i} \in X_{-i}$ for all the players except i, there exists a "large" subset Y_i of i's action space X_i, where Y_i *may depend on* (t, x_{-i}), such that each action x_i in Y_i (but not those actions outside of Y_i) eventually (*i.e.*, for large enough k) secures "virtually" the same payoff as $f^k(t_i, x_i)$ against any action profile for the rest of the players, y_{-i}, in a sufficiently small neighborhood of x_{-i} (where the neighborhood of x_{-i} can be chosen as a function of x_i and k).

The application in Sect. 4.1 illustrates how the fact that the set Y_i can be chosen to vary with (t, x_{-i}) and the neighborhood of x_{-i} can be chosen as a function of x_i and k confers meaningful flexibility to the strong uniform payoff security condition.

A continuous Bayesian game (*i.e.*, a game $(T_i, X_i, u_i, p)_{i=1}^N$ such that $u_i(t, \cdot)$ is continuous on X for each $t \in T$ and each i) is easily seen to satisfy strong uniform payoff security. In fact, it is straightforward to verify that, for continuous Bayesian games, the constant sequence (f^k) defined by $f^k(t_i, x_i) := x_i$, for each $(t_i, x_i) \in T_i \times X_i$ and each k, satisfies items (a) and (b) in Definition 9.

While item (a) in Definition 9 gives Carbonell-Nicolau and McLean's (2018) uniform payoff security condition and suffices for the normal form of a Bayesian game to be payoff secure, the extra condition, item (b), is needed to ensure that the game's Selten perturbations are also payoff secure.[6] Formally, we have the following lemma.

Lemma 1 *Suppose that the Bayesian game* $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ *satisfies strong uniform payoff security. If* p *is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$*, then there exists* $\mu \in \widehat{\mathscr{D}}$ *such that the game* $\mathfrak{G}_\Gamma^{(\alpha,\mu)}$ *defined in* (4) *is payoff secure for each* $\alpha \in [0, 1)^N$.

Lemma 1, combined with the following lemma, is instrumental in the proofs of the main results.

Lemma 2 *Given a Bayesian game* $(T_i, X_i, u_i, p)_{i=1}^N$*, suppose that for each* $t \in T$*, the map* $\sum_{i=1}^N u_i(t, \cdot) : X \to \mathbb{R}$ *is upper semicontinuous. Suppose further that* p *is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$*. Then the map* $\sum_{i=1}^N U_i(\cdot) : \mathscr{D} \to \mathbb{R}$ *is upper semicontinuous.*

If one takes the preceding two lemmas for granted, the proof of our first main existence result (Theorem 1 below) is relatively straightforward. In order to preserve the flow of the exposition, Theorem 1 is stated and proven next, while the proofs of the more technical Lemma 1 and Lemma 2 are relegated to Sect. 5.

Theorem 1 *Suppose that the Bayesian game* $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ *satisfies strong uniform payoff security and that for each* $t \in T$*, the map* $\sum_{i=1}^N u_i(t, \cdot) : X \to \mathbb{R}$ *is upper semicontinuous. If* p *is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$*, then* Γ *possesses a perfect Bayes–Nash equilibrium.*

Proof For each $n \in \mathbb{N}$, let $\alpha^n := \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. By Lemma 1, there exists $\mu \in \widehat{\mathscr{D}}$ such that for each n, the game $\mathfrak{G}_\Gamma^{(\alpha^n,\mu)}$ is payoff secure. In addition, Lemma 2 implies that the map $\sum_{i=1}^N U_i(\cdot) : \mathscr{D} \to \mathbb{R}$ is upper semicontinuous, implying that for each n the map $\sum_{i=1}^N U_i^{(\alpha^n,\mu)}(\cdot) : \mathscr{D} \to \mathbb{R}$ is upper semicontinuous. Consequently, since each $\mathscr{D}_i$ is a compact (see Milgrom and Weber (1985, p. 626)), convex subset of a topological vector space, and since the game $\mathfrak{G}_\Gamma^{(\alpha^n,\mu)}$ is quasiconcave for each n, it follows from Proposition 3.2 and Theorem 3.1 of Reny (1999) that the game $\mathfrak{G}_\Gamma^{(\alpha^n,\mu)}$ has a Nash equilibrum σ^n for each n.[7] Now, since the sequence (σ^n) lies in $\mathscr{D}$ and

[6] Indeed, it has been shown in Carbonell-Nicolau (2011b, Example 3) that, in complete information games, uniform payoff security need not imply payoff security of a game's Selten perturbations.

[7] A normal-form game $(Z_i, g_i)_{i=1}^N$ is *quasiconcave* if each Z_i is a convex subset of a topological vector space and, for each i and $z_{-i} \in Z_{-i}$, $g_i(\cdot, z_{-i})$ is quasiconcave on Z_i.

since $\mathscr{D}$ is sequentially compact, one may write (passing to a subsequence if necessary) $\sigma^n \underset{w}{\rightarrow} \sigma$ for some $\sigma \in \mathscr{D}$. It follows that σ is a perfect profile.

It remains to show that σ is a Bayes–Nash equilibrium of Γ. We shall assume that σ is not a Bayes–Nash equilibrium of Γ and derive a contradiction. Because $\sigma^n \underset{w}{\rightarrow} \sigma$ and since each U_i is bounded, we have (passing to a subsequence if necessary) $(\sigma^n, (U_1(\sigma^n), \ldots, U_N(\sigma^n))) \rightarrow (\sigma, (\beta_1, \ldots, \beta_N))$ for some $(\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$. If σ is not a Nash equilibrium of the game $\mathfrak{G}_\Gamma$ defined in (2), then, since $\mathfrak{G}_\Gamma$ satisfies better-reply security as defined in Reny (1999) (by Lemma 1, Lemma 2, and by Proposition 3.2 in Reny (1999)),[8] it follows that there exist i, $\sigma_i^* \in \mathscr{D}_i$, a neighborhood $V_{\sigma_{-i}}$ of σ_{-i}, and $\zeta > 0$ such that

$$U_i(\sigma_i^*, \sigma'_{-i}) \geq \beta_i + \zeta, \quad \text{for all } \sigma'_{-i} \in V_{\sigma_{-i}}.$$

Therefore, since $U_i(\sigma^n) \rightarrow \beta_i$, there exist $\zeta' > 0$ and $\overline{n}$ such that

$$U_i(\sigma_i^*, \sigma_{-i}^n) > U_i(\sigma^n) + \zeta', \quad \text{for all } n \geq \overline{n}.$$

Consequently, using (3), we see that there exists a large enough n' such that

$$U_i((1-\alpha_i^n)\sigma_i^* + \alpha_i^n \mu_i, ((1-\alpha_j^n)\sigma_j^n + \alpha_j^n \mu_j)_{j \neq i}) > U_i((1-\alpha_1^n)\sigma_1^n + \alpha_1^n \mu_1, \ldots, (1-\alpha_N^n)\sigma_N^n + \alpha_N^n \mu_N)$$

for all $n \geq n'$, contradicting that σ^n is a Nash equilibrium of $\mathfrak{G}_\Gamma^{(\alpha^n, \mu)}$ for each n. □

Theorem 1 is illustrated, in Sect. 4, in the context of all-pay auctions.

In the remainder of this section, we furnish a variant of Theorem 1 in which uniform payoff security (as formulated in Definition 9) is replaced by two conditions that do not require an explicit construction of the $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps f^k.

The formulation of the first condition requires some preliminaries.

Given a Bayesian game $(T_i, X_i, u_i, p)_{i=1}^N$, let A_i be the set of all accumulation points of X_i (*i.e.*, the set of all points $x_i \in X_i$ such that $(V_{x_i} \setminus \{x_i\}) \cap X_i \neq \emptyset$ for every neighborhood V_{x_i} of x_i). Since X_i is compact and metric, it can be written as a disjoint union $A_i \cup K_i$, where A_i is closed and dense in itself (*i.e.*, with no isolated points) and K_i is a countable subset of X_i whose members are isolated points (*i.e.*,

[8] Given a metric game $G = (Z_i, g_i)_{i=1}^N$, let $\text{Gr}(G)$ represent the graph of the game's vector payoff function, *i.e.*,

$$\text{Gr}(G) := \left\{(z, a) \in Z \times \mathbb{R}^N : a = g(z) = (g_1(z), \ldots, g_N(z))\right\}.$$

The closure of $\text{Gr}(G)$ in $Z \times \mathbb{R}^N$ is denoted by $\text{cl}(\text{Gr}(G))$.
The game G is said to be *better-reply secure* if, for every $(z, a) \in \text{cl}(\text{Gr}(G))$ such that z is not a Nash equilibrium of G, there exist a player j, $\beta \in \mathbb{R}$, $z_j^* \in Z_j$, and a neighborhood $V_{z_{-j}}$ of z_{-j} such that

$$g_j(z_j^*, y_{-j}) \geq \beta > a_j, \quad \text{for all } y_{-j} \in V_{z_{-j}}.$$

for each $x_i \in K_i$, there is a neighborhood of x_i, V_{x_i}, such that $(V_{x_i} \setminus \{x_i\}) \cap K_i = \emptyset$) (see, *e.g.*, Hausdorff (1962, p. 147)).

Definition 10 Suppose that $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ is a Bayesian game and, for each i, let $X_i = A_i \cup K_i$ be the decomposition from the preceding paragraph. The game Γ is said to satisfy ***generic entire payoff security*** if there exist countable subsets $C_1 \subseteq A_1, \ldots, C_N \subseteq A_N$ such that the following three conditions are satisfied:

(i) For each i, $\varepsilon > 0$, and $x_i \in A_i \setminus C_i$, and for every neighborhood V_{x_i} of x_i, there exist $y_i \in V_{x_i}$ and a neighborhood V'_{x_i} of x_i such that for every $(t, z_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$u_i(t, (y_i, y_{-i})) > u_i(t, (z'_i, z_{-i})) - \varepsilon, \quad \text{for all } (z'_i, y_{-i}) \in V'_{x_i} \times V_{z_{-i}}.$$

(ii) For each i, $\varepsilon > 0$, and $x_i \in K_i$, and for every neighborhood V_{x_i} of x_i, there exists $y_i \in V_{x_i}$ such that for every $(t, z_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$u_i(t, (y_i, y_{-i})) > u_i(t, (x_i, z_{-i})) - \varepsilon, \quad \text{for all } y_{-i} \in V_{z_{-i}}.$$

(iii) For each i, $\varepsilon > 0$, and $x_i \in C_i$, there exists $y_i \in X_i$ such that for every $(t, z_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$u_i(t, (y_i, y_{-i})) > u_i(t, (x_i, z_{-i})) - \varepsilon, \quad \text{for all } y_{-i} \in V_{z_{-i}}.$$

Note that the conditions in items (ii) and (iii) are strictly weaker than that in item (i). Roughly, generic entire payoff security requires condition (i) except for countably many x_i, *i.e.*, for the members of the countable set $C_i \cup K_i$, for which weaker conditions ((ii) for K_i and (iii) for C_i) are required.

Intuitively, for each player i and each action x_i in the player's action set X_i, an action $y_i \in X_i$ secures "virtually" the same payoff as x_i against any profile $z_{-i} \in X_{-i}$ of actions chosen by the other players, for any $t \in T$, if i's payoff $u_i(t, (y_i, y_{-i}))$ at t, when player i chooses y_i and the other players slightly deviate from z_{-i} to a nearby y_{-i}, is "virtually" as large as $u_i(t, (x_i, z_{-i}))$. Item (i) requires that the securing action y_i exist arbitrarily close to x_i and that the secured payoff $u_i(t, (y_i, y_{-i}))$ be "virtually" as large as $u_i(t, (z'_i, z_{-i}))$ when the action z'_i is a slight perturbation of x_i.

As is easily verified, a particular instance of the generic entire payoff security condition is the *equicontinuity* of the family $\{u_i(t, \cdot) : t \in T\}$, for each i, *i.e.*, the property that, for each i, and for every $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $u_i(t, y) \in N_\epsilon(u_i(t, x))$ for each $y \in N_\delta(x)$ and $t \in T$.

Definition 10, together with the *generic local equi-upper semicontinuity* condition (Definition 11 below), implies strong uniform payoff security (Definition 9) (see Lemma 3 below).

To formulate the notion of generic local equi-upper semicontinuity, we need the following terminology.

Recall that the set of all strictly positive distributional strategies in $\mathscr{D}_i$ (Definition 5) is denoted by $\widehat{\mathscr{D}}_i$. Let $\widetilde{\mathscr{D}}_i$ be the set of members σ_i of $\mathscr{D}_i$ such that for each $t_i \in T_i$,

$\sigma_i(\{x_i\}|t_i) = 0$ and $\sigma_i(N_\epsilon(x_i)|t_i) > 0$ for every $x_i \in A_i$ and $\epsilon > 0$ (where $N_\epsilon(x_i)$ denotes the ϵ-neighborhood of x_i), and $\sigma_i(\{x_i\}|t_i) > 0$ for every $x_i \in K_i$. Observe that $\widetilde{\mathscr{D}}_i \subseteq \widehat{\mathscr{D}}_i$. In addition, $\widetilde{\mathscr{D}}_i$ is nonempty (see, *e.g.*, Parthasarathy et al. (1962, Corollary 6.2)). Define $\widetilde{\mathscr{D}} := \times_{i=1}^N \widetilde{\mathscr{D}}_i$.

Definition 11 The Bayesian game $(T_i, X_i, u_i, p)_{i=1}^N$ satisfies ***generic local equi-upper semicontinuity*** if there exists $\mu = (\mu_1, \ldots, \mu_N) \in \widetilde{\mathscr{D}}$ such that for each i and $(t, x_{-i}) \in T \times X_{-i}$, there exists $Y_i \subseteq X_i$ with $\mu_i(Y_i|t_i) = 1$ satisfying the following: for each $x_i \in Y_i$ and $\varepsilon > 0$, there is a neighborhood V_{x_i} of x_i such that for every $y_i \in V_{x_i}$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (y_i, y_{-i})) < u_i(t, (x_i, y_{-i})) + \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

The generic local equi-upper semicontinuity condition requires that for each player i, and for every type profile $t \in T$ and every action profile $x_{-i} \in X_{-i}$ for the other players, there exist a "full-measure" subset of actions in X_i (which may depend on (t, x_{-i})) such that the members x_i of Y_i secure "virtually" a payoff at least as large as any other action y_i close enough to x_i against sufficiently small perturbations, y_{-i}, of x_{-i} for the rest of the players.

Note that a simple instance of the generic local equi-upper semicontinuity condition is the continuity of each $u_i(t, \cdot)$ on X for each $t \in T$.

The combination of generic entire payoff security (Definition 10) and generic local equi-upper semicontinuity (Definition 11) implies strong uniform payoff security (Definition 9).

Definition 10 and Definition 11 can be thought of as a decomposition of Definition 9 into two independent conditions, with Definition 10 (resp. Definition 11) being sufficient for item (a) (resp. item (b)) of Definition 9.

Lemma 3 *Suppose that $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ is a Bayesian game satisfying generic entire payoff security and generic local equi-upper semicontinuity. Then Γ satisfies strong uniform payoff security.*

The proof of Lemma 3 is relegated to Sect. 5.

From Theorem 1 and Lemma 3, one immediately obtains the second main existence result of the paper.

Theorem 2 *Suppose that the Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ satisfies generic entire payoff security and generic local equi-upper semicontinuity. Suppose further that for each $t \in T$, the map $\sum_{i=1}^N u_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous and p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$. Then Γ possesses a perfect Bayes–Nash equilibrium.*

Section 4 provides an illustration of Theorem 2 in the context of Cournot games.

3.1 The special case of complete information games

In this subsection, we state the main existence results in the absence of incomplete information (*i.e.*, when type spaces are singletons), obtaining Theorem 2 and Corollary 1 in Carbonell-Nicolau (2011b) as special cases of Theorem 1 and Theorem 2.

Definition 12 The ***mixed extension*** of a compact, metric, Borel game $G = (Z_i, g_i)_{i=1}^N$ is the normal-form game $\boldsymbol{G} := (\Delta(Z_i), G_i)_{i=1}^N$, where for each i, $G_i : \times_{j=1}^N \Delta(Z_j) \to \mathbb{R}$ is defined by

$$G_i(\sigma_1, \dots, \sigma_N) := \int_Z g_i(z)[\sigma_1 \otimes \cdots \otimes \sigma_N](dz).$$

Suppose that $G = (Z_i, g_i)_{i=1}^N$ is a compact, metric, Borel game. For each i, let $\widehat{\Delta}(Z_i)$ be the set of all *strictly positive* members of $\Delta(Z_i)$, *i.e.*, the set of all $\sigma_i \in \Delta(Z_i)$ such that $\sigma_i(V) > 0$ for every nonempty open set V in Z_i.

Given $\alpha = (\alpha_1, \dots, \alpha_N) \in [0, 1)^N$ and $\mu = (\mu_1, \dots, \mu_N) \in \times_{i=1}^N \widehat{\Delta}(Z_i)$, define the normal-form game

$$G^{(\alpha,\mu)} := (\Delta(Z_i), G_i^{(\alpha,\mu)})_{i=1}^N, \tag{5}$$

where $G_i^{(\alpha,\mu)} : \times_{j=1}^N \Delta(Z_j) \to \mathbb{R}$ is defined by

$$G_i^{(\alpha,\mu)}(\sigma_1, \dots, \sigma_N) := G_i((1 - \alpha_1)\sigma + \alpha_1\mu_1, \dots, (1 - \alpha_N)\sigma_N + \alpha_N\mu_N).$$

In the absence of incomplete information, the notion of perfection in Definition 7 reduces to the following:

Definition 13 Suppose that $G = (Z_i, g_i)_{i=1}^N$ is a compact, metric, Borel game. A Nash equilibrium σ of the mixed extension $\boldsymbol{G} = (\Delta(Z_i), G_i)_{i=1}^N$ is ***perfect*** if there exist sequences (α^n), (μ^n), and (σ^n) such that the following holds for each n: $\alpha^n \in (0, 1)^N$, $\mu^n \in \times_{i=1}^N \widehat{\Delta}(Z_i)$, and σ^n is a Nash equilibrium of the game $G^{(\alpha^n,\mu^n)}$ defined in (5), and in addition $\alpha^n \to 0$ and $\sigma^n \underset{w}{\to} \sigma$.

In the special case of complete information games, Definitions 9–11 can be more simply stated as follows.

Definition 14 is the analogue of Definition 9.

Definition 14 A compact, metric, Borel game $(Z_i, g_i)_{i=1}^N$ satisfies ***strong uniform payoff security*** if there exists $\mu = (\mu_1, \dots, \mu_N) \in \times_{i=1}^N \widehat{\Delta}(Z_i)$ such that for each i and $\varepsilon > 0$ there is a sequence (f^k) of $(\mathscr{B}(X_i), \mathscr{B}(X_i))$-measurable maps $f^k : X_i \to X_i$ satisfying the following:

(a) For each k and $x \in Z$, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$g_i(f^k(x_i), y_{-i}) > g_i(x) - \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

(b) For each $x_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ satisfying the following: for each $x_i \in Y_i$, there exists K such that for all $k \geq K$, there is a neighborhood $V'_{x_{-i}}$ of x_{-i} such that

$$g_i(f^k(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon, \quad \text{for all } y_{-i} \in V'_{x_{-i}}.$$

Suppose that $(Z_i, g_i)_{i=1}^N$ is a compact, metric, Borel game. Recall that Z_i can be written as a disjoint union $A_i \cup K_i$, where A_i is the set of all accumulation points of X_i, which is closed and dense in itself (*i.e.*, with no isolated points), and K_i is countable. Recall that the set of all strictly positive mixed strategies in $\Delta(Z_i)$ is denoted by $\widehat{\Delta}(Z_i)$.

In the present framework, Definition 10 reduces to Definition 15.

Definition 15 Suppose that $G = (Z_i, g_i)_{i=1}^N$ is a metric game and, for each i, let $Z_i = A_i \cup K_i$ be the decomposition from the preceding paragraph. The game G is said to satisfy ***generic entire payoff security*** if there exist countable subsets $C_1 \subseteq A_1, \ldots, C_N \subseteq A_N$ for which the following conditions are satisfied:

(i) For each i, $\varepsilon > 0$, and $x_i \in A_i \setminus C_i$, and for every neighborhood V_{x_i} of x_i, there exist $y_i \in V_{x_i}$ and a neighborhood V'_{x_i} of x_i such that for every $z_{-i} \in Z_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$g_i(y_i, y_{-i}) > g_i(z'_i, z_{-i}) - \varepsilon, \quad \text{for all } (z'_i, y_{-i}) \in V'_{x_i} \times V_{z_{-i}}.$$

(ii) For each i, $\varepsilon > 0$, and $x_i \in K_i$, and for every neighborhood V_{x_i} of x_i, there exists $y_i \in V_{x_i}$ such that, for every $z_{-i} \in Z_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$g_i(y_i, y_{-i}) > g_i(x_i, z_{-i}) - \varepsilon, \quad \text{for all } y_{-i} \in V_{z_{-i}}.$$

(iii) For each i, $\varepsilon > 0$, and $x_i \in C_i$, there exists $y_i \in Z_i$ such that, for every $z_{-i} \in Z_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$g_i(y_i, y_{-i}) > g_i(x_i, z_{-i}) - \varepsilon, \quad \text{for all } y_{-i} \in V_{z_{-i}}.$$

Let $\widetilde{\Delta}(Z_i)$ be the set of members σ_i of $\Delta(Z_i)$ such that $\sigma_i(\{x_i\}) = 0$ and $\sigma_i(N_\epsilon(x_i)) > 0$ for every $x_i \in A_i$ and $\epsilon > 0$ (where $N_\epsilon(x_i)$ denotes the ϵ-neighborhood of x_i), and $\sigma_i(\{x_i\}) > 0$ for every $x_i \in K_i$.

Definition 16 is the analogue of Definition 11.

Definition 16 A compact, metric, Borel game $(Z_i, g_i)_{i=1}^N$ satisfies ***generic local equi-upper semicontinuity*** if there exists $\mu = (\mu_1, \ldots, \mu_N) \in \widetilde{\Delta}(Z_i)$ such that for each i and $x_{-i} \in Z_{-i}$, there exists $Y_i \subseteq Z_i$ with $\mu_i(Y_i) = 1$ satisfying the following: for each $x_i \in Y_i$ and $\varepsilon > 0$, there is a neighborhood V_{x_i} of x_i such that for every $y_i \in V_{x_i}$, there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$g_i(y_i, y_{-i}) < g_i(x_i, y_{-i}) + \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

The next corollaries follow immediately from the main existence results.

Corollary 1 (to Theorem 1). *Suppose that the compact, metric, Borel game* $G = (Z_i, g_i)_{i=1}^N$ *satisfies strong uniform payoff security and that the map* $\sum_{i=1}^N g_i(\cdot) : Z \to \mathbb{R}$ *is upper semicontinuous. Then G possesses a perfect Nash equilibrium.*

Corollary 2 (to Theorem 2). *Suppose that the compact, metric, Borel game* $G = (Z_i, g_i)_{i=1}^N$ *satisfies generic entire payoff security and generic local equi-upper semicontinuity. If the map* $\sum_{i=1}^N g_i(\cdot) : Z \to \mathbb{R}$ *is upper semicontinuous, then G possesses a perfect Nash equilibrium.*

4 Applications

This section illustrates the machinery developed in Sect. 3 in the context of all-pay auctions and Cournot oligopolies.

4.1 All-pay auctions

We confine attention to a generalized version of the war of attrition considered in Krishna and Morgan (1997), but the existence result presented here extends to other all-pay auctions. An existence result is obtained, using Theorem 1, for the war of attrition with common values and interdependent types.

There are N bidders competing for a single indivisible object. After learning their types, the players simultaneously submit a sealed bid b_i from a closed and bounded subinterval $B_i := [\underline{b}, \overline{b}]$ of $\mathbb{R}_+$ (where $\underline{b} < \overline{b}$). Each B_i is endowed with the usual relative Euclidean metric, and the Cartesian product $B := \times_{i=1}^N B_i$ is equipped with the corresponding supremum metric. Let $T_1, \dots, T_N$ be the type spaces (each T_i is a compact, metric type space). The highest bidder wins the object and ties are broken via an equal probability rule. If player i wins the object when Nature chooses a type profile $t = (t_1, \dots, t_N) \in T$ and when the profile of bids chosen by the players is $b = (b_1, \dots, b_N) \in B$, then player i's payoff is given by $v(t) - \max_{j \neq i} b_j$, where $v(t) \geq 0$ represents the value of the object in state t and $\max_{j \neq i} b_j$ is the second highest bid in the action profile b. All the other players $j \neq i$ obtain a payoff of $h_j(t, b)$. The common prior over type profiles in T is represented by a probability measure p on $(T, \mathscr{B}(T))$, assumed absolutely continuous with respect to the product of its marginal probability measures, $p_1 \otimes \cdots \otimes p_N$.

Bidder i's expected payoff at $t = (t_1, \dots, t_N) \in T$ and $b = (b_1, \dots, b_N) \in B$ is given by

$$u_i(t, b) := \begin{cases} h_i(t, b) & \text{if } b_i < \max_j b_j, \\ \frac{v(t)}{\#\{j : b_j = \max_\iota b_\iota\}} + h_i(t, b) & \text{if } b_i = \max_j b_j. \end{cases}$$

Here, the map $v : T \to \mathbb{R}$ is assumed bounded and $(\mathscr{B}(T), \mathscr{B}(\mathbb{R}))$-measurable, and the maps $h_i : T \times B \to \mathbb{R}$ are bounded and $(\mathscr{B}(T \times B), \mathscr{B}(\mathbb{R}))$-measurable and satisfy the following: for each i, the family $\{h_i(t, \cdot) : t \in T\}$ is equicontinuous on B and $h_i(t, b) = -\max_{j \neq i} b_j$ whenever $t \in T$ and $b \in B$ satisfies $b_i = \max_j b_j$.

In particular, if $h_i(t, b) = -b_i$ whenever $b_i < \max_j b_j$ (and if one makes additional assumptions on the affiliation of types) one obtains the war of attrition game considered in Krishna and Morgan (1997).[9]

The associated Bayesian game is

$$\Gamma := (T_i, B_i, u_i, p)_{i=1}^N. \tag{6}$$

Lemma 4 *The game* Γ *defined in* (6) *satisfies strong uniform payoff security.*

Proof Let $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ be such that for each i and $t_i \in T_i$, $\mu_i(\cdot|t_i)$ is the normalized Lebesgue measure over $(B_i, \mathscr{B}(B_i))$.

Fix i and $\varepsilon > 0$. Because $\{h_i(t, \cdot) : t \in T\}$ is equicontinuous on the compact set B, $\{h_i(t, \cdot) : t \in T\}$ is uniformly equicontinuous on B. Therefore, there exists $\delta > 0$ such that

$$|h_i(t, b) - h_i(t, b')| < \varepsilon, \quad \text{for all } t \in T \text{ and } (b, b') \in B \times B \text{ with } d(b, b') < \delta,$$

where d is a compatible metric on B.

For each k, define $f^k : B_i \to B_i$ as follows: $f^k(b_i) := \frac{1}{k}\overline{b} + (1 - \frac{1}{k})b_i$. Let $k^* > \frac{\overline{b} - \underline{b}}{\delta}$ and observe that for $k \geq k^*$ and $b_i \in B_i$,

$$f^k(b_i) - b_i = \frac{1}{k}(\overline{b} - b_i) \leq \frac{1}{k}(\overline{b} - \underline{b}) < \delta.$$

Fix $k \geq k^*$ and $(t, b) \in T \times B$. We consider three cases:
Case 1 $b_i = \max_j b_j < \overline{b}$. Let $V_{b_{-i}}$ be a neighborhood of b_{-i} contained in $N_\varepsilon(b_{-i})$ such that $\max_{j \neq i} b'_j < f^k(b_i)$ for all $b'_{-i} \in V_{b_{-i}}$, and pick any $b'_{-i} \in V_{b_{-i}}$. Then

$$\begin{aligned} u_i(t, (f^k(b_i), b'_{-i})) &= v(t) - \max_j b'_j \\ &\geq \frac{v(t)}{\#\{j : b_j = \max_\iota b_\iota\}} - \max_j b'_j \\ &> \frac{v(t)}{\#\{j : b_j = \max_\iota b_\iota\}} - \max_j b_j - \varepsilon = u_i(t, b) - \varepsilon. \end{aligned}$$

Case 2 $b_i = \max_j b_j = \overline{b}$. Let $V'_{b_{-i}}$ be a neighborhood of b_{-i} contained in $N_\varepsilon(b_{-i})$ such that $b'_j < \overline{b}$ for each $j \neq i$ whenever $b'_{-i} \in V'_{b_{-i}}$ and $b_j < \overline{b}$. For any $b'_{-i} \in V'_{b_{-i}}$,

$$\begin{aligned} u_i(t, (f^k(b_i), b'_{-i})) = u_i(t, (b_i, b'_{-i})) &= \frac{v(t)}{1 + \#\{j : b'_j = \max_\iota b'_\iota\}} - \max_j b'_j \\ &> \frac{v(t)}{\#\{j : b_j = \max_\iota b_\iota\}} - \max_{j \neq i} b_j - \varepsilon = u_i(t, b) - \varepsilon. \end{aligned}$$

[9] While the game considered here is a special case of the model studied Carbonell-Nicolau and McLean (2018, Subsection 6.1), no additional assumptions are needed to establish the existence of a perfect Bayes–Nash equilibrium.

Case 3 $b_i < \max_j b_j$. Choose $b'_{-i} \in N_\delta(b_{-i})$. If $f^k(b_i) \geq \max_j b'_j$, then

$$u_i(t, (f^k(b_i), b'_{-i})) \geq -\max_j b'_j = h_i(t, (f^k(b_i), b'_{-i})) > h_i(t, b) - \varepsilon = u_i(t, b) - \varepsilon.$$

If $f^k(b_i) < \max_j b'_j$, then

$$u_i(t, (f^k(b_i), b'_{-i})) = h_i(t, (f^k(b_i), b'_{-i})) > h_i(t, b) - \varepsilon = u_i(t, b) - \varepsilon.$$

This establishes item (a) of Definition 9 for Γ. To see that item (b) of Definition 9 holds, fix $(t, b_{-i}) \in T \times B_{-i}$ and choose $b_i \in B_i$ with $b_i \neq \max_{j \neq i} b_j$. If $b_i > \max_{j \neq i} b_j$, then for each k and for $V_{b_{-i}}$ a neighborhood of b_{-i} such that $b'_j < b_i$ for each $j \neq i$ whenever $b'_{-i} \in V_{b_{-i}}$,

$$\begin{aligned} u_i(t, (f^k(b_i), b'_{-i})) &= v(t) - \max_j b'_j = u_i(t, (b_i, b'_{-i})) \\ &< u_i(t, (b_i, b'_{-i})) + \varepsilon, \quad \text{for all } b'_{-i} \in V_{b_{-i}}. \end{aligned}$$

If $b_i < \max_{j \neq i} b_j$, there exists K such that for $k \geq K$ one has $f^k(b_i) < \max_{j \neq i} b_j - \beta$ for some $\beta > 0$ and $f^k(b_i) - b_i < \delta$, and one can choose a neighborhood $V'_{b_{-i}}$ of b_{-i} such that for all $b'_{-i} \in V'_{b_{-i}}$, $\max_j b'_j > f^k(b_i) > b_i$. Then, for $k \geq K$,

$$\begin{aligned} u_i(t, (f^k(b_i), b'_{-i})) &= h_i(t, (f^k(b_i), b'_{-i})) < h_i(t, (b_i, b'_{-i})) + \varepsilon \\ &= u_i(t, (b_i, b'_{-i})) + \varepsilon, \quad \text{for all } b'_{-i} \in V_{b_{-i}}. \end{aligned}$$

This establishes item (b) of Definition 9. □

Proposition 1 *The game Γ defined in* (6) *possesses a perfect Bayes–Nash equilibrium.*

Proof In view of Lemma 4, the assertion is an immediate consequence of Theorem 1 once one observes that for each $t \in T$, the map $\sum_{i=1}^N u_i(t, \cdot) : B \to \mathbb{R}$ is upper semicontinuous (in fact, continuous). The continuity of this sum follows from the fact that, for every $(t, b) \in T \times B$,

$$\sum_{i=1}^N u_i(t, b) = v(t) - \#\left\{i : b_i = \max_\iota b_\iota\right\} b^* + \sum_{i : b_i < \max_\iota b_\iota} h_i(t, b),$$

where b^* represents the second highest bid in b, together with the equicontinuity of $\{h_i(t, \cdot) : t \in T\}$ on B, for each i, and the condition that, for each i, $h_i(t, b) = -\max_{j \neq i} b_j$ whenever $b_i = \max_j b_j$. □

4.2 Cournot games

This subsection considers Cournot oligopolies with incomplete information and cost discontinuities. The setup is essentially the same as that in Carbonell-Nicolau and

McLean (2018, §6.2), and the reader is referred to Carbonell-Nicolau and McLean (2018) for references on the literature.

There are N firms in a market for a single homogeneous good. The firm's type spaces, $T_1, \ldots, T_N$, are compact, metric spaces. A type profile $t = (t_1, \ldots, t_N) \in T$ determines the market inverse demand (in state t), $p(t, \cdot)$. Thus, $p(t, q)$ represents the price that clears the market in state t when aggregate output is q. Each firm i faces a cost function $c_i(t, q_i)$ on type profiles t and individual output levels q_i selected from a compact subset X_i of $\mathbb{R}_+$. The common prior is given by η, a probability measure in $\Delta(T)$, with corresponding marginal probability measures $\eta_1, \ldots, \eta_N$. It is assumed that η is absolutely continuous with respect to the product of its marginals.

The firms simultaneously choose an output level. Given an output profile $(q_1, \ldots, q_N)$, firm i's profit is given by $q_i p(t, q_1 + \cdots + q_N) - c_i(t, q_i)$.

The associated Bayesian game is

$$\Gamma = (T_i, X_i, u_i, \eta)_{i=1}^N, \tag{7}$$

where, for each i,

$$u_i(t, (q_1, \ldots, q_N)) := q_i p(t, q_1 + \cdots + q_N) - c_i(t, q_i),$$

and where the maps $p : T \times \mathbb{R}_+ \to \mathbb{R}_+$ and $c_i : T \times R_+ \to \mathbb{R}_+$ are assumed bounded and $(\mathscr{B}(T) \otimes \mathscr{B}(\mathbb{R}_+), \mathscr{B}(\mathbb{R}_+))$-measurable.

We make the following additional assumptions: (i) the family $\{p(t, \cdot) : t \in T\}$ is equicontinuous on $\{q_1 + \cdots + q_N : (q_1, \ldots, q_N) \in X\}$; and (ii) for each i the family $\{c_i(t, \cdot) : t \in T\}$ is equi-lower semicontinuous on X_i, *i.e.*, for each $q_i \in X_i$ and $\epsilon > 0$, there exists $\delta > 0$ such that $c_i(t, s_i) > c_i(t, q_i) - \epsilon$ for each $s_i \in N_\delta(q_i)$ and $t \in T$.

The following proposition establishes the existence of a perfect Bayes–Nash equilibrium in the game Γ defined in (7).

Proposition 2 *The game Γ defined in (7) possesses a perfect Bayes–Nash equilibrium.*

Proof We prove the assertion as an application of Theorem 2. By virtue of Theorem 2, it suffices to show that Γ satisfies generic entire payoff security and generic local equi-upper semicontinuity, and that, for each $t \in T$, the map $\sum_{i=1}^N u_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous.

First, note that the above assumptions imply that, for each $t \in T$, the map $p(t, \cdot)|_{\{q_1+\cdots+q_N : (q_1,\ldots,q_N) \in X\}}$ is continuous and the map $q \in X \mapsto c_1(t, q_1) + \cdots + c_N(t, q_N)$ is lower semicontinuous. This, together with the fact that

$$\sum_{i=1}^N u_i(t, q) = \left(\sum_{i=1}^N q_i\right) p\left(t, \sum_{i=1}^N q_i\right) - \sum_{i=1}^N c_i(t, q_i),$$

implies that the map $\sum_{i=1}^N u_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous for each $t \in T$.

To see that Γ satisfies generic entire payoff security (Definition 10), note first that it suffices to show that for each i, $\varepsilon > 0$, and $q_i \in X_i$, and for every neighborhood

V_{q_i} of q_i in X_i, there exist $q_i^* \in V_{q_i}$ and a neighborhood V'_{q_i} of q_i such that, for every $(t, q_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{q_{-i}}$ of q_{-i} in X_{-i} such that

$$u_i(t, (q_i^*, s_{-i})) > u_i(t, (s_i, q_{-i})) - \varepsilon, \quad \text{for all } (s_i, s_{-i}) \in V'_{q_i} \times V_{q_{-i}}.$$

Since the family $\{p(t, \cdot) : t \in T\}$ is equicontinuous on $\{q_1 + \cdots + q_N : (q_1, \ldots, q_N) \in X\}$ and $\{q_1 + \cdots + q_N : (q_1, \ldots, q_N) \in X\}$ is compact, it follows that the family $\{p(t, \cdot) : t \in T\}$ is uniformly equicontinuous on $\{q_1 + \cdots + q_N : (q_1, \ldots, q_N) \in X\}$, and so there exists $\delta > 0$ such that

$$\left| q_i p\left(t, \sum_{j=1}^{N} q_j\right) - s_i p\left(t, \sum_{j=1}^{N} s_j\right) \right| < \frac{\varepsilon}{2},$$
$$\text{for all } t \in T \text{ and all } (q, s) \in X \times X \text{ with } d(q, s) < \delta,$$

where d is a compatible metric on X. Therefore, given i, $\varepsilon > 0$, $q_i \in X_i$, and a neighborhood V_{q_i} of q_i, there is a neighborhood $V^*_{q_i}$ of q_i such that, for every $(t, q_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{q_{-i}}$ of q_{-i} in X_{-i} such that

$$q_i p\left(t, q_i + \sum_{j \neq i} s_j\right) > s_i p\left(t, s_i + \sum_{j \neq i} q_j\right) - \frac{\varepsilon}{2}, \quad \text{for all } (s_i, s_{-i}) \in V^*_{q_i} \times V_{q_{-i}}. \tag{8}$$

In addition, because the family $\{c_i(\tau, \cdot) : \tau \in T\}$ is equi-lower semicontinuous at q_i, there exists a neighborhood $\widehat{V}_{q_i}$ of q_i such that

$$c_i(t, s_i) > c_i(t, q_i) - \frac{\varepsilon}{2}, \quad \text{for all } s_i \in \widehat{V}_{q_i},$$

implying that

$$-c_i(t, s_i) - \frac{\varepsilon}{2} < -c_i(t, q_i), \quad \text{for all } s_i \in \widehat{V}_{q_i}. \tag{9}$$

Therefore, given i, $\varepsilon > 0$, $q_i \in X_i$, and a neighborhood V_{q_i} of q_i, and setting $q_i^* := q_i$ and $V'_{q_i} := V^*_{q_i} \cap \widehat{V}_{q_i}$, it follows that, for every $(t, q_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{q_{-i}}$ of q_{-i} such that, for all $(s_i, s_{-i}) \in V'_{q_i} \times V_{q_{-i}}$,

$$\begin{aligned}
u_i(t, (q_i^*, s_{-i})) &= u_i(t, (q_i, s_{-i})) \\
&= q_i p\left(t, q_i + \sum_{j \neq i} s_j\right) - c_i(t, q_i) \\
&> s_i p\left(t, s_i + \sum_{j \neq i} q_j\right) - \frac{\varepsilon}{2} - c_i(t, s_i) - \frac{\varepsilon}{2}
\end{aligned}$$

$$= s_i p\left(t, s_i + \sum_{j \neq i} q_j\right) - c_i(t, s_i) - \varepsilon$$
$$= u_i(t, (s_i, q_{-i})) - \varepsilon,$$

where the inequality follows from (8) and (9). We conclude that Γ satisfies generic entire payoff security.

It only remains to show that Γ satisfies generic local equi-upper semicontinuity (Definition 11). Note that it suffices to show that for each i, $(t, q) \in T \times X$, and $\varepsilon > 0$, there is a neighborhood V_{q_i} of q_i such that for every $s_i \in V_{q_i}$, there is a neighborhood $V_{q_{-i}}$ of q_{-i} such that

$$u_i(t, (s_i, s_{-i})) < u_i(t, (q_i, s_{-i})) + \varepsilon, \quad \text{for all } s_{-i} \in V_{q_{-i}}.$$

Fix i, $(t, q) \in T \times X$, and $\varepsilon > 0$. By the continuity of $p(t, \cdot)$ on $\{q_1 + \cdots + q_N : (q_1, \ldots, q_N) \in X\}$, there are neighborhoods V'_{q_i} and $V_{q_{-i}}$ of q_i and q_{-i}, respectively, such that

$$s_i p\left(t, \sum_{j=1}^{N} s_j\right) < q_i p\left(t, q_i + \sum_{j \neq i} s_j\right) + \frac{\varepsilon}{2}, \quad \text{for all } (s_i, s_{-i}) \in V'_{q_i} \times V_{q_{-i}}. \tag{10}$$

In addition, because $c_i(t, \cdot)$ is lower semicontinuous at q_i, there exists a neighborhood V''_{q_i} of q_i such that

$$c_i(t, s_i) > c_i(t, q_i) - \frac{\varepsilon}{2}, \quad \text{for all } s_i \in V''_{q_i},$$

implying that

$$-c_i(t, s_i) < -c_i(t, q_i) + \frac{\varepsilon}{2}, \quad \text{for all } s_i \in V''_{q_i}. \tag{11}$$

Consequently, setting $V_{q_i} := V'_{q_i} \cap V''_{q_i}$, one obtains, for all $(s_i, s_{-i}) \in V_{q_i} \times V_{q_{-i}}$,

$$u_i(t, (s_i, s_{-i})) = s_i p\left(t, \sum_{j=1}^{N} s_j\right) - c_i(t, s_i)$$
$$< q_i p\left(t, q_i + \sum_{j \neq i} s_j\right) + \frac{\varepsilon}{2} - c_i(t, q_i) + \frac{\varepsilon}{2}$$
$$= q_i p\left(t, q_i + \sum_{j \neq i} s_j\right) - c_i(t, q_i) + \varepsilon$$
$$= u_i(t, (q_i, s_{-i})) + \varepsilon,$$

where the inequality follows from (10) and (11). We conclude that Γ satisfies generic local equi-upper semicontinuity. □

5 Proofs of Lemma 1, Lemma 2, and Lemma 3

5.1 Proof of Lemma 1

The proof of Lemma 1 relies on the following technical lemma, whose proof is relegated to Appendix A.

Lemma 5 *Suppose that the Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ satisfies strong uniform payoff security. Suppose that p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$. Then there exists $(\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$, there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying the following:*

(I) For each $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j \neq i}{\otimes} \sigma'_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$> \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j \neq i}{\otimes} \sigma_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$- \varepsilon, \text{ for all } \sigma'_{-i} \in V_{\sigma_{-i}},$$

where g is a density of p with respect to $p_1 \otimes \cdots \otimes p_N$.

(II) For each $\sigma_{-i} \in \mathscr{D}_{-i}$, there exists K such that for each $k \geq K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i^k, \sigma'_{-i}) < U_i(\mu_i, \sigma'_{-i}) + \varepsilon, \quad \text{for all } \sigma'_{-i} \in V'_{\sigma_{-i}},$$

where $\mu_i^k(\cdot|t_i)$ is defined by

$$\mu_i^k(B|t_i) := \mu_i \left(\left\{ x_i \in X_i : f^k(t_i, x_i) \in B \right\} \middle| t_i \right). \tag{12}$$

See Step 1 in the proof of Lemma 5 for a proof that the conditional probability $\mu_i^k(\cdot|t_i)$ defined in (12) is well-defined.

We are now ready to prove Lemma 1, which is restated here for the convenience of the reader.

Lemma 1 *Suppose that the Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ satisfies strong uniform payoff security. If p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$, then there exists $\mu \in \widehat{\mathscr{D}}$ such that the game $\mathfrak{G}_\Gamma^{(\alpha,\mu)}$ defined in (4) is payoff secure for each $\alpha \in [0, 1)^N$.* □

Proof Let $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ be the measure profile given by Lemma 5. Fix $\alpha \in [0, 1)^N$, $\varepsilon > 0$, $\sigma \in \mathscr{D}$, and i. We must show that there exist $\sigma_i^* \in \mathscr{D}_i$ and a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} in $\mathscr{D}_{-i}$ such that

$$U_i^{(\alpha,\mu)}(\sigma_i^*, \sigma'_{-i}) > U_i^{(\alpha,\mu)}(\sigma) - \varepsilon, \quad \text{for all } \sigma'_{-i} \in V_{\sigma_{-i}}. \tag{13}$$

Define

$$\widehat{\sigma} = (\widehat{\sigma}_1, \ldots, \widehat{\sigma}_N) := ((1-\alpha_1)\sigma_1 + \alpha_1\mu_1, \ldots, (1-\alpha_N)\sigma_N + \alpha_N\mu_N).$$

Lemma 5 gives a a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying the following:

(i) For each $(t_i, x_i) \in T_i \times X_i$ and k, there is a neighborhood $V_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ such that

$$\begin{aligned}\int_{T_{-i}} \int_{X_{-i}} &[u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j\neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i}) \\ &> \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i}) \left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i}) \\ &- \frac{\varepsilon}{4}, \quad \text{for all } \sigma'_{-i} \in V_{\widehat{\sigma}_{-i}},\end{aligned}$$

where g is a density of p with respect to $p_1 \otimes \cdots \otimes p_N$.

(ii) There exists K such that for each $k \geq K$, there is a neighborhood $V'_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ such that

$$U_i(\mu_i^k, \sigma'_{-i}) < U_i(\mu_i, \sigma'_{-i}) + \frac{\varepsilon}{2}, \quad \text{for all } \sigma'_{-i} \in V'_{\widehat{\sigma}_{-i}}, \tag{14}$$

where $\mu_i^k(\cdot|t_i)$ is defined by

$$\mu_i^k(B|t_i) := \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right).$$

Define $\widehat{\sigma}_i^k \in \mathscr{D}_i$ and $\sigma_i^k \in \mathscr{D}_i$ via their corresponding regular conditional probability measures as follows:

$$\begin{aligned}\widehat{\sigma}_i^k(B|t_i) &:= \widehat{\sigma}_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right) \quad \text{and} \\ \sigma_i^k(B|t_i) &:= \sigma_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right).\end{aligned}$$

Note that

$$\widehat{\sigma}_i^k = (1-\alpha_i)\sigma_i^k + \alpha_i\mu_i^k. \tag{15}$$

Indeed, given $t_i \in T_i$ and $B \in \mathscr{B}(X_i)$, one has

$$\begin{aligned}\widehat{\sigma}_i^k(B|t_i) &= \widehat{\sigma}_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right) \\ &= (1-\alpha_i)\sigma_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right) \\ &\quad + \alpha_i\mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right) \\ &= (1-\alpha_i)\sigma_i^k(B|t_i) + \alpha_i\mu_i^k(B|t_i).\end{aligned}$$

Below we show that for each k, there exists a neighborhood $V''_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ in $\mathscr{D}_{-i}$ such that for each $\sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}$,

$$\int_T \int_X [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\widehat{\sigma}_i(\cdot|t_i) \otimes \left[\underset{j \neq i}{\otimes} \sigma'_j(\cdot|t_j)\right]\right](dx) \left[\overset{N}{\underset{j=1}{\otimes}} p_j\right](dt)$$
$$> U_i(\widehat{\sigma}) - \frac{\varepsilon}{2}, \tag{16}$$

implying that

$$U_i(\widehat{\sigma}_i^k, \sigma'_{-i}) > U_i(\widehat{\sigma}) - \frac{\varepsilon}{2}, \quad \text{for all } \sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}. \tag{17}$$

By item (ii), there exists K such that for each $k \geq K$, there is a neighborhood $V'_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ such that (14) holds. Consequently, for $k \geq K$, and for $\sigma'_{-i} \in V'_{\widehat{\sigma}_{-i}}$,

$$\begin{aligned} U_i((1-\alpha_i)\sigma_i^k + \alpha_i \mu_i, \sigma'_{-i}) &= (1-\alpha_i)U_i(\sigma_i^k, \sigma'_{-i}) + \alpha_i U_i(\mu_i, \sigma'_{-i}) \\ &> (1-\alpha_i)U_i(\sigma_i^k, \sigma'_{-i}) + \alpha_i U_i(\mu_i^k, \sigma'_{-i}) - \frac{\varepsilon}{2} \\ &= U_i(\widehat{\sigma}_i^k, \sigma'_{-i}) - \frac{\varepsilon}{2}, \end{aligned}$$

where the last equality uses (15). This, together with (17), yields, for $k \geq K$,

$$U_i((1-\alpha_i)\sigma_i^k + \alpha_i \mu_i, \sigma'_{-i}) > U_i(\widehat{\sigma}) - \varepsilon$$

for all σ'_{-i} in some neighborhood of $\widehat{\sigma}_{-i}$. In particular, (13) holds for some $\sigma_i^* \in \mathscr{D}_i$ and some neighborhood $V_{\sigma_{-i}}$ of σ_{-i} in $\mathscr{D}_{-i}$.

It remains to show that for each k, there exists a neighborhood $V''_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ such that (16) holds for each $\sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}$. The proof of this assertion proceeds in five steps (Step 1–Step 5 below).

We begin with the following definitions. For each k and $n \in \mathbb{N}$, define the map $\phi^{(k,n)} : T_i \times X_i \to \mathbb{R}$ by

$$\begin{aligned} \phi^{(k,n)}(t_i, x_i) &:= \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j \neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j\right](dt_{-i}) \\ &- \inf_{\sigma'_{-i} \in N_{\frac{1}{n}}(\widehat{\sigma}_{-i})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j \neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \\ &\left[\underset{j \neq i}{\otimes} p_j\right](dt_{-i}) \end{aligned}$$

(here $N_{\frac{1}{n}}(\widehat{\sigma}_{-i})$ denotes the $\frac{1}{n}$-neighborhood of $\widehat{\sigma}_{-i}$ in $\mathscr{D}_{-i}$).

Define $\psi : T_i \times X_i \to \mathbb{R}$ and $\vartheta^{(k,n)} : T_i \times X_i \to \mathbb{R}$ by

$$\psi(t_i, x_i) := \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j \neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j\right](dt_{-i}) \tag{18}$$

and

$$\vartheta^{(k,n)}(t_i, x_i) := \inf_{\sigma'_{-i} \in N_{\frac{1}{n}}(\widehat{\sigma}_{-i})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j) \right] (dx_{-i}) \left[\bigotimes_{j \neq i} p_j \right] (dt_{-i}), \tag{19}$$

so that $\phi^{(k,n)} = \psi^{(k,n)} - \vartheta^{(k,n)}$. □

Step 1 *The Radon-Nikodym derivative, g, may be taken bounded.*

Proof of Step 1 See the proof of Step 2 in the proof of Lemma 5, on p. 1628. □

Step 2 *The map ψ defined in (18) is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable.*

Proof of Step 2 Define $\boldsymbol{\psi} : \Delta(T \times X) \to \mathbb{R}$ by

$$\boldsymbol{\psi}(\varrho) := \int_{T \times X} [u_i(t, x)g(t)]\varrho(d(t, x)). \tag{20}$$

Since u_i is bounded and $(\mathscr{B}(T \times X), \mathscr{B}(\mathbb{R}))$-measurable, the map $\boldsymbol{\psi}$ is $(\mathscr{B}(\Delta(T \times X)), \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (2006, Theorem 15.13)).

Let $\Delta^p(T \times X)$ be the set of all product measures in $\Delta(T \times X)$ (*i.e.*, $\nu \in \Delta^p(T \times X)$ if and only if $\nu = \nu_1 \otimes \cdots \otimes \nu_N$ for some $(\nu_1, \ldots, \nu_N) \in \times_j \Delta(T_j \times X_j)$).

The set $\Delta^p(T \times X)$ (with the relative w-topology (Definition 1)) is closed in $\Delta(T \times X)$. To see this, let $(\nu^n = \nu_1^n \otimes \cdots \otimes \nu_N^n)$ be a sequence in $\Delta^p(T \times X)$ with $\nu^n \underset{w}{\to} \nu \in \Delta(T \times X)$. Then $\nu^n(A_1 \times \cdots \times A_N) \to \nu(A_1 \times \cdots \times A_N)$, where for each j, A_j is any ν_j-continuity subset of $T_j \times X_j$ and ν_j denotes the marginal projection of ν into $T_j \times X_j$ (see, *e.g.*, Billingsley (1999, Theorem 2.8(i)))). In particular, letting ν_j^n represent the marginal projection of ν^n into the factor $T_j \times X_j$, we have $\nu_j^n(A_j) \to \nu_j(A_j)$ for every ν_j-continuity set A_j, and so it follows from the Portmanteau Theorem (*e.g.*, see Billingsley (1999, Theorem 2.1)) that $\nu_j^n \underset{w}{\to} \nu_j$ for each j. Therefore, applying Theorem 2.8(ii) in Billingsley Billingsley (1999), we see that $\nu = \nu_1 \otimes \cdots \otimes \nu_N \in \Delta^p(T \times X)$.

Since $\Delta^p(T \times X)$ is closed in $\Delta(T \times X)$, and since the map $\boldsymbol{\psi}$ defined in (20) is $(\mathscr{B}(\Delta(T \times X)), \mathscr{B}(\mathbb{R}))$-measurable, it follows that the map $\boldsymbol{\psi}|_{\Delta^p(T \times X)}$ is $(\mathscr{B}(\Delta^p(T \times X)), \mathscr{B}(\mathbb{R}))$-measurable. Hence, because the map $(\nu_1, \ldots, \nu_N) \in \times_j \Delta(T_j \times X_j) \mapsto \nu_1 \otimes \cdots \otimes \nu_N \in \Delta^p(T \times X)$ is continuous (by Theorem 2.8(ii) in Billingsley (1999)), it follows that the map

$$(\nu_1, \ldots, \nu_N) \in \times_j \Delta(T_j \times X_j) \mapsto \boldsymbol{\psi}(\nu_1 \otimes \cdots \otimes \nu_N)$$

is $(\mathscr{B}(\times_j \Delta(T_j \times X_j)), \mathscr{B}(\mathbb{R}))$-measurable, and hence $(\mathscr{B}(\Delta(T_i \times X_i)) \otimes \mathscr{B}(\times_{j \neq i} \Delta (T_j \times X_j)), \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (2006, Theorem

4.44)). Therefore, the map $\nu_i \in \Delta(T_i \times X_i) \mapsto \boldsymbol{\psi}(\nu_i \otimes [\otimes_{j\neq i}\widehat{\sigma}_j])$ is $(\mathscr{B}(\Delta(T_i \times X_i)), \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (Aliprantis and Border 2006, Theorem 4.48)). Now let $\delta_{(t_i,x_i)}$ denote the Dirac measure in $\Delta(T_i \times X_i)$ with support $\{(t_i, x_i)\}$. The set $\{\delta_{(t_i,x_i)} : (t_i, x_i) \in T_i \times X_i\}$ is closed in $\Delta(T_i \times X_i)$ (see, *e.g.*, Aliprantis and Border (2006, Theorem 15.8)), and so the map $\nu_i \in \{\delta_{(t_i,x_i)} : (t_i, x_i) \in T_i \times X_i\} \mapsto \boldsymbol{\psi}(\nu_i \otimes [\otimes_{j\neq i}\widehat{\sigma}_j])$ is $(\mathscr{B}(\{\delta_{(t_i,x_i)} : (t_i, x_i) \in T_i \times X_i\}), \mathscr{B}(\mathbb{R}))$-measurable. Because the map $(t_i, x_i) \in T_i \times X_i \mapsto \delta_{(t_i,x_i)}$ is an embedding (Aliprantis and Border (Aliprantis and Border 2006, Theorem 15.8)), it follows that ψ is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable. □

Step 3 *There exist a* $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$*-measurable map* $\widehat{\vartheta}^{(k,n)} : T_i \times X_i \to \mathbb{R}$ *and* $\widehat{A} \in \mathscr{B}(T_i \times X_i)$ *such that*

$$\widehat{\sigma}_i(\widehat{A}) = 0 \quad \textit{and} \quad \widehat{\vartheta}^{(k,n)}(t_i, x_i) = \vartheta^{(k,n)}(t_i, x_i) \textit{ for all } (t_i, x_i) \in (T_i \times X_i) \setminus \widehat{A} \tag{21}$$

(where $\vartheta^{(k,n)}$ *is the map defined in* (19)*).*

Proof of Step 3 Define $\boldsymbol{\vartheta}^k : \Delta(T \times X) \to \mathbb{R}$ by

$$\boldsymbol{\vartheta}^k(\varrho) := \int_{T\times X} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)]\varrho(d(t, x)).$$

Reasoning as in the proof of Step 2, one can show that the map

$$((t_i, x_i), \nu_{-i}) \in (T_i \times X_i) \times \left[\bigtimes_{j\neq i} \Delta(T_j \times X_j)\right] \mapsto \boldsymbol{\vartheta}^k\left(\delta_{(t_i,x_i)} \otimes \left[\bigotimes_{j\neq i} \nu_j\right]\right)$$

is $(\mathscr{B}(T_i \times X_i) \otimes \mathscr{B}(\times_{j\neq i}\Delta(T_j \times X_j)), \mathscr{B}(\mathbb{R}))$-measurable.

For each j, $\mathscr{D}_j$ is closed in $\Delta(T_j \times X_j)$ (with the w-topology (Definition 1)) (see, *e.g.*, Milgrom and Weber (Milgrom and Weber 1985, p. 626)). Hence, $\mathscr{D}_{-i}$ is closed in $\times_{j\neq i}\Delta(T_j \times X_j)$. Consequently, the map

$$((t_i, x_i), \nu_{-i}) \in (T_i \times X_i) \times \mathscr{D}_{-i} \mapsto \boldsymbol{\vartheta}^k\left(\delta_{(t_i,x_i)} \otimes \left[\bigotimes_{j\neq i} \nu_j\right]\right) \tag{22}$$

is $(\mathscr{B}(T_i \times X_i) \otimes \mathscr{B}(\mathscr{D}_{-i}), \mathscr{B}(\mathbb{R}))$-measurable.

Let $\mathscr{B}^{\widehat{\sigma}_i}(T_i \times X_i)$ be the $\widehat{\sigma}_i$-completion of $\mathscr{B}(T_i \times X_i)$. Then the map in (22) is $(\mathscr{B}^{\widehat{\sigma}_i}(T_i \times X_i) \otimes \mathscr{B}(\mathscr{D}_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and since $\mathscr{B}^{\widehat{\sigma}_i}(T_i \times X_i)$ equals its universal completion, it follows from the proof of the Theorem in Carbonell-Nicolau (2014a) that the map

$$((t_i, x_i), \nu_{-i}) \in (T_i \times X_i) \times \mathscr{D}_{-i} \mapsto \inf_{\nu'_{-i}\in N_{\frac{1}{n}}(\nu_{-i})} \boldsymbol{\vartheta}^k\left(\delta_{(t_i,x_i)} \otimes \left[\bigotimes_{j\neq i} \nu'_j\right]\right)$$

is $(\mathscr{B}^{\widehat{\sigma}_i}(T_i \times X_i) \otimes \mathscr{B}(\mathscr{D}_{-i}), \mathscr{B}(\mathbb{R}))$-measurable (here $N_{\frac{1}{n}}(\nu_{-i})$ denotes the $\frac{1}{n}$-neighborhood of ν_{-i} in $\mathscr{D}_{-i}$), and consequently the map

$$(t_i, x_i) \in (T_i \times X_i) \mapsto \inf_{\nu'_{-i} \in N_{\frac{1}{n}}(\widehat{\sigma}_{-i})} \boldsymbol{\vartheta}^k \left(\delta_{(t_i, x_i)} \otimes \left[\bigotimes_{j \neq i} \nu'_j \right] \right)$$

is $(\mathscr{B}^{\widehat{\sigma}_i}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable (Aliprantis and Border (Aliprantis and Border 2006, Theorem 4.48)). Now Theorem 10.35 in Aliprantis and Border (2006) gives a $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable map $\widehat{\vartheta}^{(k,n)} : T_i \times X_i \to \mathbb{R}$ and $\widehat{A} \in \mathscr{B}(T_i \times X_i)$ satisfying (21). □

Step 4 *There exist a* $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$*-measurable map* $\widehat{\phi}^{(k,n)} : T_i \times X_i \to \mathbb{R}$ *and* $\widehat{A} \in \mathscr{B}(T_i \times X_i)$ *such that*

$$\widehat{\sigma}_i(\widehat{A}) = 0 \quad \text{and} \quad \widehat{\phi}^{(k,n)}(t_i, x_i) = \phi^{(k,n)}(t_i, x_i) \text{ for all } (t_i, x_i) \in (T_i \times X_i) \setminus \widehat{A}. \quad (23)$$

Proof of Step 4 Because $\phi^{(k,n)} = \psi^{(k,n)} - \vartheta^{(k,n)}$, the assertion follows immediately from Step 2 and Step 3. □

Step 5 *Let* $\widehat{A}$ *be the set given in Step 4. There exist* $A \subseteq (T_i \times X_i) \setminus \widehat{A}$ *with*

$$\widehat{\sigma}_i(A) \left(\sup_{(t,x) \in T \times X} |u_i(t, x) g(t)| \right) < \frac{\varepsilon}{16} \quad (24)$$

and a neighborhood $V''_{\widehat{\sigma}_{-i}}$ *of* $\widehat{\sigma}_{-i}$ *such that, for all* $\sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}$,

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i})) g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot | t_j) \right] (dx_{-i}) \left[\bigotimes_{j \neq i} p_j \right] (dt_{-i})$$
$$> \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x) g(t)] \left[\bigotimes_{j \neq i} \widehat{\sigma}_j(\cdot | t_j) \right] (dx_{-i}) \left[\bigotimes_{j \neq i} p_j \right] (dt_{-i}) - \frac{3\varepsilon}{8} \quad (25)$$

for all $(t_i, x_i) \in (T_i \times X_i) \setminus (A \cup \widehat{A})$.

Proof of Step 5 First, note that the left-hand side of (24) is well-defined because u_i is bounded by assumption and g may be taken bounded (Step 1).

Define $\varphi^k : T_i \times X_i \to \mathbb{R}$ by

$$\varphi^k(t_i, x_i) := \lim_{n \to \infty} \left[\sup_{n' \geq n} \widehat{\phi}^{(k,n')}(t_i, x_i) \right], \quad (26)$$

where $\widehat{\phi}^{(k,n)} : T_i \times X_i \to \mathbb{R}$ is the map given in Step 4.

For each (k, n), the map $\widehat{\phi}^{(k,n)} : T_i \times X_i \to \mathbb{R}$ is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable (Step 4). Therefore, the map $(t_i, x_i) \in T_i \times X_i \mapsto \sup_{n' \geq n} \widehat{\phi}^{(k,n')}(t_i, x_i)$ is also

$(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable. In addition, the map φ^k defined in (26), being the pointwise limit of a sequence of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable maps,

$$\left((t_i, x_i) \in T_i \times X_i \mapsto \sup_{n' \geq n} \widehat{\phi}^{(k,n')}(t_i, x_i) \right)_{n=1}^{\infty}, \tag{27}$$

is itself a $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable map. Consequently, thanks to Egorov's Theorem (see, *e.g.*, Dudley (Dudley 2004, Theorem 7.5.1)), there exists $A \subseteq (T_i \times X_i) \setminus \widehat{A}$ such that (24) holds and the sequence of maps in (27) converges, as $n \to \infty$, to the map φ^k (defined in (26)) uniformly on $(T_i \times X_i) \setminus (A \cup \widehat{A})$. Therefore, there exists $\overline{n}$ such that, for all $n \geq \overline{n}$,

$$\left| \sup_{n' \geq n} \widehat{\phi}^{(k,n')}(t_i, x_i) - \varphi^k(t_i, x_i) \right| < \frac{\varepsilon}{8}, \quad \text{for all } (t_i, x_i) \in (T_i \times X_i) \setminus (A \cup \widehat{A}). \tag{28}$$

Next, observe that, by item (i) on p. 17, we have

$$\varphi^k(t_i, x_i) \leq \frac{\varepsilon}{4}, \quad \text{for each } (t_i, x_i) \in (T_i \times X_i).$$

Combined with (28), this yields, for all $n \geq \overline{n}$,

$$\sup_{n' \geq n} \widehat{\phi}^{(k,n')}(t_i, x_i) < \frac{3\varepsilon}{8}, \quad \text{for all } (t_i, x_i) \in (T_i \times X_i) \setminus (A \cup \widehat{A}).$$

Consequently, there exists a neighborhood $V''_{\widehat{\sigma}_{-i}}$ of $\widehat{\sigma}_{-i}$ such that, for all $\sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}$, (25) holds. □

Let $V''_{\widehat{\sigma}_{-i}}$ be the neighborhood given in Step 5. For all $\sigma'_{-i} \in V''_{\widehat{\sigma}_{-i}}$, one has

$$\begin{aligned}
&\int_T \int_X [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\widehat{\sigma}_i(\cdot|t_i) \otimes \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j)\right]\right](dx) \left[\bigotimes_{j=1}^N p_j\right](dt) \\
&= \int_{(T_i \times X_i) \setminus (A \cup \widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \\
&\quad \left[\bigotimes_{j \neq i} p_j\right](dt_{-i}) \widehat{\sigma}_i(d(t_i, x_i)) \\
&\quad + \int_{(A \cup \widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \\
&\quad \left[\bigotimes_{j \neq i} p_j\right](dt_{-i}) \widehat{\sigma}_i(d(t_i, x_i)) \\
&= \int_{(T_i \times X_i) \setminus (A \cup \widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j)\right](dx_{-i})
\end{aligned}$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$+\int_A \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j\neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$> \int_{(T_i\times X_i)\setminus(A\cup\widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i)) - \frac{3\varepsilon}{8}$$

$$+\int_A \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j\neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$+\int_{(A\cup\widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$-\int_{(A\cup\widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$= U_i(\widehat{\sigma}) + \int_A \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j\neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$-\frac{3\varepsilon}{8} - \int_{(A\cup\widehat{A})} \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$= U_i(\widehat{\sigma}) + \int_A \int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\underset{j\neq i}{\otimes} \sigma'_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$-\frac{3\varepsilon}{8} - \int_A \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j\neq i}{\otimes} \widehat{\sigma}_j(\cdot|t_j)\right](dx_{-i})$$

$$\left[\underset{j\neq i}{\otimes} p_j\right](dt_{-i})\widehat{\sigma}_i(d(t_i, x_i))$$

$$\geq U_i(\widehat{\sigma}) - \widehat{\sigma}_i(A)\left(\sup_{(t,x)\in T\times X}|u_i(t,x)g(t)|\right) - \frac{3\varepsilon}{8} - \widehat{\sigma}_i(A)$$
$$\left(\sup_{(t,x)\in T\times X}|u_i(t,x)g(t)|\right)$$
$$= U_i(\widehat{\sigma}) - \frac{\varepsilon}{2},$$

where the second and fourth equalities follow from the fact that $\widehat{\sigma}_i(\widehat{A}) = 0$ (see (23) in Step 4), the first inequality uses the fact that (25) holds for all $(t_i, x_i) \in (T_i \times X_i) \setminus (A \cup \widehat{A})$, and the last equality uses (24).

This finishes the proof of Lemma 1. □

5.2 Proof of Lemma 2

We restate Lemma 2 here for the convenience of the reader.

Lemma 2 *Given a Bayesian game $(T_i, X_i, u_i, p)_{i=1}^N$, suppose that for each $t \in T$, the map $\sum_{i=1}^N u_i(t,\cdot) : X \to \mathbb{R}$ is upper semicontinuous. Suppose further that p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$. Then the map $\sum_{i=1}^N U_i(\cdot) : \mathscr{D} \to \mathbb{R}$ is upper semicontinuous.* □

Proof The map $\sum_{i=1}^N U_i(\cdot) : \mathscr{D} \to \mathbb{R}$ is upper semicontinuous with respect to the so-called *weak-strong topology* (*ws-topology* for short) (see Balder (2001, Definition 1.1)). More precisely, if each $\mathscr{D}_i$ is endowed with the relative *ws*-topology, and $\mathscr{D}$ is endowed with the corresponding product topology, then the map $\sum_{i=1}^N U_i(\cdot) : \mathscr{D} \to \mathbb{R}$ is upper semicontinuous (see Carbonell-Nicolau and McLean (2018, §5.2)). It only remains to observe that, by an argument analogous to that in the proof of Step 19 (on p. 1641, in the proof of Lemma 5), the relative product w-topology on $\mathscr{D}$ is equivalent to the relative product *ws*-topology on $\mathscr{D}$. □

5.3 Proof of Lemma 3

Lemma 3 *Suppose that $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ is a Bayesian game satisfying generic entire payoff security and generic local equi-upper semicontinuity. Then Γ satisfies strong uniform payoff security.* □

Proof Let $\mu = (\mu_1, \ldots, \mu_N) \in \widetilde{\mathscr{D}}$ be the profile of measures given by the generic local equi-upper semicontinuity condition (see Definition 11).

Since $\widetilde{\mathscr{D}} \subseteq \widehat{\mathscr{D}}$, it suffices to show the following (recall Definition 9): For each i and $\varepsilon > 0$ there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying the following:

(a) For each k and $(t,x) \in T \times X$, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) > u_i(t,x) - \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

(b) For each $(t, x_{-i}) \in T \times X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i|t_i) = 1$ satisfying the following: for each $x_i \in Y_i$, there exists K such that for all $k \geq K$, there is a neighborhood $V'_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) < u_i(t, (x_i, y_{-i})) + \varepsilon, \quad \text{for all } y_{-i} \in V'_{x_{-i}}.$$

Fix $\varepsilon > 0$ and i. By the generic entire payoff security condition (Definition 10), for each $x_i \in X_i$ and $k \in \mathbb{N}$, there exist $h^k(x_i) \in X_i$ and $\gamma^k(x_i) > 0$ such that for every $(t, z_{-i}) \in T \times X_{-i}$, there is a neighborhood $V_{z_{-i}}$ of z_{-i} such that

$$\begin{aligned} &u_i(t, (h^k(x_i), y_{-i})) > u_i(t, (x_i, z_{-i})) - \varepsilon \quad \text{for all } y_{-i} \in V_{z_{-i}}, \text{ if } x_i \in K_i \cup C_i, \\ &h^k(x_i) \in N_{\frac{1}{k}}(x_i) \text{ and } u_i(t, (h^k(x_i), y_{-i})) \\ &> u_i(t, (z'_i, z_{-i})) - \varepsilon \quad \text{for all } (z'_i, y_{-i}) \in N_{\gamma^k(x_i)}(x_i) \times V_{z_{-i}}, \\ &\qquad \text{if } x_i \in A_i \setminus C_i, \end{aligned}$$

where C_i is a countable subset of A_i. In addition, there is no loss of generality in assuming that $\gamma^k(x_i) < \frac{1}{k}$, and, since the members of K_i are isolated points, one may take $h^k(x_i) = x_i$ for $x_i \in K_i$.

Now, since $A_i \setminus C_i \subseteq X_i$ and X_i is compact and metric, $A_i \setminus C_i$ is separable, hence Lindelöf, and so, for each k, there is a countable subset $\{x_i^{(k,1)}, x_i^{(k,2)}, \ldots\}$ of $A_i \setminus C_i$ such that

$$\bigcup_{l=1}^{\infty} \left(N_{\gamma^k(x_i^{(k,l)})}(x_i^{(k,l)}) \cap (A_i \setminus C_i) \right) = \bigcup_{x_i \in A_i \setminus C_i} \left(N_{\gamma^k(x_i)}(x_i) \cap (A_i \setminus C_i) \right).$$

Now define $V^{(k,1)}, V^{(k,2)}, \ldots$ recursively as follows:

$$V^{(k,1)} := N_{\gamma^k(x_i^{(k,1)})}(x_i^{(k,1)}) \cap (A_i \setminus C_i)$$

and

$$V^{(k,l)} := \left(N_{\gamma^k(x_i^{(k,l)})}(x_i^{(k,l)}) \cap (A_i \setminus C_i) \right) \setminus \left(\bigcup_{m=1}^{l-1} V^{(k,m)} \right), \quad l \in \{2, 3, \ldots\}.$$

Next, define, for each k, $f^k : T_i \times X_i \to X_i$ by

$$f^k(t_i, x_i) := \begin{cases} h^k(x_i^{(k,l)}) & \text{if } x_i \in V^{(k,l)}, \\ h^k(x_i) & \text{if } x_i \in C_i \cup K_i. \end{cases}$$

Observe that

$$f^k(T_i \times X_i) = f^k(T_i \times (A_i \setminus C_i)) \cup f^k(T_i \times (C_i \cup K_i))$$

$$= \left\{h^k(x^{(k,1)}), h^k(x^{(k,2)}), \ldots\right\} \cup f^k(T_i \times (C_i \cup K_i)),$$

and so $f^k(T_i \times X_i)$ is countable. Therefore, given $B \in \mathscr{B}(X_i)$, $B \cap f^k(T_i \times X_i)$ is countable, and

$$\begin{aligned} f^{k^{-1}}(B) &= f^{k^{-1}}(B \cap f^k(T_i \times X_i)) \\ &= f^{k^{-1}}\left(\left\{h^k(x^{(k,1)}), h^k(x^{(k,2)}), \ldots\right\} \cap B\right) \\ &\quad \cup \left(T_i \times \left\{x_i \in C_i \cup K_i : h^k(x_i) \in B\right\}\right) \\ &= \left(T_i \times \left(\bigcup_{m=1}^{\infty} V^{(k,l_m)}\right)\right) \cup \left(T_i \times \left\{x_i \in C_i \cup K_i : h^k(x_i) \in B\right\}\right) \end{aligned}$$

for some subsequence (l_m) of (l). Thus, $f^{k^{-1}}(B)$ is expressible as a union of Borel subsets of $T_i \times X_i$, and we see that f^k is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable.

To see that item (a) holds, fix k and $(t, x) \in T \times X$. If $x_i \in C_i \cup K_i$, it is clear that there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) > u_i(t, x) - \varepsilon, \quad \text{for all } y_{-i} \in V_{x_{-i}}. \tag{29}$$

Now suppose that $x_i \in A_i \setminus C_i$. Then $x_i \in V^{(k,l)}$ for some l and $f^k(t_i, x_i) = h^k(x_i^{(k,l)})$. Therefore, since there is a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$\begin{aligned} &u_i(t, (h^k(x_i^{(k,l)}), y_{-i})) \\ &\quad > u_i(t, (x_i', x_{-i})) - \varepsilon, \quad \text{for all } (x_i', y_{-i}) \in N_{\gamma^k(x_i^{(k,l)})}(x_i^{(k,l)}) \times V_{x_{-i}}, \end{aligned}$$

and because $x_i \in N_{\gamma^k(x_i^{(k,l)})}(x_i^{(k,l)})$, one obtains (29).

To see that item (b) holds, fix $(t, x_{-i}) \in T \times X_{-i}$ and let Y_i be the set given by the generic local equi-upper semicontinuity condition (Definition 11). Set $Y_i' := Y_i \setminus C_i$. Then $\mu_i(Y_i'|t_i) = 1$. In addition, given $x_i \in Y_i'$, $f^k(t_i, x_i) = x_i$ if $x_i \in K_i$ and $f^k(t_i, x_i) = h^k(x_i^{(k,l)}), h^k(x_i^{(k,l)}) \in N_{\frac{1}{k}}(x_i^{(k,l)})$, and $x_i \in V^{(k,l)} \subseteq N_{\gamma^k(x_i^{(k,l)})}(x_i^{(k,l)}) \subseteq N_{\frac{1}{k}}(x_i^{(k,l)})$ if $x_i \in A_i \setminus C_i$. Consequently, $f^k(t_i, x_i) \to x_i$ for every $x_i \in Y_i'$. Now given $x_i \in Y_i'$, the generic local equi-upper semicontinuity condition (Definition 11) gives a neighborhood V_{x_i} of x_i such that for every $y_i \in V_{x_i}$, there is a neighborhood $V'_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (y_i, y_{-i})) < u_i(t, (x_i, y_{-i})) + \varepsilon, \quad \text{for all } y_{-i} \in V'_{x_{-i}}.$$

Since $f^k(t_i, x_i) \to x_i$, there exists K such that for all $k \geq K$, $f^k(t_i, x_i) \in V_{x_i}$, and so, for all $k \geq K$, there is a neighborhood $V'_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k(t_i, x_i), y_{-i})) < u_i(t, (x_i, y_{-i})) + \varepsilon, \quad \text{for all } y_{-i} \in V'_{x_{-i}},$$

as desired. □

A Proof of Lemma 5

In this appendix, we prove Lemma 5, which is instrumental for the proof of Lemma 1. Since the proof of Lemma 5 is rather technical, the formal details are preceded by a preliminary sketch of the proof's main argument.

Lemma 5 *Suppose that the Bayesian game $\Gamma = (T_i, X_i, u_i, p)_{i=1}^N$ satisfies strong uniform payoff security. Suppose that p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$. Then there exists $(\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$, there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying the following:*

(I) For each $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that

$$\begin{aligned}
&\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j \neq i} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \left[\bigotimes_{j \neq i} p_j\right](dt_{-i}) \\
&\quad > \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j)\right](dx_{-i}) \left[\bigotimes_{j \neq i} p_j\right](dt_{-i}) \\
&\quad\quad - \varepsilon, \text{ for all } \sigma'_{-i} \in V_{\sigma_{-i}},
\end{aligned}$$

where g is a density of p with respect to $p_1 \otimes \cdots \otimes p_N$.

(II) For each $\sigma_{-i} \in \mathscr{D}_{-i}$, there exists K such that for each $k \geq K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i^k, \sigma'_{-i}) < U_i(\mu_i, \sigma'_{-i}) + \varepsilon, \quad \text{for all } \sigma'_{-i} \in V'_{\sigma_{-i}},$$

where $\mu_i^k(\cdot|t_i)$ is defined by

$$\mu_i^k(B|t_i) := \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right). \tag{12}$$

A.1 Sketch of the proof of Lemma 5

We first verify (in Step 1 of the proof of Lemma 5) that the conditional probability $\mu_i^k(\cdot|t_i)$ defined in (12) is well-defined.

We must show that there exists $(\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$, there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying items (I) and (II) in the statement of Lemma 5.

Strong uniform payoff security (Definition 9) immediately gives $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\eta > 0$ there is a sequence $(f^k_{(i,\eta)})_{k=1}^{\infty}$ of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k_{(i,\eta)} : T_i \times X_i \to X_i$ satisfying the following:

(a) For each k and $(t, x) \in T \times X$, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k_{(i,\eta)}(t_i, x_i), y_{-i}))g(t) \geq [u_i(t, x) - \eta]g(t), \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

(b) For each $(t, x_{-i}) \in T \times X_{-i}$, there is a subset $Y_{(i,\eta,t,x_{-i})}$ of X_i with $\mu_i(Y_{(i,\eta,t,x_{-i})}|t_i) = 1$ satisfying the following: for each $x_i \in Y_{(i,\eta,t,x_{-i})}$, there exists $K_{(i,\eta,t,x)}$ such that for all $k \geq K_{(i,\eta,t,x)}$, there exists $n_{(i,\eta,t,x,k)}$ such that

$$\begin{aligned} &u_i(t, (f^k_{(i,\eta)}(t_i, x_i), y_{-i}))g(t) \\ &\quad \leq [u_i(t, (x_i, y_{-i})) + \eta]g(t), \quad \text{for all } y_{-i} \in N_{1/n_{(i,\eta,t,x,k)}}(x_{-i}). \end{aligned}$$

Fix (i, ε). To prove item (II) in the statement of Lemma 5, it suffices to show that there exists η such that, letting $f^k := f^k_{(i,\eta)}$ for each k, and given $\sigma_{-i} \in \mathscr{D}_{-i}$, there exists K such that, for each $k \geq K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i^k, \sigma'_{-i}) < U_i(\mu_i, \sigma'_{-i}) + \varepsilon, \quad \text{for all } \sigma'_{-i} \in V'_{\sigma_{-i}},$$

where $\mu_i^k(\cdot|t_i)$ is defined by

$$\mu_i^k(B|t_i) := \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right).$$

Choose any $\eta < \frac{\varepsilon}{12}$ and $\sigma_{-i} \in \mathscr{D}_{-i}$. Define $\psi^k_{(i,\eta)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\begin{aligned} &\psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \\ &\quad := \int_{T_i \times X_i} \left[u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i})) - u_i(t, (x_i, x_{-i}))\right] g(t)\mu_i(d(t_i, x_i)) \end{aligned}$$

and $\overline{\psi}^k_{(i,\eta)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) := \inf_n \sup_{y_{-i} \in N_{\frac{1}{n}}(x_{-i})} \psi^k_{(i,\eta)}(t_{-i}, y_{-i}).$$

Define $p_{-i} \in \Delta(T_{-i})$ by

$$p_{-i} := \otimes_{j \neq i} p_j,$$

and let $\mathscr{B}^*(T_{-i})$ be the p_{-i}-completion of $\mathscr{B}(T_{-i})$. Next, let p^*_{-i} be the complete extension of p_{-i}, and let $\mathscr{P}^*_{-i}$ be the space of all probability measures ν in $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ with

$$\nu(A \times X_{-i}) = p^*_{-i}(A), \quad \text{for all } A \in \mathscr{B}^*(T_{-i}).$$

Let $p^*_{-i} \otimes \sigma_{-i}$ be a probability measure in $\mathscr{P}^*_{-i}$ defined by

$$[p^*_{-i} \otimes \sigma_{-i}](A^* \times B) := \int_{A^*} \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right](B) p^*_{-i}(dt_{-i}),$$

for $A^* \in \mathscr{B}^*(T_{-i})$ and $B \in \mathscr{B}(X_{-i})$.

Endow the space $\mathscr{P}^*_{-i}$ with the relative w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$, and recall that each $\mathscr{D}_j$ is endowed with the relative w-topology (Definition 1) on $\Delta(T_j \times X_j)$, and that $\mathscr{D}_{-i}$ is provided with the corresponding product topology.

For each k, there is a neighborhood V^k_η of $p^*_{-i} \otimes \sigma_{-i}$ in $\mathscr{P}^*_{-i}$ such that

$$\begin{aligned} &\int_{T_{-i} \times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) \nu(d(t_{-i}, x_{-i})) \\ &\quad < \int_{T_{-i} \times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) [p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) + \frac{\varepsilon}{2}, \quad \text{for all } \nu \in V^k_\eta \end{aligned}$$

(this is proven in Step 19 of the proof of Lemma 5). Since $\psi^k_{(i,\eta)} \leq \overline{\psi}^k_{(i,\eta)}$, it follows that

$$\begin{aligned} &\int_{T_{-i} \times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \nu(d(t_{-i}, x_{-i})) \\ &\quad < \int_{T_{-i} \times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) [p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) + \frac{\varepsilon}{2}, \quad \text{for all } \nu \in V^k_\eta. \end{aligned}$$

Now since there exists k_η such that, for all $k \geq k_\eta$,

$$\int_{T_{-i} \times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) [p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) < 5\eta$$

(see Step 15 in the proof of Lemma 5), we see that, for all $k \geq k_\eta$,

$$\int_{T_{-i} \times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \nu(d(t_{-i}, x_{-i})) < \frac{\varepsilon}{2} + 5\eta, \quad \text{for all } \nu \in V^k_\eta.$$

Next, since V^k_η is open in $\mathscr{P}^*_{-i}$ and the map $h : \mathscr{D}_{-i} \to \mathscr{P}^*_{-i}$ defined by

$$h(\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_N) := p^*_{-i} \otimes \nu_{-i} \tag{30}$$

is continuous (see Step 17 in the proof of Lemma 5), it follows that $U_\eta^k := h^{-1}(V_\eta^k)$ is open in $\mathscr{D}_{-i}$. Because U_η^k contains $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \sigma_N)$, and since, for all $\sigma'_{-i} \in U_\eta^k$, we have $h(\sigma'_{-i}) \in V_\eta^k$ and

$$\int_{T_{-i}\times X_{-i}} \psi_{(i,\eta)}^k(t_{-i}, x_{-i}) \left[\bigotimes_{j\neq i} \sigma'_j\right] (d(t_{-i}, x_{-i})) \\ = \int_{T_{-i}\times X_{-i}} \psi_{(i,\eta)}^k(t_{-i}, x_{-i}) h(\sigma'_{-i})(d(t_{-i}, x_{-i}))$$

(the map $\psi_{(i,\varepsilon)}^k$ is $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable by Step 8 in the proof of Lemma 5), it follows that, for all $k \geq k_\eta$,

$$\int_{T_{-i}\times X_{-i}} \psi_{(i,\eta)}^k(t_{-i}, x_{-i}) \left[\bigotimes_{j\neq i} \sigma'_j\right] (d(t_{-i}, x_{-i})) < \frac{\varepsilon}{2} + 5\eta, \quad \text{for all } \sigma'_{-i} \in U_\eta^k.$$

But since $\eta \in (0, \frac{\varepsilon}{12})$, one obtains K such that, for all $k \geq K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i^k, \sigma'_{-i}) - U_i(\mu_i, \sigma'_{-i}) \\ = \int_{T_{-i}\times X_{-i}} \psi_{(i,\eta)}^k(t_{-i}, x_{-i}) \left[\bigotimes_{j\neq i} \sigma'_j\right] (d(t_{-i}, x_{-i})) < \varepsilon, \text{ for all } \sigma'_{-i} \in V'_{\sigma_{-i}}.$$

This establishes item (II) in the statement of Lemma 5.

To prove item (I) in the statement of Lemma 5, fix i and ε, and note that it suffices to show that there exists η (which may depend on i and ε) such that, given $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f_{(i,\eta)}^k(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j\neq i} \sigma'_j(\cdot|t_j)\right] (dx_{-i}) \left[\bigotimes_{j\neq i} p_j\right] (dt_{-i}) \\ > \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\bigotimes_{j\neq i} \sigma_j(\cdot|t_j)\right] (dx_{-i}) \left[\bigotimes_{j\neq i} p_j\right] (dt_{-i}) \\ - \varepsilon, \text{ for all } \sigma'_{-i} \in V_{\sigma_{-i}}, \tag{31}$$

Choose $\eta < \frac{\varepsilon}{2}$. Fix $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k. Define $\zeta_\eta : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\zeta_\eta(t_{-i}, x_{-i}) := \sup_{n\in\mathbb{N}} \inf_{y_{-i}\in N_{\frac{1}{n}}(x_{-i})} u_i(t, (f_{(i,\eta)}^k(t_i, x_i), y_{-i}))g(t).$$

There exists a neighborhood V_η^* of $p_{-i}^* \otimes \sigma_{-i}$ in $\mathscr{P}_{-i}^*$ such that

$$\int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i}))$$

$$> \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta, \text{ for all } \nu \in V^*_\eta$$

(see Step 23 in the proof of Lemma 5). From item (a) (on page 1622) and from the definition of ζ_η, we see that, for every $(t_{-i}, x_{-i}) \in T_{-i} \times X_{-i}$,

$$u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t) \geq \zeta_\eta(t_{-i}, x_{-i}) \geq [u_i(t, x) - \eta]g(t).$$

Consequently, for all $\nu \in \mathscr{P}^*_{-i}$,

$$\int_{T_{-i}\times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)\nu(d(t_{-i}, x_{-i})) \geq \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i}))$$

and

$$\int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \geq \int_{T_{-i}\times X_{-i}} [u_i(t, x) - \eta]g(t)\nu(d(t_{-i}, x_{-i})),$$

and so one obtains, for every $\nu \in V^*_\eta$,

$$\begin{aligned}
&\int_{T_{-i}\times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)\nu(d(t_{-i}, x_{-i})) \\
&\geq \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \\
&> \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta \\
&\geq \int_{T_{-i}\times X_{-i}} [u_i(t, x) - \eta]g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta \\
&= \int_{T_{-i}\times X_{-i}} u_i(t, x)g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta - \eta \\
&> \int_{T_{-i}\times X_{-i}} u_i(t, x)g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \varepsilon,
\end{aligned}$$

where the last inequality follows from the inequality $\eta < \frac{\varepsilon}{2}$.

Next, because the map h defined in (30) is continuous (see Step 17 in the proof of Lemma 5), and since V^*_η is open in $\mathscr{P}^*_{-i}$, it follows that $V_{\sigma_{-i}} := h^{-1}(V^*_\eta)$ is open in $\mathscr{D}_{-i}$. Since $V_{\sigma_{-i}}$ contains σ_{-i}, and since, for all $\sigma'_{-i} \in V_{\sigma_{-i}}$, one has $h(\sigma'_{-i}) \in V^*_\eta$ and

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)] \left[\bigotimes_{j\neq i} \sigma'_j(\cdot|t_j)\right](dx_{-i}) \left[\bigotimes_{j\neq i} p_j\right](dt_{-i})$$

$$= \int_{T_{-i}\times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)h(\sigma'_{-i})(d(t_{-i}, x_{-i}))$$

and

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\bigotimes_{j\neq i} \sigma_j(\cdot|t_j)\right](dx_{-i}) \left[\bigotimes_{j\neq i} p_j\right](dt_{-i})$$
$$= \int_{T_{-i}\times X_{-i}} [u_i(t, x)g(t)][p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})),$$

it follows that (31) holds.

This establishes item (I) in the statement of Lemma 5 and completes the argument.

We are now ready for the formal proof of Lemma 5.

Proof of Lemma 5 The proof is organized in a number of steps. To begin, we verify that the conditional probability $\mu_i^k(\cdot|t_i)$ defined in (12) is well-defined.

Step 1 *The map $t_i \in T_i \mapsto \mu_i^k(B|t_i) \in [0, 1]$ is $(\mathscr{B}(T_i), \mathscr{B}([0, 1]))$-measurable for each $B \in \mathscr{B}(X_i)$, and $\mu_i^k(\cdot|t_i) \in \Delta(X_i)$ for each $t_i \in T_i$.*

Proof of Step 1 For $t_i \in T_i$, it is clear that $\mu_i^k(B|t_i) \in [0, 1]$ for each $B \in \mathscr{B}(X_i)$, and that $\mu_i^k(X_i|t_i) = \mu_i(X_i|t_i) = 1$ and $\mu_i^k(\emptyset|t_i) = \mu_i(\emptyset|t_i) = 0$. To see that each $\mu_i^k(\cdot|t_i)$ is countably additive, choose a countable collection $(B^l)_{l=1}^{\infty}$ of pairwise disjoint sets in $\mathscr{B}(X_i)$ and note that

$$\begin{aligned}
\mu_i^k\left(\bigcup_{l=1}^{\infty} B^l \middle| t_i\right) &= \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in \bigcup_{l=1}^{\infty} B^l\right\} \middle| t_i\right) \\
&= \mu_i\left(\bigcup_{l=1}^{\infty} \left\{x_i \in X_i : f^k(t_i, x_i) \in B^l\right\} \middle| t_i\right) \\
&= \sum_{l=1}^{\infty} \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B^l\right\} \middle| t_i\right) \\
&= \sum_{l=1}^{\infty} \mu_i^k(B^l|t_i).
\end{aligned}$$

Thus, $\mu_i^k(\cdot|t_i) \in \Delta(X_i)$ for each $t_i \in T_i$.[10]

[10] The sets

$$\left\{x_i \in X_i : f^k(t_i, x_i) \in \bigcup_{l=1}^{\infty} B^l\right\} \quad \text{and} \quad \left\{x_i \in X_i : f^k(t_i, x_i) \in B^l\right\}$$

are in $\mathscr{B}(X_i)$ because the map f^k, being jointly measurable, is separately measurable (see, *e.g.*, Aliprantis and Border (2006, Theorem 4.48)).

Next, we show that the map $t_i \in T_i \mapsto \mu_i^k(B|t_i) \in [0, 1]$ is $(\mathscr{B}(T_i), \mathscr{B}([0, 1]))$-measurable for each $B \in \mathscr{B}(X_i)$.

Fix $B \in \mathscr{B}(X_i)$. Because the map $f^k : T_i \times X_i \to X_i$ is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable, one has

$$A := f^{k^{-1}}(B) \in \mathscr{B}(T_i \times X_i).$$

Now define the maps $\vartheta : T_i \to T_i \times \Delta(X_i)$ and $\zeta : T_i \times \Delta(X_i) \to [0, 1]$ as follows:

$$\vartheta(t_i) := (t_i, \mu_i(\cdot|t_i)) \quad \text{and} \quad \zeta(t_i, \nu) := \nu(A_{t_i}), \tag{32}$$

where A_{t_i} denotes the t_i-section of A in X_i: $A_{t_i} := \{x_i \in X_i : (t_i, x_i) \in A\}$.

We proceed in four sub-steps.

Step 1.1 *The map ϑ defined in (32) is $(\mathscr{B}(T_i), \mathscr{B}(T_i \times \Delta(X_i)))$-measurable.*

Proof of Step 1.1 Because the map $t_i \in T_i \mapsto \mu_i(\hat{B}|t_i) \in [0, 1]$ is $(\mathscr{B}(T_i), \mathscr{B}([0, 1]))$-measurable for each $\hat{B} \in \mathscr{B}(X_i)$, Proposition 7.26 in Bertsekas and Shreve (1996) implies that the map $t_i \in T_i \mapsto \mu(\cdot|t_i) \in \Delta(X_i)$ is $(\mathscr{B}(T_i), \mathscr{B}(\Delta(X_i)))$-measurable. Therefore, because the map $t_i \in T_i \mapsto t_i \in T_i$ is $(\mathscr{B}(T_i), \mathscr{B}(T_i))$-measurable and the map $t_i \in T_i \mapsto \mu(\cdot|t_i) \in \Delta(X_i)$ is $(\mathscr{B}(T_i), \mathscr{B}(\Delta(X_i)))$-measurable, it follows from Lemma 4.49 in Aliprantis and Border (2006) that the map $\vartheta : T_i \to T_i \times \Delta(X_i)$ is $(\mathscr{B}(T_i), \mathscr{B}(T_i) \otimes \mathscr{B}(\Delta(X_i)))$-measurable, and hence $(\mathscr{B}(T_i), \mathscr{B}(T_i \times \Delta(X_i)))$-measurable. □

Step 1.2 *The map ζ defined in (32) is $(\mathscr{B}(T_i \times \Delta(X_i)), \mathscr{B}([0, 1]))$-measurable.*

Proof of Step 1.2 Since $\mathscr{B}(T_i) \otimes \mathscr{B}(\Delta(X_i)) = \mathscr{B}(T_i \times \Delta(X_i))$, the assertion is an immediate consequence of Theorem 17.25 in Kechris (1995). □

Step 1.3 *The composition map $\zeta \circ \vartheta : T_i \to [0, 1]$ is $(\mathscr{B}(T_i), \mathscr{B}([0, 1]))$-measurable.*

Proof of Step 1.3 The assertion follows from Step 1.1 and Step 1.2, together with the fact that compositions of measurable functions are measurable. □

Step 1.4 *The map $t_i \in T_i \mapsto \mu_i^k(B|t_i) \in [0, 1]$ is $(\mathscr{B}(T_i), \mathscr{B}([0, 1]))$-measurable.*

Proof of Step 1.4 In light of Step 1.3, it suffices to show that the map $t_i \in T_i \mapsto \mu_i^k(B|t_i) \in [0, 1]$ is identical to the composition map $\zeta \circ \vartheta : T_i \to [0, 1]$. To see this, fix $t_i \in T_i$ and note that

$$\begin{aligned}[\zeta \circ \vartheta](t_i) &= \zeta(\vartheta(t_i)) = \zeta(t_i, \mu_i(\cdot|t_i)) = \mu_i(A_{t_i}|t_i) \\ &= \mu_i(\{x_i \in X_i : (t_i, x_i) \in A\}\,|t_i) = \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\}\middle|t_i\right) \\ &= \mu_i^k(B|t_i).\end{aligned}$$

□

Step 1.4 gives the desired conclusion. This finishes the proof of Step 1. □

Next, we must show that there exists $(\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$, there is a sequence (f^k) of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k : T_i \times X_i \to X_i$ satisfying items (I) and (II) in the statement of Lemma 5.

Let $g : T \to [0, \infty)$ be a $((\mathscr{B}(T), \mathscr{B}([0, \infty)))$-measurable) density of p with respect to $p_1 \otimes \cdots \otimes p_N$ (*i.e.*, a Radon-Nikodym derivative of p with respect to $p_1 \otimes \cdots \otimes p_N$). First, we remark that there is no loss of generality in assuming that g is bounded.

Step 2 *The Radon-Nikodym derivative, g, may be taken bounded.*

Proof of Step 2 This follows from the fact that g is bounded up to sets of $p_1 \otimes \cdots \otimes p_N$-measure zero if and only if there exists $C \in \mathbb{R}$ such that $p(B) \le C[p_1 \otimes \cdots \otimes p_N](B)$ for all $B \in \mathscr{B}(T)$. The proof of this assertion is straightforward. □

Strong uniform payoff security (Definition 9) immediately gives $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{\mathscr{D}}$ such that for each i and $\varepsilon > 0$ there is a sequence $(f^k_{(i,\varepsilon)})_{k=1}^\infty$ of $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable maps $f^k_{(i,\varepsilon)} : T_i \times X_i \to X_i$ satisfying the following:

(a) For each k and $(t, x) \in T \times X$, there exists a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i}))g(t) \ge [u_i(t, x) - \varepsilon]g(t), \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

(b) For each $(t, x_{-i}) \in T \times X_{-i}$, there is a subset $Y_{(i,\varepsilon,t,x_{-i})}$ of X_i with $\mu_i(Y_{(i,\varepsilon,t,x_{-i})}|t_i) = 1$ satisfying the following: for each $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$, there exists $K_{(i,\varepsilon,t,x)}$ such that for all $k \ge K_{(i,\varepsilon,t,x)}$, there exists $n_{(i,\varepsilon,t,x,k)}$ such that

$$\begin{aligned} &u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i}))g(t) \\ &\quad \le [u_i(t, (x_i, y_{-i})) + \varepsilon]g(t), \quad \text{for all } y_{-i} \in N_{1/n_{(i,\varepsilon,t,x,k)}}(x_{-i}). \end{aligned}$$

First, we prove item (II) in the statement of Lemma 5. To this end, we first prove a number of preliminary facts, Step 3–Step 12 below.

We begin with the following definition. Given $(i, \varepsilon, t_{-i}, x_{-i})$ and $\{k, n\} \subseteq \mathbb{N}$, define $\xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})} : T_i \times X_i \to \mathbb{R}$ by

$$\begin{aligned} &\xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \\ &\quad := \sup_{y_{-i} \in N_{\frac{1}{n}}(x_{-i})} \Big[[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i}))]g(t)\Big]. \end{aligned} \tag{33}$$

Step 3 *Given $(i, \varepsilon, t_{-i}, x_{-i})$, there exist a $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable map $\widehat{\xi}^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})} : T_i \times X_i \to \mathbb{R}$ and $\widehat{A} \in \mathscr{B}(T_i \times X_i)$ such that*

$$\begin{aligned} &\mu_i(\widehat{A}) = 0 \quad \textit{and} \quad \widehat{\xi}^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \\ &\quad = \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \textit{ for all } (t_i, x_i) \in (T_i \times X_i) \setminus \widehat{A}. \end{aligned} \tag{34}$$

Proof of Step 3 Because the map $f^k_{(i,\varepsilon)} : T_i \times X_i \to X_i$ is $(\mathscr{B}(T_i \times X_i), \mathscr{B}(X_i))$-measurable, it is clear that the map $(\tau_i, y) \in T_i \times X \mapsto f^k_{(i,\varepsilon)}(\tau_i, y_i) \in X_i$ is $(\mathscr{B}(T_i \times X), \mathscr{B}(X_i))$-measurable. Therefore, applying Lemma 4.49 and Theorem 4.44 in Aliprantis and Border (2006), we see that the map $(\tau_i, y) \in T_i \times X \mapsto ((\tau_i, t_{-i}), (f^k_{(i,\varepsilon)}(\tau_i, y_i), y_{-i})) \in T_i \times X$ is $(\mathscr{B}(T_i \times X), \mathscr{B}(T_i \times X))$-measurable. Consequently, the map

$$\begin{aligned}&(\tau_i, y) \in T_i \times X \\ &\quad \mapsto \left[u_i((\tau_i, t_{-i}), (f^k_{(i,\varepsilon)}(\tau_i, y_i), y_{-i})) - u_i((\tau_i, t_{-i}), (y_i, y_{-i}))\right] g(\tau_i, t_{-i})\end{aligned}$$

is $(\mathscr{B}(T_i \times X), \mathscr{B}(\mathbb{R}))$-measurable, and hence (by Theorem 4.44 in Aliprantis and Border (2006)) the map

$$\begin{aligned}&((\tau_i, z_i), z_{-i}) \in T_i \times X_i \times X_{-i} \\ &\quad \mapsto \left[u_i((\tau_i, t_{-i}), (f^k_{(i,\varepsilon)}(\tau_i, y_i), y_{-i})) - u_i((\tau_i, t_{-i}), y)\right] g(\tau_i, t_{-i})\end{aligned} \tag{35}$$

is $(\mathscr{B}(T_i \times X_i) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. Letting $\mathscr{B}^{\mu_i}(T_i \times X_i)$ denote the μ_i-completion of $\mathscr{B}(T_i \times X_i)$, it follows that the map in (35) is $(\mathscr{B}^{\mu_i}(T_i \times X_i) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and since $\mathscr{B}^{\mu_i}(T_i \times X_i)$ equals its universal completion, it follows from the proof of the Theorem in Carbonell-Nicolau (2014a) that the map

$$\begin{aligned}&((\tau_i, z_i), z_{-i}) \in T_i \times X_i \times X_{-i} \\ &\quad \mapsto \sup_{y_{-i} \in N_{\frac{1}{n}}(z_{-i})} \left[[u_i((\tau_i t_{-i}), (f^k_{(i,\varepsilon)}(\tau_i, z_i), y_{-i})) - u_i((\tau_i, t_{-i}), (z_i, y_{-i}))] g(\tau_i, t_{-i})\right]\end{aligned}$$

is $(\mathscr{B}^{\mu_i}(T_i \times X_i) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. Consequently, the map $\xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}$ defined in (33) is $(\mathscr{B}^{\mu_i}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (2006, Theorem 4.48)). Applying Theorem 10.35 in Aliprantis and Border (2006), we see that there exist a $(\mathscr{B}(T_i \times X_i), \mathscr{B}(\mathbb{R}))$-measurable map $\widehat{\xi}^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})} : T_i \times X_i \to \mathbb{R}$ and $\widehat{A} \in \mathscr{B}(T_i \times X_i)$ satisfying (34), as we sought. □

Now let $\xi_{(i,\varepsilon,t_{-i},x_{-i})} : T_i \times X_i \to \mathbb{R}$ be defined by

$$\begin{aligned}\xi_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) &:= \limsup_{k\to\infty} \left[\inf_n \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i)\right] \\ &= \lim_{k\to\infty} \left[\sup_{k' \geq k} \left[\inf_n \xi^{(k',n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i)\right]\right].\end{aligned} \tag{36}$$

Step 4 *Given $(i, \varepsilon, t_{-i}, x_{-i})$, there exists a sequence (n_k) such that*

$$\xi_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i) = \limsup_{k\to\infty} \xi^{(k,n_k)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i)$$
$$= \lim_{k\to\infty}\left[\sup_{k'\geq k}\left[\xi^{(k',n_{k'})}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i)\right]\right], \quad \textit{for each } (t_i,x_i)\in T_i\times X_i. \quad (37)$$

Proof of Step 4 For each $(t_i, x_i) \in T_i \times X_i$ and k, there exists n_k such that

$$\inf_n \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i) \leq \xi^{(k,n_k)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i) < \inf_n \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i) + \frac{1}{k}.$$

Consequently,

$$\limsup_{k\to\infty}\left[\inf_n \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i)\right] \leq \limsup_{k\to\infty}\left[\xi^{(k,n_k)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i)\right]$$
$$\leq \limsup_{k\to\infty}\left[\inf_n \xi^{(k,n)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i)\right],$$

and (in light of (36)) this yields (37). □

Step 5 *Given $(i, \varepsilon, t_{-i}, x_{-i})$ and $t_i \in T_i$, let $Y_{(i,\varepsilon,t,x_{-i})}$ be the subset of X_i given in item (b) on page 1628. Then, for $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$, one has*

$$\xi_{(i,\varepsilon,t_{-i},x_{-i})}(t_i,x_i) \leq \varepsilon g(t). \quad (38)$$

Proof of Step 5 By item (b), for each $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$, there exists $K_{(i,\varepsilon,t,x)}$ such that for all $k \geq K_{(i,\varepsilon,t,x)}$, there exists $n_{(i,\varepsilon,t,x,k)}$ such that

$$\left[u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))\right] g(t)$$
$$\leq \varepsilon g(t), \quad \text{for all } y_{-i} \in N_{1/n_{(i,\varepsilon,t,x,k)}}(x_{-i}).$$

Therefore, using (33) and (36), we see that (38) holds. □

Step 6 *Given $(i, \varepsilon, t_{-i}, x_{-i})$, let $\widehat{A}$ be the subset of $T_i \times X_i$ given by Step 3 and let (n_k) be the sequence given by Step 4. Then there exist $A \subseteq (T_i \times X_i) \setminus \widehat{A}$ with*

$$\mu_i(A)\left[\sup_{t\in T} g(t)\right]\left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X}[u_i(\tau,z)-u_i(\tau',z')]\right] < \varepsilon \quad (39)$$

and $\overline{k}$ satisfying the following: for each $k \geq \overline{k}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$,

$$[u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t) < \varepsilon + \varepsilon g(t) \quad (40)$$

for all $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$ with $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$.

Proof of Step 6 First, note that the left-hand side of (39) is well-defined because u_i is bounded by assumption and g may be taken bounded (Step 2).

Now, given (37), and applying Egorov's Theorem (*e.g.*, see Dudley (2004, Theorem 7.5.1)), there exists $A \subseteq (T_i \times X_i) \setminus \widehat{A}$ satisfying (39) such that the map

$$(t_i, x_i) \in T_i \times X_i \mapsto \sup_{k' \geq k} \left[\xi^{(k', n_{k'})}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \right]$$

converges to $\xi_{(i,\varepsilon,t_{-i},x_{-i})}$ uniformly on $(T_i \times X_i) \setminus (\widehat{A} \cup A)$. Therefore, there exists $\overline{k}$ such that for all $k \geq \overline{k}$ and $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$,

$$\left| \sup_{k' \geq k} \left[\xi^{(k', n_{k'})}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \right] - \xi_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \right| < \varepsilon,$$

implying that for all $k \geq \overline{k}$ and $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$,

$$\sup_{k' \geq k} \left[\xi^{(k', n_{k'})}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \right] < \xi_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) + \varepsilon.$$

Consequently, in light of Step 5 and (38), we see that for all $k \geq \overline{k}$,

$$\xi^{(k, n_k)}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \leq \sup_{k' \geq k} \left[\xi^{(k', n_{k'})}_{(i,\varepsilon,t_{-i},x_{-i})}(t_i, x_i) \right] < \varepsilon + \varepsilon g(t)$$

for all $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$ with $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$, and so for each $k \geq \overline{k}$,

$$\sup_{y_{-i} \in N_{\frac{1}{n_k}}(x_{-i})} \left[[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i}))] g(t) \right] < \varepsilon + \varepsilon g(t)$$

for all $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$ with $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$, whence for each $k \geq \overline{k}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$, (40) holds for all $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$ with $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$. □

Step 7 *Given* $(i, \varepsilon, t_{-i}, x_{-i})$, *there exists* $\overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ *such that for each* $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ *and* $y_{-i} \in N_{1/n_k}(x_{-i})$,

$$\int_{T_i \times X_i} \left[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i})) \right] g(t) \mu_i(d(t_i, x_i)) < 2\varepsilon + \varepsilon \int_{T_i} g(t) p_i(dt_i). \tag{41}$$

Proof of Step 7 Given $(i, \varepsilon, t_{-i}, x_{-i})$, Step 6 gives $\overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ such that, for each $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$, (40) holds for all $(t_i, x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)$

with $x_i \in Y_{(i,\varepsilon,t,x_{-i})}$. Therefore, for each $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$, and given $t_i \in T_i$, we have

$$\int_{Y_{(i,\varepsilon,t,x_{-i})}\cap X_{t_i}} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i) < \varepsilon + \varepsilon g(t),$$

where X_{t_i} denotes the t_i-section of $(T_i \times X_i) \setminus (\widehat{A} \cup A)$ in X_i, *i.e.*,

$$X_{t_i} := \left\{x_i \in X_i : (t_i,x_i) \in (T_i \times X_i) \setminus (\widehat{A} \cup A)\right\}.$$ [11]

Since $\mu_i(Y_{(i,\varepsilon,t,x_{-i})}|t_i) = 1$ (see (b)), it follows that, for each $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$, and given $t_i \in T_i$, we have

$$\int_{X_{t_i}} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i) < \varepsilon + \varepsilon g(t).$$

Note that the last inequality is expressible as

$$\begin{aligned}
&\int_{X_i} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i)\\
&\quad < \varepsilon + \varepsilon g(t) + \int_{X_i\setminus X_{t_i}} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i)\\
&\quad = \varepsilon + \varepsilon g(t) + \int_{(A\cup\widehat{A})_{t_i}} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i),
\end{aligned}$$

where $(A \cup \widehat{A})_{t_i}$ denotes the t_i-section of $A \cup \widehat{A}$ in X_i (*i.e.*, $(A \cup \widehat{A})_{t_i} := \left\{x_i \in X_i : (t_i,x_i) \in A \cup \widehat{A}\right\}$). Now since

$$\begin{aligned}
&\int_{(A\cup\widehat{A})_{t_i}} [u_i(t,(f^k_{(i,\varepsilon)}(t_i,x_i),y_{-i})) - u_i(t,(x_i,y_{-i}))]g(t)\mu_i(dx_i|t_i)\\
&\quad \leq \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau,z) - u_i(\tau',z')]\right] g(t) \int_{(A\cup\widehat{A})_{t_i}} \mu_i(dx_i|t_i)\\
&\quad = \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau,z) - u_i(\tau',z')]\right] g(t)\mu_i((A\cup\widehat{A})_{t_i}|t_i)\\
&\quad = \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau,z) - u_i(\tau',z')]\right] g(t)\mu_i((A)_{t_i}\cup(\widehat{A})_{t_i}|t_i)\\
&\quad \leq \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau,z) - u_i(\tau',z')]\right]
\end{aligned}$$

[11] Since $(T_i \times X_i) \setminus (\widehat{A} \cup A) \in \mathscr{B}(T_i \times X_i)$, it follows from Halmos (1974, Theorem A, §34, p. 141) that $X_{t_i} \in \mathscr{B}(X_i)$.

$$\left[\sup_{t\in T} g(t)\right]\left(\mu_i((A)_{t_i}|t_i) + \mu_i((\widehat{A})_{t_i}|t_i)\right),$$

where $(A)_{t_i}$ and $(\widehat{A})_{t_i}$ denote, respectively, the t_i-sections of A and $\widehat{A}$ in X_i, we conclude that for each $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$, and given $t_i \in T_i$, we have

$$\begin{aligned}
&\int_{X_i} [u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i}))]g(t)\mu_i(dx_i|t_i) \\
&\quad < \varepsilon + \varepsilon g(t) + \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau, z) - u_i(\tau', z')]\right] \\
&\quad\quad \left[\sup_{t\in T} g(t)\right]\left(\mu_i((A)_{t_i}|t_i) + \mu_i((\widehat{A})_{t_i}|t_i)\right).
\end{aligned}$$

Consequently, letting

$$C := \left[\sup_{((\tau,z),(\tau',z'))\in T\times X\times T\times X} [u_i(\tau, z) - u_i(\tau', z')]\right]\left[\sup_{t\in T} g(t)\right],$$

we have, for each $k \geq \overline{k}_{(i,\varepsilon,t_{-i},x_{-i})}$ and $y_{-i} \in N_{1/n_k}(x_{-i})$,

$$\begin{aligned}
&\int_{T_i\times X_i} [u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i}))]g(t)\mu_i(d(t_i, x_i)) \\
&= \int_{T_i}\int_{X_i} [u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), y_{-i})) - u_i(t, (x_i, y_{-i}))]g(t)\mu_i(dx_i|t_i)p_i(dt_i) \\
&< \varepsilon + \varepsilon \int_{T_i} g(t)p_i(dt_i) + C\left(\int_{T_i} \mu_i((A)_{t_i}|t_i)p_i(dt_i) + \int_{T_i} \mu_i((\widehat{A})_{t_i}|t_i)p_i(dt_i)\right) \\
&= \varepsilon + \varepsilon \int_{T_i} g(t)p_i(dt_i) + C(\mu_i(A) + \mu_i(\widehat{A})) \\
&= \varepsilon + \varepsilon \int_{T_i} g(t)p_i(dt_i) + C\mu_i(A) < 2\varepsilon + \varepsilon \int_{T_i} g(t)p_i(dt_i),
\end{aligned}$$

where the last equality uses the fact that $\mu_i(\widehat{A}) = 0$ (see Step 3) and the last inequality follows from (39) (see Step 6). This establishes (41) and finishes the proof of Step 7. □

Next, define $\psi^k_{(i,\varepsilon)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\begin{aligned}
&\psi^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) \\
&\quad := \int_{T_i\times X_i} \left[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), x_{-i})) - u_i(t, (x_i, x_{-i}))\right] g(t)\mu_i(d(t_i, x_i)) \quad (42)
\end{aligned}$$

and $\overline{\psi}^k_{(i,\varepsilon)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\overline{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) := \inf_n \sup_{y_{-i} \in N_{\frac{1}{n}}(x_{-i})} \psi^k_{(i,\varepsilon)}(t_{-i}, y_{-i}). \tag{43}$$

Step 8 *The map $\psi^k_{(i,\varepsilon)}$ is $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable.*

Proof of Step 8 Since the map

$$\begin{aligned}&((t_i, x_i), (t_{-i}, x_{-i})) \in T_i \times X_i \times T_{-i} \times X_{-i}\\ &\quad\mapsto \left[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), x_{-i})) - u_i(t, (x_i, x_{-i}))\right] g(t)\end{aligned}$$

is $(\mathscr{B}(T_i \times X_i) \otimes \mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, it follows from Theorem 17.25 in Kechris (1995) that the map

$$\begin{aligned}&(\nu_i, (t_{-i}, x_{-i})) \in \Delta(T_i \times X_i) \times T_{-i} \times X_{-i}\\ &\quad\mapsto \int_{T_i \times X_i} \left[u_i(t, (f^k_{(i,\varepsilon)}(t_i, x_i), x_{-i})) - u_i(t, (x_i, x_{-i}))\right] g(t)\nu_i(d(t_i, x_i))\end{aligned}$$

is $(\mathscr{B}(\Delta(T_i \times X_i)) \otimes \mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. Consequently, by Theorem 4.48 in Aliprantis and Border (2006), the map $\psi^k_{(i,\varepsilon)}$ is $(\mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and hence $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. □

Given i, define $p_{-i} \in \Delta(T_{-i})$ by

$$p_{-i} := \otimes_{j \neq i} p_j. \tag{44}$$

Define $\mathscr{B}^*(T_{-i})$ as the p_{-i}-completion of $\mathscr{B}(T_{-i})$.

Step 9 *The map $\overline{\psi}^k_{(i,\varepsilon)}$ is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable.*

Proof of Step 9 Because $\psi^k_{(i,\varepsilon)}$ is $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable (Step 8), and hence $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and since $\mathscr{B}^*(T_{-i})$ coincides with its universal completion, the assertion follows from the Theorem in Carbonell-Nicolau (2014a). □

Define $\widehat{\psi}^k_{(i,\varepsilon)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\widehat{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) := \sup_{k' \geq k} \overline{\psi}^{k'}_{(i,\varepsilon)}(t_{-i}, x_{-i}). \tag{45}$$

Given i and $\sigma_{-i} \in \mathscr{D}_{-i}$, let p^*_{-i} be the complete extension of p_{-i} (which was defined in (44)), and define $p_{-i} \otimes \sigma_{-i} \in \Delta(T_{-i} \times X_{-i})$ by

$$[p_{-i} \otimes \sigma_{-i}](A \times B) := \int_A \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j)\right](B) p_{-i}(dt_{-i}) \tag{46}$$

for $A \in \mathscr{B}(T_{-i})$ and $B \in \mathscr{B}(X_{-i})$, and let $p^*_{-i} \otimes \sigma_{-i}$ be a probability measure in $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ defined by

$$[p^*_{-i} \otimes \sigma_{-i}](A^* \times B) := \int_{A^*} \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}), \tag{47}$$

for $A^* \in \mathscr{B}^*(T_{-i})$ and $B \in \mathscr{B}(X_{-i})$.

Given i and $\sigma_{-i} \in \mathscr{D}_{-i}$, let $\mathscr{A}_{-i}(\sigma_{-i})$ denote the $p_{-i} \otimes \sigma_{-i}$-completion of $\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})$, and let $\mathscr{A}^*_{-i}(\sigma_{-i})$ represent the $p^*_{-i} \otimes \sigma_{-i}$-completion of $\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})$.

Step 10 *We have* $\mathscr{A}_{-i}(\sigma_{-i}) = \mathscr{A}^*_{-i}(\sigma_{-i})$.

Proof of Step 10 Let $\mathscr{B}(T_{-i}) \times \mathscr{B}(X_{-i})$ denote the product semiring of $\mathscr{B}(T_{-i})$ and $\mathscr{B}(X_{-i})$, and similarly for $\mathscr{B}^*(T_{-i}) \times \mathscr{B}(X_{-i})$. Let ν and ν^* denote the Carathéodory extensions of $p_{-i} \otimes \sigma_{-i}$ and $p^*_{-i} \otimes \sigma_{-i}$, respectively (which were defined in (46) and (47)).

We claim that $\nu = \nu^*$. To see this, note first that, because $\mathscr{B}(T_{-i}) \subseteq \mathscr{B}^*(T_{-i})$ and $p^*_{-i}|_{\mathscr{B}(T_{-i})} = p_{-i}$,

$$\begin{aligned}
&\nu^*(E) \\
&= \inf \left\{ \sum_{n=1}^{\infty} [p^*_{-i} \otimes \sigma_{-i}](A^n \times B^n) : A^n \in \mathscr{B}^*(T_{-i}), B^n \in \mathscr{B}(X_{-i}), E \subseteq \bigcup_{n=1}^{\infty} (A^n \times B^n) \right\} \\
&\leq \inf \left\{ \sum_{n=1}^{\infty} [p^*_{-i} \otimes \sigma_{-i}](A^n \times B^n) : A^n \in \mathscr{B}(T_{-i}), B^n \in \mathscr{B}(X_{-i}), E \subseteq \bigcup_{n=1}^{\infty} (A^n \times B^n) \right\} \\
&= \inf \left\{ \sum_{n=1}^{\infty} [p_{-i} \otimes \sigma_{-i}](A^n \times B^n) : A^n \in \mathscr{B}(T_{-i}), B^n \in \mathscr{B}(X_{-i}), E \subseteq \bigcup_{n=1}^{\infty} (A^n \times B^n) \right\} \\
&= \nu(E)
\end{aligned}$$

for each $E \subseteq T_{-i} \times X_{-i}$. In addition, since for each $A \in \mathscr{B}^*(T_{-i})$ and $B \in \mathscr{B}(X_{-i})$ there exists $C \in \mathscr{B}^*(T_{-i})$ with $p^*_{-i}(C) = 0$, $A \cap C = \emptyset$, and $A \cup C \in \mathscr{B}(T_{-i})$ (see, *e.g.*, Aliprantis and Border (2006, Theorem 10.23(7))), so that

$$\begin{aligned}
[p^*_{-i} \otimes \sigma_{-i}](A \times B) &= \int_A \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}) \\
&= \int_A \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}) \\
&\quad + \int_C \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}) \\
&= \int_{A \cup C} \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}) \\
&= \int_{A \cup C} \left[\bigotimes_{j \neq i} \sigma_j(\cdot|t_j) \right] (B) p_{-i}(dt_{-i}) \\
&= [p_{-i} \otimes \sigma_{-i}]((A \cup C) \times B),
\end{aligned}$$

it follows that $\nu^*(E) \geq \nu(E)$ for each $E \subseteq T_{-i} \times X_{-i}$. Thus, $\nu^*(E) = \nu(E)$ for each $E \subseteq T_{-i} \times X_{-i}$. It follows from Definition 10.34 and Theorem 10.23 in Aliprantis and Border (2006) that $\mathscr{A}_{-i} = \mathscr{A}^*_{-i}$. □

Step 11 *For every $\sigma_{-i} \in \mathscr{D}_{-i}$, the map $\widehat{\psi}^k_{(i,\varepsilon)}$ is $(\mathscr{A}^*(\sigma_{-i}), \mathscr{B}(\mathbb{R}))$-measurable.*

Proof of Step 11 By Step 9 and the definition of $\widehat{\psi}^k_{(i,\varepsilon)}$ in (45), we see that the map $\widehat{\psi}^k_{(i,\varepsilon)}$ is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and hence $(\mathscr{A}^*(\sigma_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. □

Now define $\widehat{\psi}_{(i,\varepsilon)} : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\widehat{\psi}_{(i,\varepsilon)}(t_{-i}, x_{-i}) := \lim_{k\to\infty} \widehat{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}). \tag{48}$$

Step 12 *Given (i, ε) and $\sigma_{-i} \in \mathscr{D}_{-i}$, there exists $\widehat{B}_{(i,\varepsilon,\sigma_{-i})} \subseteq T_{-i} \times X_{-i}$ such that*

$$[p^*_{-i} \otimes \sigma_{-i}](\widehat{B}_{(i,\varepsilon,\sigma_{-i})}) \left[\sup_k \left(\sup_{(t_{-i},x_{-i})\in T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) \right) \right] < \varepsilon$$

and $\widehat{\psi}^k_{(i,\varepsilon)}$ converges uniformly to $\widehat{\psi}_{(i,\varepsilon)}$ on $(T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\varepsilon,\sigma_{-i})}$, i.e., there exists $\overline{k}_{(i,\varepsilon,\sigma_{-i})}$ such that for all $k \geq \overline{k}_{(i,\varepsilon,\sigma_{-i})}$ and $(t_{-i}, x_{-i}) \in (T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\varepsilon,\sigma_{-i})}$,

$$\left| \widehat{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) - \widehat{\psi}_{(i,\varepsilon)}(t_{-i}, x_{-i}) \right| < \varepsilon.$$

Proof of Step 12 To lighten notation, let $\mathscr{A}^* = \mathscr{A}^*(\sigma_{-i})$. Let ν^* denote the Carathéodory extension of $p^*_{-i} \otimes \sigma_{-i}$ (which was defined in (47)), and let $\nu^*|_{\mathscr{A}^*}$ be the restriction of ν^* to $\mathscr{A}^*$.

Given the definition of $\widehat{\psi}_{(i,\varepsilon)}$ in (48), and since each $\widehat{\psi}^k_{(i,\varepsilon)}$ is $(\mathscr{A}^*, \mathscr{B}(\mathbb{R}))$-measurable (Step 11) and $\widehat{\psi}_{(i,\varepsilon)}$, being the pointwise limit of a sequence of $(\mathscr{A}^*, \mathscr{B}(\mathbb{R}))$-measurable functions, is itself $(\mathscr{A}^*, \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (2006, Lemma 4.29)), Egorov's Theorem (*e.g.*, see Dudley (2004, Theorem 7.5.1)) implies that there exists $B_{(i,\varepsilon,\sigma_{-i})} \in \mathscr{A}^*$ such that

$$\nu^*|_{\mathscr{A}^*}(B_{(i,\varepsilon,\sigma_{-i})}) \left[\sup_k \left(\sup_{(t_{-i},x_{-i})\in T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i}) \right) \right] < \varepsilon$$

and $\widehat{\psi}^k_{(i,\varepsilon)}$ converges uniformly to $\widehat{\psi}_{(i,\varepsilon)}$ on $(T_{-i} \times X_{-i}) \setminus B_{(i,\varepsilon,\sigma_{-i})}$.

It only remains to show that there exists $\widehat{B}_{(i,\varepsilon,\sigma_{-i})} \in \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})$ such that $B_{(i,\varepsilon,\sigma_{-i})} \subseteq \widehat{B}_{(i,\varepsilon,\sigma_{-i})}$ and

$$\nu^*|_{\mathscr{A}^*}(B_{(i,\varepsilon,\sigma_{-i})}) = [p_{-i} \otimes \sigma_{-i}](\widehat{B}_{(i,\varepsilon,\sigma_{-i})}).$$

But this follows from Theorem 10.23(6) in Aliprantis and Border (2006). □

We are now ready to prove item (II) in the statement of Lemma 5.

Fix (i, ε) and, for each η, let $(f^k_{(i,\eta)})_{k=1}^{\infty}$ be the sequence given on page 1628. It suffices to show that there exists η such that, letting $f^k := f^k_{(i,\eta)}$ for each k, and given $\sigma_{-i} \in \mathscr{D}_{-i}$, there exists K such that, for each $k \geq K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i^k, \sigma'_{-i}) < U_i(\mu_i, \sigma'_{-i}) + \varepsilon, \quad \text{for all } \sigma'_{-i} \in V'_{\sigma_{-i}}, \tag{49}$$

where $\mu_i^k(\cdot|t_i)$ is defined by

$$\mu_i^k(B|t_i) := \mu_i\left(\left\{x_i \in X_i : f^k(t_i, x_i) \in B\right\} \middle| t_i\right).$$

Choose $\eta < \frac{\varepsilon}{12}$ and $\sigma_{-i} \in \mathscr{D}_{-i}$. We proceed in nine additional steps (Step 13–Step 21).

Step 13 *We have*

$$\widehat{\psi}_{(i,\eta)}(t_{-i}, x_{-i}) \leq 2\eta + \eta \int_{T_i} g(t) p_i(dt_i), \quad \textit{for all } (t_{-i}, x_{-i}) \in T_{-i} \times X_{-i}. \tag{50}$$

Proof of Step 13 Given the definitions in (48), (45), (43), and (42), (50) follows from (41) (see Step 7). □

Step 14 *There exist* $\overline{k}_{(i,\eta,\sigma_{-i})}$ *and* $\widehat{B}_{(i,\eta,\sigma_{-i})}$ *with*

$$[p^*_{-i} \otimes \sigma_{-i}](\widehat{B}_{(i,\eta,\sigma_{-i})}) \left[\sup_k \left(\sup_{(t_{-i},x_{-i}) \in T_{-i} \times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})\right)\right] < \eta \tag{51}$$

such that for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$,

$$\begin{aligned}&\overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) \\ &\quad < 3\eta + \eta \int_{T_i} g(t) p_i(dt_i), \quad \textit{for all } (t_{-i}, x_{-i}) \in (T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\eta,\sigma_{-i})}.\end{aligned} \tag{52}$$

Proof of Step 14 Recall that i and $\sigma_{-i} \in \mathscr{D}_{-i}$ have been fixed, and choose η. By Step 12, there exists $\widehat{B}_{(i,\eta,\sigma_{-i})} \subseteq T_{-i} \times X_{-i}$ such that (51) holds and $\widehat{\psi}^k_{(i,\eta)}$ converges uniformly to $\widehat{\psi}_{(i,\eta)}$ on $(T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\eta,\sigma_{-i})}$, *i.e.*, there exists $\overline{k}_{(i,\eta,\sigma_{-i})}$ such that for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$ and $(t_{-i}, x_{-i}) \in (T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\eta,\sigma_{-i})}$,

$$\left|\widehat{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) - \widehat{\psi}_{(i,\eta)}(t_{-i}, x_{-i})\right| < \eta.$$

Consequently, in light of (50), we see that, for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$,

$$\widehat{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i}) < 3\eta + \eta \int_{T_i} g(t) p_i(dt_i), \quad \text{for all } (t_{-i}, x_{-i}) \in (T_{-i} \times X_{-i}) \setminus \widehat{B}_{(i,\eta,\sigma_{-i})},$$

and so, recalling the definition in (45), it follows that, for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$, (52) holds. □

Step 15 *There exists $\overline{k}_{(i,\eta,\sigma_{-i})}$ such that, for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$,*

$$\int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) < 5\eta \tag{53}$$

(recall that $p^*_{-i} \otimes \sigma_{-i}$ was defined in (47)).

Proof of Step 15 First, recall from Step 9 that the map $\overline{\psi}^k_{(i,\varepsilon)}$ is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, implying that the integral on the left-hand side of (53) is well-defined.

By Step 14, there there exist $\overline{k}_{(i,\eta,\sigma_{-i})}$ and $\widehat{B}_{(i,\eta,\sigma_{-i})}$ satisfying (51) such that for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$, (52) holds. Consequently, for all $k \geq \overline{k}_{(i,\eta,\sigma_{-i})}$,

$$\begin{aligned}
&\int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) \\
&\quad= \int_{(T_{-i}\times X_{-i})\setminus \widehat{B}_{(i,\eta,\sigma_{-i})}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) \\
&\qquad+ \int_{\widehat{B}_{(i,\eta,\sigma_{-i})}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) \\
&\quad< 4\eta + [p^*_{-i} \otimes \sigma_{-i}](\widehat{B}_{(i,\eta,\sigma_{-i})}) \left[\sup_{k'} \left(\sup_{(t_{-i},x_{-i})\in T_{-i}\times X_{-i}} \overline{\psi}^{k'}_{(i,\eta)}(t_{-i}, x_{-i})\right)\right] \\
&\quad< 5\eta,
\end{aligned}$$

as desired. □

Next, let $\mathscr{P}^*_{-i}$ be the space of all probability measures ν in $\Delta(T_{-i}\times X_{-i}, \mathscr{B}^*(T_{-i})\otimes \mathscr{B}(X_{-i}))$ with

$$\nu(A \times X_{-i}) = p^*_{-i}(A), \quad \text{for all } A \in \mathscr{B}^*(T_{-i}),$$

where, recall, p^*_{-i} denotes the complete extension of p_{-i} (which was defined in (44)), and where $\mathscr{B}^*(T_{-i})$ denotes the p_{-i}-completion of $\mathscr{B}(T_{-i})$.

Endow the space $\mathscr{P}^*_{-i}$ with the relative w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$.

Recall that each $\mathscr{D}_j$ is endowed with the relative w-topology (Definition 1) on $\Delta(T_j \times X_j)$, and that $\mathscr{D}_{-i}$ is provided with the corresponding product topology.

Define the map $h : \mathscr{D}_{-i} \to \mathscr{P}^*_{-i}$ by

$$h(\nu_1, \ldots, \nu_{i-1}, \nu_{i+1}, \ldots, \nu_N) := p^*_{-i} \otimes \nu_{-i}, \tag{54}$$

where $p^*_{-i} \otimes \nu_{-i}$ is a member of $\mathscr{P}^*_{-i}$ defined as follows:

$$[p^*_{-i} \otimes \nu_{-i}](A \times B) := \int_A \left[\underset{j \neq i}{\otimes} \nu_j(\cdot|t_j) \right] (B) p^*_{-i}(dt_{-i}),$$

for $A \in \mathscr{B}^*(T_{-i})$ and $B \in \mathscr{B}(X_{-i})$.

Step 16 *The space $\mathscr{P}^*_{-i}$ (with the relative w-topology) is homeomorphic to the space $\mathscr{P}_{-i}$ of all probability measures ν in $\Delta(T_{-i} \times X_{-i}, \mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})) = \Delta(T_{-i} \times X_{-i})$ with*

$$\nu(A \times X_{-i}) = p_{-i}(A), \quad \textit{for all } A \in \mathscr{B}(T_{-i})$$

(with the relative w-topology) (where, recall, p_{-i} was defined in (44)).

Proof of Step 16 Define $H : \mathscr{P}^*_{-i} \to \mathscr{P}_{-i}$ by

$$H(\nu) := \nu|_{\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})},$$

where $\nu|_{\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})}$ denotes the restriction of ν to $\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})$. We claim that H is a homeomorphism of $\mathscr{P}^*_{-i}$ onto $\mathscr{P}_{-i}$.

To see that H is one-to-one, fix ν and ν' in $\mathscr{P}^*_{-i}$ and suppose that

$$H(\nu) = \nu|_{\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})} = \nu'|_{\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})} = H(\nu').$$

Recall that $\mathscr{A}_{-i}(\sigma_{-i})$ (resp. $\mathscr{A}^*_{-i}(\sigma_{-i})$) is the $p_{-i} \otimes \sigma_{-i}$-completion of $\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})$ (resp., the $p^*_{-i} \otimes \sigma_{-i}$-completion of $\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})$). Since $\mathscr{A}_{-i}(\sigma_{-i}) = \mathscr{A}^*_{-i}(\sigma_{-i})$ (Step 10), and since $\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}) \subseteq \mathscr{A}^*(\sigma_{-i})$, it follows from Theorem 10.23(8) in Aliprantis and Border (2006) that there is a unique extension of $H(\nu) = H(\nu')$ to $\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})$, implying that $\nu = \nu'$.

To see that H is onto, pick $\nu \in \mathscr{P}_{-i}$. Let ν^* be the (unique) extension of ν to $\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})$. Then $H(\nu^*) = \nu$.

It remains to show that H and H^{-1} are continuous maps. First, note that the w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ can be viewed as the *initial topology* on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ generated by the family of maps $(F_f)_{f \in C^b(T_{-i} \times X_{-i})}$, where $F_f : \Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i})) \to \mathbb{R}$ is defined by

$$F_f(\nu) := \int_{T_{-i} \times X_{-i}} f(t_{-i}, x_{-i}) \nu(d(t_{-i}, x_{-i}))$$

(recall that $C^b(T_{-i} \times X_{-i})$ denotes the set of all bounded, continuous, real-valued functions on $T_{-i} \times X_{-i}$), *i.e.*, the coarsest topology on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ that makes all the functions F_f continuous. By Lemma 2.52 in Aliprantis and Border (2006), a net (ν^α) w-converges to ν in $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ if and only if $F_f(\nu^\alpha) \to F_f(\nu)$ for all $f \in C^b(T_{-i} \times X_{-i})$, *i.e.*, if and only if

$$\int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})\nu^\alpha(d(t_{-i}, x_{-i})) \\ \to \int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})), \text{ for all } f \in C^b(T_{-i} \times X_{-i}). \tag{55}$$

A similar argument can be made for the w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i})) = \Delta(T_{-i} \times X_{-i})$. Thus, a net (ν^α) w-converges to ν in $\Delta(T_{-i} \times X_{-i})$ if and only if (55) holds.

To see that H is continuous, let (ν^α) be a weakly convergent net in $\mathscr{P}^*_{-i}$ with limit point $\nu \in \mathscr{P}^*_{-i}$. Then, since the members of $C^b(T_{-i} \times X_{-i})$ are $(\mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable,

$$\int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})H(\nu^\alpha)(d(t_{-i}, x_{-i})) \\ = \int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})\nu^\alpha(d(t_{-i}, x_{-i})) \\ \to \int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \\ = \int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i})H(\nu)(d(t_{-i}, x_{-i})), \text{ for all } f \in C^b(T_{-i} \times X_{-i}).$$

The continuity of H^{-1} can be proven analogously. □

Step 17 *The map h defined in* (54) *is continuous.*

Proof of Step 17 Let $(\nu^n_j)_{j\neq i}$ be a weakly convergent sequence in $\mathscr{D}_{-i}$ with limit point $(\nu_j)_{j\neq i} \in \mathscr{D}_{-i}$. Applying Theorem 2.8 in Billingsley (1999), it follows that $\otimes_{j\neq i}\nu^n_j \underset{w}{\to} \otimes_{j\neq i}\nu_j$. Therefore, by the Portmanteau Theorem,

$$\int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i}) \left[\underset{j\neq i}{\otimes} \nu^n_j\right](d(t_{-i}, x_{-i})) \\ \to \int_{T_{-i}\times X_{-i}} f(t_{-i}, x_{-i}) \left[\underset{j\neq i}{\otimes} \nu_j\right](d(t_{-i}, x_{-i})) \tag{56}$$

for all bounded, continuous $f : T_{-i} \times X_{-i} \to \mathbb{R}$. Because $\mathscr{P}^*_{-i}$ is metrizable (Step 18 below), it suffices to show that $h((\nu^n_j)_{j\neq i}) \underset{w}{\to} h((\nu_j)_{j\neq i})$. By Step 16 and the Portmanteau Theorem, it suffices to prove that for all bounded, continuous $f : T_{-i} \times X_{-i} \to \mathbb{R}$,

$$\int_{T_{-i}\times X_{-i}} f(t_{-i},x_{-i})\left[p^*_{-i}\otimes \nu^n_{-i}\right](d(t_{-i},x_{-i}))$$
$$\to \int_{T_{-i}\times X_{-i}} f(t_{-i},x_{-i})\left[p^*_{-i}\otimes \mu_{-i}\right](d(t_{-i},x_{-i})). \tag{57}$$

For every $(\rho_j)_{j\neq i}\in\mathscr{D}_{-i}$ and bounded continuous $f:T_{-i}\times X_{-i}\to\mathbb{R}$, one has

$$\begin{aligned}
&\int_{T_{-i}\times X_{-i}} f(t_{-i},x_{-i})\left[\bigotimes_{j\neq i}\rho_j\right](d(t_{-i},x_{-i}))\\
&=\int_{T_1\times X_1}\cdots\int_{T_{i-1}\times X_{i-1}}\int_{T_{i+1}\times X_{i+1}}\\
&\quad\cdots\int_{T_N\times X_N} f(t_{-i},x_{-i})\rho_N(d(t_N,x_N))\cdots\rho_{i+1}(d(t_{i+1},x_{i+1}))\rho_{i-1}\\
&\quad(d(t_{i-1},x_{i-1}))\cdots\rho_1(d(t_1,x_1))\\
&=\int_{T_1}\int_{X_1}\cdots\int_{T_{i-1}}\int_{X_{i-1}}\int_{T_{i+1}}\int_{X_{i+1}}\\
&\quad\cdots\int_{T_N}\int_{X_N} f(t_{-i},x_{-i})\rho_N(dx_N|t_N)p_N(dt_N)\\
&\quad\cdots\rho_{i+1}(dx_{i+1}|t_{i+1})p_{i+1}(dt_{i+1})\rho_{i-1}(dx_{i-1}|t_{i-1})p_{i-1}(dt_{i-1})\cdots\rho_1(dx_1|t_1)p_1(dt_1)\\
&=\int_{T_{-i}\times X_{-i}} f(t_{-i},x_{-i})\left[p_{-i}\otimes\rho_{-i}\right](d(t_{-i},x_{-i}))\\
&=\int_{T_{-i}\times X_{-i}} f(t_{-i},x_{-i})\left[p^*_{-i}\otimes\rho_{-i}\right](d(t_{-i},x_{-i})).
\end{aligned}$$

Consequently, because (56) holds for all bounded, continuous $f:T_{-i}\times X_{-i}\to\mathbb{R}$, it follows that (57) holds for all bounded, continuous $f:T_{-i}\times X_{-i}\to\mathbb{R}$. □

Step 18 *The space $\mathscr{P}^*_{-i}$ with the relative w-topology is metrizable.*

Proof of Step 18 Because a topological space is metrizable if and only if it is homeomorphic to a subspace of some metric space, and since $\Delta(T_{-i}\times X_{-i})$ is metrizable, the assertion follows from Step 16. □

Recall that (i,ε) and $\sigma_{-i}\in\mathscr{D}_{-i}$ have been fixed.

Step 19 *For each k, there is a neighborhood V^k_η of $p^*_{-i}\otimes\sigma_{-i}$ in $\mathscr{P}^*_{-i}$ such that*

$$\int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i},x_{-i})\nu(d(t_{-i},x_{-i}))$$
$$<\int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i},x_{-i})[p^*_{-i}\otimes\sigma_{-i}](d(t_{-i},x_{-i}))+\frac{\varepsilon}{2},\quad \textit{for all } \nu\in V^k_\eta.$$

Proof of Step 19 First, note that the map $\overline{\psi}^k_{(i,\eta)}:T_{-i}\times X_{-i}\to\mathbb{R}$ (recall the definition in (43)) is $(\mathscr{B}^*(T_{-i})\otimes\mathscr{B}(X_{-i}),\mathscr{B}(\mathbb{R}))$-measurable (Step 9) and satisfies the following: the map $x_{-i} \in X_{-i} \mapsto \overline{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i})$ is upper semicontinuous for each $t_{-i} \in T_{-i}$ (see, *e.g.*, Ash (1972, Theorem A6.5)).

Next, recall that the space $\mathscr{P}^*_{-i}$ is endowed with the relative w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$.

Let (ν^α) be a weakly convergent net in $\mathscr{P}^*_{-i}$ with limit point $\nu \in \mathscr{P}^*_{-i}$. Then (ν^α) converges to ν in the *weak-strong topology* (*ws-topology* for short) (see Balder (2001, Definition 1.1)).[12] To see this, note that the net $(\nu^\alpha(\cdot \times X_{-i})) = (p^*_{-i})$ is constant, and so, because $\nu^\alpha \underset{w}{\to} \nu$, Theorem 3.7(viii) in Schäl (1975) implies that (ν^α) *ws*-converges to ν in $\mathscr{P}^*_{-i}$. Now suppose that (ν^α) *ws*-converges to ν in $\mathscr{P}^*_{-i}$. Again applying Theorem 3.7(viii) in Schäl (1975), it is clear that $\nu^\alpha \underset{w}{\to} \nu$. We have seen that the relative w-topology on $\mathscr{P}^*_{-i}$ is equivalent to the relative *ws*-topology on $\mathscr{P}^*_{-i}$. In other words, $\mathscr{P}^*_{-i}$ with the relative w-topology is homeomorphic to $\mathscr{P}^*_{-i}$ with the relative *ws*-topology. Consequently, because $\mathscr{P}^*_{-i}$ with the relative w-topology is metrizable (Step 18), it follows that $\mathscr{P}^*_{-i}$ with the relative *ws*-topology is metrizable.

Now suppose that (ν^n) is a *ws*-convergent sequence in $\mathscr{P}^*_{-i}$ with limit point $p^*_{-i} \otimes \sigma_{-i}$. Then, because the map $\overline{\psi}^k_{(i,\eta)} : T_{-i} \times X_{-i} \to \mathbb{R}$ is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, and since the map $x_{-i} \in X_{-i} \mapsto \overline{\psi}^k_{(i,\varepsilon)}(t_{-i}, x_{-i})$ is upper semicontinuous for each $t_{-i} \in T_{-i}$, Theorem 3.1 in Balder (2001) implies that

$$\limsup_{n\to\infty} \int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})\nu^n(d(t_{-i}, x_{-i})) \\ \leq \int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})).$$

Consequently, there is a *ws*-open neighborhood V of $p^*_{-i} \otimes \sigma_{-i}$ for which

$$\int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \\ < \int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) + \frac{\varepsilon}{2} \tag{58}$$

for all $\nu \in V$, and so there exists a w-open neighborhood V^* of $p^*_{-i} \otimes \sigma_{-i}$ such that (58) holds for all $\nu \in V^*$. □

Step 20 *There exists k_η such that, for all $k \geq k_\eta$, there is a neighborhood U^k_η of σ_{-i} in $\mathscr{D}_{-i}$ such that*

$$\int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \left[\bigotimes_{j\neq i} \sigma'_j\right](d(t_{-i}, x_{-i})) < \frac{\varepsilon}{2} + 5\eta, \quad \textit{for all } \sigma'_{-i} \in U^k_\eta. \tag{59}$$

[12] The *ws*-topology was introduced in Schäl (1975). See also Balder (2001).

Proof of Step 20 Since $\psi^k_{(i,\eta)} \le \overline{\psi}^k_{(i,\eta)}$, Step 19 gives a neighborhood V^k_η of $p^*_{-i} \otimes \sigma_{-i}$ in $\mathscr{P}^*_{-i}$ such that

$$\int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \\ < \int_{T_{-i}\times X_{-i}} \overline{\psi}^k_{(i,\eta)}(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) + \frac{\varepsilon}{2}, \quad \text{for all } \nu \in V^k_\eta.$$

By Step 15, there exists k_η such that, for all $k \ge k_\eta$,

$$\int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) < \frac{\varepsilon}{2} + 5\eta, \quad \text{for all } \nu \in V^k_\eta.$$

Now, since V^k_η is open in $\mathscr{P}^*_{-i}$ and the map $h : \mathscr{D}_{-i} \to \mathscr{P}^*_{-i}$ defined in (54) is continuous (Step 17), it follows that $U^k_\eta := h^{-1}(V^k_\eta)$ is open in $\mathscr{D}_{-i}$. Since U^k_η contains $(\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \sigma_N)$, and since, for all $\sigma'_{-i} \in U^k_\eta$, we have $h(\sigma'_{-i}) \in V^k_\eta$ and

$$\int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \left[\bigotimes_{j\ne i} \sigma'_j\right](d(t_{-i}, x_{-i})) \\ = \int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) h(\sigma'_{-i})(d(t_{-i}, x_{-i}))$$

(recall that the map $\psi^k_{(i,\varepsilon)}$ is $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable by Step 8), it follows that, for all $k \ge k_\eta$, (59) holds. □

Step 21 *There exists K such that, for each $k \ge K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that* (49) *holds.*

Proof of Step 21 In light of Step 20, and since $\eta \in (0, \frac{\varepsilon}{12})$, one obtains K such that, for all $k \ge K$, there is a neighborhood $V'_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu^k_i, \sigma'_{-i}) - U_i(\mu_i, \sigma'_{-i}) \\ = \int_{T_{-i}\times X_{-i}} \psi^k_{(i,\eta)}(t_{-i}, x_{-i}) \left[\bigotimes_{j\ne i} \sigma'_j\right](d(t_{-i}, x_{-i})) < \varepsilon, \text{ for all } \sigma'_{-i} \in V'_{\sigma_{-i}}. \quad \square$$

Step 21 establishes item (II) in the statement of Lemma 5.

It remains to prove item (I) in the statement of Lemma 5.

Fix i and ε, and, given η, let $(f^k_{(i,\eta)})^\infty_{k=1}$ be the sequence given on page 1628. It suffices to show that there exists η (which may depend on i and ε) such that, given $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)] \left[\underset{j \neq i}{\otimes} \sigma'_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$> \int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j \neq i}{\otimes} \sigma_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$- \varepsilon, \text{ for all } \sigma'_{-i} \in V_{\sigma_{-i}}, \tag{60}$$

Choose $\eta < \frac{\varepsilon}{2}$. Fix $(t_i, x_i) \in T_i \times X_i$, $\sigma_{-i} \in \mathscr{D}_{-i}$, and k. Define $\zeta_\eta : T_{-i} \times X_{-i} \to \mathbb{R}$ by

$$\zeta_\eta(t_{-i}, x_{-i}) := \sup_{n \in \mathbb{N}} \inf_{y_{-i} \in N_{\frac{1}{n}}(x_{-i})} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), y_{-i}))g(t). \tag{61}$$

Recall that $\mathscr{P}^*_{-i}$ represents the space of all probability measures ν in $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ with

$$\nu(A \times X_{-i}) = p^*_{-i}(A), \quad \text{for all } A \in \mathscr{B}^*(T_{-i}),$$

where p^*_{-i} denotes the complete extension of p_{-i} (which was defined in (44)), and where $\mathscr{B}^*(T_{-i})$ denotes the p_{-i}-completion of $\mathscr{B}(T_{-i})$.

Endow the space $\mathscr{P}^*_{-i}$ with the relative w-topology (Definition 1) on $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$.

Let $p^*_{-i} \otimes \sigma_{-i}$ be the member of $\Delta(T_{-i} \times X_{-i}, \mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}))$ defined in (47).

We proceed in four steps (Step 22–Step 25).

Step 22 *The map ζ_η defined in (61) is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable.*

Proof of Step 22 Because u_i is $(\mathscr{B}(T_i \times X_i) \otimes \mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable, the map

$$(t_{-i}, x_{-i}) \in T_{-i} \times X_{-i} \mapsto u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)$$

is $(\mathscr{B}(T_{-i} \times X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable (see, *e.g.*, Aliprantis and Border (2006, Theorem 4.48)), and hence $(\mathscr{B}(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable and $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. Consequently, since $\mathscr{B}^*(T_{-i})$ coincides with its universal completion, the Theorem in Carbonell-Nicolau (2014a) implies that ζ_η is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable. □

Step 23 *There exists a neighborhood V^*_η of $p^*_{-i} \otimes \sigma_{-i}$ in $\mathscr{P}^*_{-i}$ such that*

$$\int_{T_{-i} \times X_{-i}} \zeta_\eta(t_{-i}, x_{-i}) \nu(d(t_{-i}, x_{-i}))$$
$$> \int_{T_{-i} \times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta, \textit{ for all } \nu \in V^*_\eta.$$

Proof of Step 23 First, note that the map $\zeta_\eta : T_{-i} \times X_{-i} \to \mathbb{R}$ defined in (61) is $(\mathscr{B}^*(T_{-i}) \otimes \mathscr{B}(X_{-i}), \mathscr{B}(\mathbb{R}))$-measurable (Step 22) and satisfies the following: the map $x_{-i} \in X_{-i} \mapsto \zeta_\eta(t_{-i}, x_{-i})$ is lower semicontinuous for every $t_{-i} \in T_{-i}$ (see, *e.g.*, Ash (1972, Theorem A6.5)). The rest of the proof is an almost verbatim transcription of the proof of Step 19. □

Step 24 *Let V^*_η be the neighborhood from Step 23. For all $\nu \in V^*_\eta$,*

$$\int_{T_{-i}\times X_{-i}} [u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)]\nu(d(t_{-i}, x_{-i})) > \int_{T_{-i}\times X_{-i}} [u_i(t, x)g(t)][p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \varepsilon.$$

Proof of Step 24 For every $(t_{-i}, x_{-i}) \in T_{-i} \times X_{-i}$, one has

$$u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t) \geq \zeta_\eta(t_{-i}, x_{-i}) \geq [u_i(t, x) - \eta]g(t).$$

Indeed, these inequalities follow from item (a) (on page 1628) and from the definition of $\zeta_\eta : T_{-i} \times X_{-i} \to \mathbb{R}$ in (61). Consequently, for all $\nu \in \mathscr{P}^*_{-i}$,

$$\int_{T_{-i}\times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)\nu(d(t_{-i}, x_{-i})) \geq \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i}))$$

and

$$\int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \geq \int_{T_{-i}\times X_{-i}} [u_i(t, x) - \eta]g(t)\nu(d(t_{-i}, x_{-i})),$$

and so, applying Step 23, one obtains, for every $\nu \in V^*_\eta$,

$$\begin{aligned}
&\int_{T_{-i}\times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)\nu(d(t_{-i}, x_{-i})) \\
&\geq \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})\nu(d(t_{-i}, x_{-i})) \\
&> \int_{T_{-i}\times X_{-i}} \zeta_\eta(t_{-i}, x_{-i})[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta \\
&\geq \int_{T_{-i}\times X_{-i}} [u_i(t, x) - \eta]g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta \\
&= \int_{T_{-i}\times X_{-i}} u_i(t, x)g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \eta - \eta \\
&> \int_{T_{-i}\times X_{-i}} u_i(t, x)g(t)[p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})) - \varepsilon,
\end{aligned}$$

where the last inequality follows from the inequality $\eta < \frac{\varepsilon}{2}$. □

Step 25 *There is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} in $\mathscr{D}_{-i}$ such that (60) holds.*

Proof of Step 25 Recall the definition of the map $h : \mathscr{D}_{-i} \to \mathscr{P}^*_{-i}$ in (54). Because h is continuous (Step 17), and since V^*_η is open in $\mathscr{P}^*_{-i}$, it follows that $V_{\sigma_{-i}} := h^{-1}(V^*_\eta)$ is open in $\mathscr{D}_{-i}$. Since $V_{\sigma_{-i}}$ contains σ_{-i}, and since, for all $\sigma'_{-i} \in V_{\sigma_{-i}}$, one has $h(\sigma'_{-i}) \in V^*_\eta$ and

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)] \left[\underset{j \neq i}{\otimes} \sigma'_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$= \int_{T_{-i} \times X_{-i}} u_i(t, (f^k_{(i,\eta)}(t_i, x_i), x_{-i}))g(t)h(\sigma'_{-i})(d(t_{-i}, x_{-i}))$$

and

$$\int_{T_{-i}} \int_{X_{-i}} [u_i(t, x)g(t)] \left[\underset{j \neq i}{\otimes} \sigma_j(\cdot|t_j) \right] (dx_{-i}) \left[\underset{j \neq i}{\otimes} p_j \right] (dt_{-i})$$
$$= \int_{T_{-i} \times X_{-i}} [u_i(t, x)g(t)][p^*_{-i} \otimes \sigma_{-i}](d(t_{-i}, x_{-i})),$$

it follows that (60) holds. □

Step 25 gives item (I) in the statement of Lemma 5 and completes the proof of Lemma 5. □

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