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On the existence of pure-strategy perfect equilibrium in discontinuous games ☆

Oriol Carbonell-Nicolau *

Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick, NJ 08901, USA

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ABSTRACT

We provide sufficient conditions for a (possibly) discontinuous normal-form game to possess a pure-strategy trembling-hand perfect equilibrium. We first show that compactness, continuity, and quasiconcavity of a game are too weak to warrant the existence of a pure-strategy perfect equilibrium. We then identify two classes of games for which the existence of a pure-strategy perfect equilibrium can be established: (1) the class of compact, metric, concave games satisfying upper semicontinuity of the sum of payoffs and a strengthening of payoff security; and (2) the class of compact, metric games satisfying upper semicontinuity of the sum of payoffs, strengthenings of payoff security and quasiconcavity, and a notion of local concavity and boundedness of payoff differences on certain subdomains of a player's payoff function. Various economic games illustrate our results.
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1. Introduction

The notion of perfect equilibrium (Selten, 1975) adds a sense of robustness to the Nash equilibrium concept by requiring that Nash equilibria be immune to slight trembles of the players' actions. Nash equilibria survive perfectness if they are good approximations of equilibrium behavior in some perturbed game in which the players make slight mistakes in the execution of their strategies.

Selten (1975) defines his equilibrium concept for games with finite strategy spaces. Simon and Stinchcombe (1995) extend the notion to continuous normal-form games with (possibly) infinite action spaces, and Carbonell-Nicolau (2010a, 2010b) allows for infinite actions and payoff discontinuities. While Simon and Stinchcombe (1995) and Carbonell-Nicolau (2010a) provide existence results for mixed-strategy perfect equilibria, the existence of pure-strategy perfect equilibrium in infinite normal-form games remains an open question.

A classic result in the literature on the existence of Nash equilibrium states that compact, continuous, and quasiconcave normal-form games possess a pure-strategy Nash equilibrium (cf. Fan, 1952; Glicksberg, 1952; Berge, 1957, §23, p. 72; Debreu, 1952, and Friedman, 1977, p. 160). We first show that this result is tight in the sense that continuity and quasiconcavity are not strong enough to warrant the existence of a pure-strategy perfect equilibrium. Example 1 illustrates this point. In light of Example 1, we seek suitable strengthenings of continuity or quasiconcavity that ensure the existence of a pure-strategy perfect equilibrium.

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* Fax: +1 732 932 7416.
E-mail address: carbonell@econ.rutgers.edu.

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As pointed out in Carbonell-Nicolau (2010a), the existence of (mixed-strategy) perfect equilibria crucially relies on the existence of Nash equilibria in slight Selten perturbations of the original game (i.e., perturbations in which every player is forced to choose any one of her actions with (small) positive probability). In compact, metric games, the existence of pure-strategy Nash equilibria in Selten perturbations leads to the existence of pure-strategy perfect equilibria (Proposition 1). It is therefore natural to ask if the machinery developed within the literature on the existence of a pure-strategy Nash equilibrium can be used to establish the existence of pure-strategy Nash equilibria in Selten perturbations.

Compact, quasiconcave, and better-reply secure games possess a pure-strategy Nash equilibrium (Reny, 1999).[1] While standard strengthenings of better-reply security do not generally give better-reply security (or some of its weak forms) in Selten perturbations (cf. Carbonell-Nicolau, 2010a), certain strengthening of Reny's (1999) payoff security—a condition termed Condition (A) in Carbonell-Nicolau (2010a)—along with upper semicontinuity of the sum of payoffs, does imply better-reply security, defined over pure strategies, of some Selten perturbations of a game.

We show that, unlike quasiconcavity of a game, concavity of a game ensures quasiconcavity of the game's Selten perturbations (Lemma 2). This fact can be combined with the previous observations to derive our first result: a compact, metric, and concave game satisfying upper semicontinuity of the sum of payoffs and Condition (A) possesses a pure-strategy perfect equilibrium (Theorem 3).

A variant of this result, which relies on a strengthening of Condition (A) in terms of two independent conditions—generic entire payoff security and generic local equi-upper semicontinuity—is also stated (Corollary 1). This result provides an alternative hypothesis in terms of two independent conditions that are met by several economic games. Furthermore, for concave games whose action spaces have a nonempty interior, generic local equi-upper semicontinuity can be dropped (Corollary 2).

These results are illustrated in the context of Bertrand competition (Example 4), rank-order tournaments (Example 5), rent-seeking games (Example 6), and probabilistic voting models (Example 7). These examples appear in Section 4. While concave games need not have continuous payoffs, the economic games of Section 4 happen to be continuous, and for such games Condition (A) is trivially met.

Our second result handles nonconcave games. We first argue that in order to relax concavity in the statement of our first result, one must strengthen other aspects of the result's hypothesis. We then introduce a condition requiring local concavity and boundedness of payoff differences on certain neighborhoods of a player's domain of actions—Condition (LC). Condition (LC), which is strong enough to ensure that strictly quasiconcave games have quasiconcave Selten perturbations, cannot be combined with quasiconcavity to replace concavity in the statement of our first result. An additional condition is needed. We use a strengthening of quasiconcavity, called strong quasiconcavity, and implied by strict quasiconcavity. Together with Condition (LC), strong quasiconcavity gives a pure-strategy perfect equilibrium in compact, metric games satisfying Condition (A) and upper semicontinuity of the sum of payoffs (Theorem 4). An analogue of this result, in terms of generic entire payoff security and generic local equi-upper semicontinuity is also stated (Corollary 3). Two nonconcave games—a timing game (Example 8) and a Cournot game with discontinuous costs (Example 9)—illustrate these results.

2. Preliminaries

A ***metric game*** is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, each X_i is a nonempty metric space, and each $u_i : X \to \mathbb{R}$ is bounded and Borel measurable, with $X := \bigtimes_{i=1}^N X_i$. If in addition each X_i is compact, G is called a ***compact metric game***.

The symbol X_{-i} designates the set $\bigtimes_{j\neq i} X_j$, and, given i, $x_i \in X_i$, and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_{-i}$, we slightly abuse notation and represent the point $(x_1, \ldots, x_N)$ as (x_i, x_{-i}).

If each X_i is a convex subset of a vector space and, for each i and every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasiconcave on X_i, we say that G is ***quasiconcave***. The game G is ***strictly quasiconcave*** if each X_i is a convex subset of a vector space and $u_i(\cdot, x_{-i})$ is strictly quasiconcave on X_i for each i and every $x_{-i} \in X_{-i}$, i.e., if each X_i is a convex subset of a vector space, and for each i and every $x_{-i} \in X_{-i}$, and for any $\{x_i, y_i\} \subseteq X_i$, we have

$$u_i\big(\alpha x_i + (1-\alpha) y_i, x_{-i}\big) > \min\big\{u_i(x_i, x_{-i}), u_i(y_i, x_{-i})\big\}, \quad \text{for all } \alpha \in (0, 1).$$

Similarly, the game is ***concave*** if each X_i is a convex subset of a vector space and $u_i(\cdot, x_{-i})$ is concave on X_i for each i and every $x_{-i} \in X_{-i}$.

The ***mixed extension*** of G is the game $\overline{G} = (M_i, U_i)_{i=1}^N$, where each M_i stands for the set of Borel probability measures on X_i, endowed with the weak* topology, and $U_i : M \to \mathbb{R}$ is defined by

$$U_i(\mu) := \int_X u_i \,\mathrm{d}\mu,$$

where $M := \bigtimes_{i=1}^N M_i$.

[1] Barelli and Soza (2010), Carmona (2010), and McLennan et al. (2009) provide existence results in terms of weaker forms of better-reply security.

Given $x \in X$ (respectively, $x_i \in X_i$), let δ_x (respectively, δ_{x_i}) be the Dirac measure on X (respectively, X_i) with support $\{x\}$ (respectively, $\{x_i\}$). We sometimes write, by a slight abuse of notation, x (respectively, x_i) in place of δ_x (respectively, δ_{x_i}).

The set M_{-i} denotes the Cartesian product $\bigtimes_{j \neq i} M_j$, and given i, $\mu_i \in M_i$, and

$$\mu_{-i} = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N) \in M_{-i},$$

we sometimes represent the point $(\mu_1, \ldots, \mu_N)$ as (μ_i, μ_{-i}).

For $\emptyset \neq I = \{i_1, \ldots, i_M\} \subseteq \{1, \ldots, N\}$, $\emptyset \neq \{1, \ldots, N\} \setminus I = \{i_{M+1}, \ldots, i_N\}$, $\mu_I = (\mu_{i_1}, \ldots, \mu_{i_M}) \in \bigtimes_{j \in I} M_j$, and $\mu_{-I} = (\mu_{i_{M+1}}, \ldots, \mu_{i_N}) \in \bigtimes_{j \in \{1,\ldots,N\} \setminus I} M_j$, and given a point $(x_1, \ldots, x_N) \in X$, represented as (x_i, x_{-i}), for some $i \in \{1, \ldots, N\}$, according to the convention introduced above, if $\mu_j = \delta_{x_j}$ for every $j \in I$, we write $((x_i, x_{-i})_I, \mu_{-I})$ for the point $(\mu_1, \ldots, \mu_N)$.

For $I = \emptyset$ and $\mu = (\mu_1, \ldots, \mu_N) \in M$, we adopt the convention of writing $((x_i, x_{-i})_I, \mu_{-I})$ for μ. Similarly, for $I = \{1, \ldots, N\}$, if $\mu_j = \delta_{x_j}$ for every $j \in I$, μ is denoted as $((x_i, x_{-i})_I, \mu_{-I})$.

For $I \subseteq \{1, \ldots, N\}$, $\#I$ represents the cardinality of I, and by convention we set $\#I := 0$ if and only if $I = \emptyset$.

Given $A, B \subseteq \mathbb{R} \ni \varepsilon$, we write

$$A > \varepsilon \quad \text{and} \quad A > B - \varepsilon$$

for $a > \varepsilon$, for all $a \in A$, and $a > b - \varepsilon$, for all $(a, b) \in A \times B$, respectively. The definitions of $A \geqslant \varepsilon$ and $A \geqslant B - \varepsilon$ are analogous.

For A a subset of a vector space, the *convex hull* of A is denoted as $\operatorname{co} A$.

A probability measure $\mu_i \in M_i$ is said to be ***strictly positive*** if $\mu_i(O) > 0$ for every nonempty open set O in X_i.

For each i, let $\widehat{M}_i$ denote the set of all strictly positive members of M_i. Set $\widehat{M} := \bigtimes_{i=1}^N \widehat{M}_i$. For $\mu_i \in \widehat{M}_i$ and $\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N$, define

$$M_i(\delta_i \mu_i) := \{\nu_i \in M_i \colon \nu_i \geqslant \delta_i \mu_i\}$$

and $M(\delta_1 \mu_1, \ldots, \delta_N \mu_N) := \bigtimes_{i=1}^N M_i(\delta_i \mu_i)$. Throughout the sequel, we write $M(\delta\mu)$ for $M(\delta_1 \mu_1, \ldots, \delta_N \mu_N)$. Given $\delta = (\delta_1, \ldots, \delta_N) \in [0, 1)^N$ and $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{M}$, the game

$$\overline{G}_{\delta\mu} = \bigl(M_i(\delta_i \mu_i), U_i|_{M(\delta\mu)}\bigr)_{i=1}^N$$

is called a ***Selten perturbation*** of G. We often work with perturbations $\overline{G}_{\delta\mu}$ satisfying $\delta_1 = \cdots = \delta_N$. When referring to these objects, we simply write $\overline{G}_{\delta\mu}$ with $\delta = \delta_1 = \cdots = \delta_N$.

For $\delta \in [0, 1]$ and $(\mu_i, \nu_i) \in M_i^2$,

$$(1 - \delta)\nu_i + \delta\mu_i$$

denotes the member σ_i of M_i for which $\sigma_i(B) = (1 - \delta)\nu_i(B) + \delta\mu_i(B)$ for every Borel set B. When $\nu_i = \delta_{x_i}$ for some $x_i \in X_i$, we slightly abuse notation and write $(1 - \delta)x_i + \delta\mu_i$ for $(1 - \delta)\nu_i + \delta\mu_i$. Similarly, given $(\nu, \mu) = ((\nu_1, \ldots, \nu_N), (\mu_1, \ldots, \mu_N)) \in M^2$,

$$(1 - \delta)\nu + \delta\mu$$

denotes the point

$$\bigl((1 - \delta)\nu_1 + \delta\mu_1, \ldots, (1 - \delta)\nu_N + \delta\mu_N\bigr).$$

Given i, if $\nu_j = \delta_{x_j}$ for all $j \in \{1, \ldots, N\} \setminus \{i\}$, $(1 - \delta)\nu + \delta\mu$ is sometimes represented as

$$\bigl((1 - \delta)\nu_i + \delta\mu_i, (1 - \delta)x_{-i} + \delta\mu_{-i}\bigr).$$

It is convenient to introduce the following variant of $\overline{G}_{\delta\mu}$. Given $(\delta, \mu) \in [0, 1) \times \widehat{M}$, let $G_{(\delta,\mu)}$ be defined as

$$G_{(\delta,\mu)} = \bigl(X_i, u_i^{(\delta,\mu)}\bigr)_{i=1}^N,$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := U_i\bigl((1 - \delta)x_1 + \delta\mu_1, \ldots, (1 - \delta)x_N + \delta\mu_N\bigr).$$

Observe that if $\delta = 0$ we have $G_{(\delta,\mu)} = G$.

Each $G_{(\delta,\mu)}$ is the result of perturbing the payoffs of G in a certain way. The game $G_{(\delta,\mu)}$ can be interpreted as a perturbed version of G in which each pure strategy x_i of i in G is replaced by the mixed strategy $(1 - \delta)x_i + \delta\mu_i$. Note the similarities between $G_{(\delta,\mu)}$ and $\overline{G}_{\delta\mu}$: the latter game corresponds to a perturbation of the strategy spaces of $\overline{G}$, and the mixed extension of $G_{(\delta,\mu)}$ is equivalent to $\overline{G}_{\delta\mu}$.[2]

The *graph* of G is the set

$$\Gamma_G := \bigl\{(x, u) \in X \times \mathbb{R}^N \colon u_i(x) = u_i, \text{ for all } i\bigr\}.$$

[2] For instance, there exists a homeomorphism between the Nash equilibrium sets of the two games.

The graph of the mixed extension, $\Gamma_{\overline{G}}$, is defined analogously. The closures of Γ_G and $\Gamma_{\overline{G}}$ are denoted $\overline{\Gamma}_G$ and $\overline{\Gamma}_{\overline{G}}$ respectively.

Definition 1. A strategy profile $\mu = (\mu_1, \ldots, \mu_N) \in M$ is a ***Nash equilibrium*** of G if for each i, $U_i(\mu) \geqslant U_i(\nu_i, \mu_{-i})$ for every $\nu_i \in M_i$.

A strategy profile $x \in X$ is a ***pure-strategy Nash equilibrium*** of G if δ_x is a Nash equilibrium of G.

Definition 2. A strategy profile $\mu \in M$ is a ***trembling-hand perfect* (*thp*) *equilibrium*** of G if there are sequences (δ^n), (ν^n), and (μ^n) such that $(0,1)^N \ni \delta^n \to 0$, $\nu^n \in \widehat{M}$, $\mu^n \to \mu$, and each μ^n is a pure-strategy Nash equilibrium of the perturbed game $\overline{G}_{\delta^n \nu^n}$.

In words, μ is a thp equilibrium of G if it is the limit of some sequence of exact equilibria of neighboring Selten perturbations of G. If Selten perturbations of G are viewed as "models of mistakes" in which any player may "tremble" in the execution of her strategy, the requirement that μ be the limit of some sequence of equilibria of perturbations of G says that there exists at least one model of (low-probability) mistakes that has at least one equilibrium close to μ, so that each μ_i reflects approximate behavior (at the said equilibrium) were the players to interact in the perturbed game.

Remark 1. Note that Definition 2 does not require that μ be a Nash equilibrium of G. For continuous games, the fact that a strategy profile μ is the limit of some sequence of equilibria of Selten perturbations of G guarantees that μ is a Nash equilibrium of G. While we do not impose continuity of a game's payoff functions, our conditions also ensure that the limit point is an equilibrium. See Proposition 1, Theorem 3, and Theorem 4.

Definition 3 (below) adapts Simon and Stinchcombe's (1995) notion of *strong perfectness* to potential discontinuities in the payoff functions of a game.

For $\mu \in M$, let $Br_i(\mu)$ denote player i's set of best responses in M_i to the vector of strategies μ:

$$Br_i(\mu) := \left\{\sigma_i \in M_i\colon\ U_i(\sigma_i, \mu_{-i}) \geqslant \sup_{\varrho_i \in M_i} U_i(\varrho_i, \mu_{-i})\right\}.$$

Consider the following distance function between members of M_i:

$$\rho_i^s(\mu, \nu) := \sup_B \left|\mu(B) - \nu(B)\right|.$$

Definition 3. (See Simon and Stinchcombe, 1995.) A ***strong ϵ-perfect equilibrium*** of G is a vector $\mu^\epsilon \in \widehat{M}$ such that for each i,

$$\rho_i^s\left(\mu_i^\epsilon, Br_i\left(\mu^\epsilon\right)\right) < \epsilon.$$

A strategy profile in G is a ***strong perfect equilibrium*** of G if it is the weak* limit as $\epsilon^n \to 0$ of strong ϵ^n-perfect equilibria.

Carbonell-Nicolau (2010b) proves the following analogue of the standard three-way characterization of perfectness (e.g., van Damme, 2002, p. 28).

Theorem 1. *For a metric game, the following three conditions are equivalent*:

1. *μ is a trembling-hand perfect equilibrium of G.*
2. *μ is a strong perfect equilibrium of G.*
3. *μ is the limit of a sequence (μ^n) in $\widehat{M}$ with the property that for each i and every $\epsilon > 0$,*

$$\mu_i^n\left(\left\{x_i \in X_i\colon\ U_i\left(x_i, \mu_{-i}^n\right) \geqslant \sup_{y_i \in X_i} U_i\left(y_i, \mu_{-i}^n\right)\right\}\right) \geqslant 1 - \epsilon,$$

for any sufficiently large n.

3. Pure-strategy perfect equilibrium

In this paper we shall be concerned with strategy profiles in which each player's action is pure.

Definition 4. A strategy profile $x \in X$ is a ***pure-strategy trembling-hand perfect* (*thp*) *equilibrium*** of G if δ_x is a trembling-hand perfect equilibrium of G.

We say that a game $G = (X_i, u_i)_{i=1}^N$ is *continuous* if each u_i is continuous. It is well known that compact, continuous, and quasiconcave games possess a pure-strategy Nash equilibrium (cf. Fan, 1952; Glicksberg, 1952; Berge, 1957, §23, p. 72; Debreu, 1952, and Friedman, 1977, p. 160).[3] For metric games, the formal statement is as follows:

Theorem 2. *Suppose that G is compact, metric, continuous, and quasiconcave. Then G possesses a pure-strategy Nash equilibrium.*

In light of this result, it is only natural to ask if continuity and quasiconcavity are strong enough to warrant the existence of pure-strategy thp equilibria. Example 1 presents a compact, continuous, and quasiconcave game that has no pure-strategy thp equilibrium.

Example 1. Consider the two-player game $G = ([0,1], [0,1], u_1, u_2)$, where

$$u_1(x_1, x_2) := \begin{cases} 0 & \text{if } x_2 = 0 \text{ and } x_1 \in [0, \frac{7}{8}), \\ -\frac{7}{4} + 2x_1 & \text{if } x_2 = 0 \text{ and } x_1 \in [\frac{7}{8}, 1], \\ 0 & \text{if } x_2 = \frac{1}{2}, \\ \alpha u_1(x_1, 0) + (1-\alpha) u_1(x_1, \frac{1}{2}) & \text{if } \alpha \in (0,1) \text{ and } x_2 = (1-\alpha)(\frac{1}{2}), \\ x_1 & \text{if } x_1 \in [0, \frac{1}{2}] \text{ and } x_2 = 1, \\ -4x_1 + \frac{5}{2} & \text{if } x_1 \in (\frac{1}{2}, \frac{5}{8}] \text{ and } x_2 = 1, \\ 0 & \text{if } x_1 \in (\frac{5}{8}, 1] \text{ and } x_2 = 1, \\ \alpha u_1(x_1, \frac{1}{2}) + (1-\alpha) u_1(x_1, 1) & \text{if } \alpha \in (0,1) \text{ and } x_2 = \alpha(\frac{1}{2}) + 1 - \alpha, \end{cases}$$

and

$$u_2(x_1, x_2) := \begin{cases} -|x_2 - \frac{1}{2}| & \text{if } x_1 = 0, \\ 0 & \text{if } x_1 = \frac{1}{8}, \\ \alpha u_2(0, x_2) + (1-\alpha) u_2(\frac{1}{8}, x_2) & \text{if } \alpha \in (0,1) \text{ and } x_1 = (1-\alpha)(\frac{1}{8}), \\ (x_2 - 1)^2 & \text{if } x_1 = \frac{1}{4}, \\ \alpha u_2(\frac{1}{8}, x_2) + (1-\alpha) u_2(\frac{1}{4}, x_2) & \text{if } \alpha \in (0,1) \text{ and } x_1 = \alpha(\frac{1}{8}) + (1-\alpha)(\frac{1}{4}), \\ 0 & \text{if } x_1 = \frac{3}{4}, \\ \alpha u_2(\frac{1}{4}, x_2) + (1-\alpha) u_2(\frac{3}{4}, x_2) & \text{if } \alpha \in (0,1) \text{ and } x_1 = \alpha(\frac{1}{4}) + (1-\alpha)(\frac{3}{4}), \\ x_2^2 & \text{if } x_1 = 1, \\ \alpha u_2(\frac{3}{4}, x_2) + (1-\alpha) u_2(1, x_2) & \text{if } \alpha \in (0,1) \text{ and } x_1 = \alpha(\frac{3}{4}) + 1 - \alpha. \end{cases}$$

It is routine to verify that G is compact, continuous, and quasiconcave.

We show that G has no pure-strategy thp equilibrium. The set of pure-strategy Nash equilibria is

$$\left\{\left(x_1, \tfrac{1}{2}\right) : x_1 \in \left[0, \tfrac{1}{8}\right]\right\} \cup \left\{\left(\tfrac{3}{4}, \tfrac{1}{2}\right)\right\}.$$

To see this, note that if $x_2 \in [0, \frac{1}{2})$, the map $u_1(\cdot, x_2)$ attains its maximum at $x_1 = 1$, and yet $u_2(1, \cdot)$ is maximized at $x_2 = 1$, so in equilibrium player 2 cannot choose a member of $[0, \frac{1}{2})$. Similarly, if $x_2 \in (\frac{1}{2}, 1]$, $u_1(\cdot, x_2)$ attains its maximum at $x_1 = \frac{1}{2}$, while $u_2(\frac{1}{2}, \cdot)$ has a unique maximizer at $x_2 = 0$, so there is no equilibrium in which player 2 chooses an action in $(\frac{1}{2}, 1]$. Therefore, (x_1^*, x_2^*) is a pure-strategy Nash equilibrium only if $x_2^* = \frac{1}{2}$. Now, given that any $x_1 \in [0,1]$ is a best response to $\frac{1}{2}$ for player 1, x_1^* can be any point x_1 of $[0,1]$ such that $\frac{1}{2}$ is a best response to x_1 for player 2. It is easy to see that only the members of the set $[0, \frac{1}{8}] \cup \{\frac{3}{4}\}$ have this property.

We now show that none of the elements of $\{(x_1, \frac{1}{2}) : x_1 \in [0, \frac{1}{8}]\}$ are trembling-hand perfect. In fact, we have, by construction of u_1,

$$u_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 \in [0, \frac{1}{2}], \\ (2x_2 - 1)x_1 & \text{if } x_2 \in (\frac{1}{2}, 1], \end{cases}$$

[3] Glicksberg (1952) assumes that a player's action space is a compact subset of a locally convex Hausdorff topological vector space. Fan's (1952) and Berge's (1957) action spaces are compact convex subsets of a locally convex topological vector space. Debreu (1952) assumes contractibility of the best-reply correspondence of each player instead of quasiconcavity of G.

for all $x_1 \in [0, \frac{1}{2}]$. Clearly, $u_1(\frac{1}{2}, x_2) \geqslant u_1(x_1, x_2)$, for all $(x_1, x_2) \in [0, \frac{1}{8}] \times [0, 1]$, and there exists $\alpha \in \mathbb{R}$ such that $u_1(\frac{1}{2}, x_2) > \alpha \geqslant u_1(x_1, x_2)$, for all $(x_1, x_2) \in [0, \frac{1}{8}] \times (\frac{1}{2}, 1]$. Therefore, every strategy in $[0, \frac{1}{8}]$ is (weakly) dominated by $\frac{1}{2}$, and $\frac{1}{2}$ gives player 1 a higher payoff than any point in $[0, \frac{1}{8}]$ against any x_2 in $(\frac{1}{2}, 1]$. This implies that player 1 cannot select any element of $[0, \frac{1}{8}]$ at a thp equilibrium.

To see that $(\frac{3}{4}, \frac{1}{2})$ is not trembling-hand perfect, let $\mu_2 \in \widehat{M}_2$ be a tremble for player 2, and take $\delta \in (0, 1)$. Let $N_\epsilon(\frac{3}{4})$ be the ϵ-neighborhood of $\frac{3}{4}$. For some sufficiently small $\epsilon > 0$ (in fact, for any $0 < \epsilon \leqslant \frac{1}{8}$), we have, for every $x_1 \in N_\epsilon(\frac{3}{4})$,

$$\begin{aligned} U_1\big(\tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}1, (1-\delta)\tfrac{1}{2} + \delta\mu_2\big) &= \tfrac{1}{2}(1-\delta)U_1\big(\tfrac{1}{2}, \tfrac{1}{2}\big) + \tfrac{1}{2}\delta U_1\big(\tfrac{1}{2}, \mu_2\big) + \tfrac{1}{2}(1-\delta)U_1\big(1, \tfrac{1}{2}\big) + \tfrac{1}{2}\delta U_1(1, \mu_2) \\ &= \tfrac{1}{2}\delta U_1\big(\tfrac{1}{2}, \mu_2\big) + \tfrac{1}{2}\delta U_1(1, \mu_2) \\ &> 0 \\ &= \delta U_1(x_1, \mu_2) \\ &= U_1\big(x_1, (1-\delta)\tfrac{1}{2} + \delta\mu_2\big). \end{aligned}$$

It follows that no $x_1 \in N_\epsilon(\frac{3}{4})$ can be assigned positive probability at any Nash equilibrium of any Selten perturbation, which implies that there cannot exist a sequence (σ^n) such that $\sigma_1^n \to \frac{3}{4}$ and each σ^n is a Nash equilibrium of a Selten perturbation.

In light of Example 1, in this paper we seek suitable strengthenings of the hypothesis of Theorem 2 that guarantee the existence of a pure-strategy thp equilibrium.

The following definition appears in Reny (1999).

Definition 5. The game G is ***better-reply secure*** if, for every $(x, u) \in \overline{\Gamma}_G$ such that x is not a Nash equilibrium of G, there exist i, $y_i \in X_i$, a neighborhood $O_{x_{-i}}$ of x_{-i}, and $\alpha \in \mathbb{R}$ such that $u_i(y_i, O_{x_{-i}}) \geqslant \alpha > u_i$.

Our first result imposes quasiconcavity and better-reply security on slight perturbations of G of the form $G_{(\delta,\mu)}$ and assumes that $\overline{G}$ is better-reply secure.[4] This allows us to establish the existence of pure-strategy thp equilibria in G.

Proposition 1. *If G is compact and metric, $\overline{G}$ is better-reply secure, and there exists $(\alpha, \mu) \in (0, 1) \times \widehat{M}$ such that $G_{(\delta,\mu)}$ is quasiconcave and better-reply secure for every $0 \leqslant \delta \leqslant \alpha$, then G possesses a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*[5]

Proof. Suppose that G is compact and metric, and let $(\alpha, \mu) \in (0, 1) \times \widehat{M}$ be such that $G_{(\delta,\mu)}$ is quasiconcave and better-reply secure for every $0 \leqslant \delta \leqslant \alpha$. By Theorem 3.1 of Reny (1999), each $G_{(\frac{1}{n},\mu)}$ (n large enough) possesses a pure-strategy Nash equilibrium $x^n = (x_1^n, \dots, x_N^n)$. Because $x^n \in X$ and X is sequentially compact, we may write (passing to a subsequence if necessary) $x^n \to x$ for some $x \in X$. Therefore, we have

$$\sigma^n := \big(1 - \tfrac{1}{n}\big)x^n + \tfrac{1}{n}\mu \to x. \tag{1}$$

Now consider the sequence of perturbed games $(\overline{G}_{\frac{1}{n}\mu})$. Each σ^n is a Nash equilibrium of $\overline{G}_{\frac{1}{n}\mu}$. In fact, because x^n is a Nash equilibrium of $G_{(\frac{1}{n},\mu)}$, we have, for each i,

$$u_i^{(\frac{1}{n},\mu)}\big(x^n\big) \geqslant u_i^{(\frac{1}{n},\mu)}\big(x_i, x_{-i}^n\big), \quad \text{for all } x_i \in X_i.$$

Hence, for each i,

$$u_i^{(\frac{1}{n},\mu)}\big(x^n\big) \geqslant \int_{X_i} u_i^{(\frac{1}{n},\mu)}\big(\cdot, x_{-i}^n\big)\,\mathrm{d}\nu_i, \quad \text{for all } \nu_i \in M_i,$$

which can be written as

$$U_i\big(\big(1 - \tfrac{1}{n}\big)x^n + \tfrac{1}{n}\mu\big) \geqslant U_i\big(\big(1 - \tfrac{1}{n}\big)\nu_i + \tfrac{1}{n}\mu_i, \big(1 - \tfrac{1}{n}\big)x_{-i}^n + \tfrac{1}{n}\mu_{-i}\big),$$

[4] As pointed out by Reny (1999), better-reply security of a game neither implies nor is implied by better-reply security of the game's mixed extension. Consequently, better-reply security of $G_{(\delta,\mu)}$ neither implies nor is implied by better-reply security of $\overline{G}_{\delta\mu}$.

[5] It is worth noting that the hypothesis of Proposition 1 can be weakened. In fact, it suffices that $G_{(\delta,\mu)}$ be quasiconcave and that $G_{(\delta,\mu)}$ and $\overline{G}$ satisfy Barelli and Soza's (2010) generalized better-reply security or Carmona's (2010) weak better-reply security.

for all $\nu_i \in M_i$. Consequently, since for given $p_i \in M_i(\frac{1}{n}\mu_i)$ there exists $\nu_i \in M_i$ such that $p_i = (1-\frac{1}{n})\nu_i + \frac{1}{n}\mu_i$, we have

$$U_i\big((1-\tfrac{1}{n})x^n + \tfrac{1}{n}\mu\big) \geqslant U_i\big(p_i, (1-\tfrac{1}{n})x^n_{-i} + \tfrac{1}{n}\mu_{-i}\big), \quad \text{for all } p_i \in M_i\big(\tfrac{1}{n}\mu_i\big),$$

so σ^n is a Nash equilibrium of $\overline{G}_{\frac{1}{n}\mu}$. Hence, given (1), x is a pure-strategy thp equilibrium of G.

It remains to show that all thp equilibria of G are Nash. Suppose that ϱ is a thp equilibrium of G, and let (ϱ^n) be the corresponding sequence of equilibria in Selten perturbations, i.e., each ϱ^n is a pure-strategy Nash equilibrium of $\overline{G}_{\delta^n\mu^n}$, where $\delta^n \to 0$ and $\mu^n \in \widehat{M}$. We wish to show that ϱ is a Nash equilibrium of G. To this end, we assume that ϱ is not an equilibrium and derive a contradiction.

Because $\varrho^n \to \varrho$ and each u_i is bounded, we may write (passing to a subsequence if necessary)

$$\big(\varrho^n, \big(U_1(\varrho^n), \dots, U_N(\varrho^n)\big)\big) \to \big(\varrho, (\alpha_1, \dots, \alpha_N)\big) \tag{2}$$

for some $\alpha := (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$. Consequently, $(\varrho, \alpha) \in \overline{\Gamma}_{\overline{G}}$, and hence, if ϱ is not a Nash equilibrium of G, then (since $\overline{G}$ is better-reply secure) some player i can secure a payoff strictly above α_i at ϱ. That is, for some $\sigma_i \in M_i$, some neighborhood $O_{\varrho_{-i}}$ of ϱ_{-i}, and some $\gamma > 0$,

$$U_i(\sigma_i, \sigma_{-i}) \geqslant \alpha_i + \gamma, \quad \text{for all } \sigma_{-i} \in O_{\varrho_{-i}}.$$

We therefore have, in view of (2),

$$U_i\big(\sigma_i, \varrho^n_{-i}\big) > U_i\big(\varrho^n\big) + \beta,$$

for any sufficiently large n and for some $\beta > 0$. Consequently, for large enough n,

$$U_i\big((1-\delta^n_i)\sigma_i + \delta^n_i\mu_i, \varrho^n_{-i}\big) > U_i\big(\varrho^n\big),$$

thereby contradicting that ϱ^n is a (pure-strategy) Nash equilibrium in $\overline{G}_{\delta^n\mu^n}$. □

Proposition 1 has limited applicability. In fact, verifying the statement's hypothesis entails dealing with expected payoffs (defined over mixed strategies) and the weak* convergence of measures. In this regard, a functional result needs to rely on conditions defined directly on the payoffs of the original game. Before introducing conditions on G that lead to the existence of pure-strategy thp equilibria via Proposition 1, we make a few remarks on certain properties of G that are too weak to sustain the hypothesis of Proposition 1.

First, quasiconcavity of G does not generally give quasiconcavity in $G_{(\delta,\mu)}$. For instance, the game G from Example 1 is continuous (so $\overline{G}$ and $G_{(\delta,\mu)}$ are better-reply secure) and quasiconcave. Consequently, since G lacks a thp equilibrium, Proposition 1 implies that given $\mu \in \widehat{M}$ and $\alpha \in (0,1)$ there exists $\delta \in [0,\alpha]$ such that $G_{(\delta,\mu)}$ fails quasiconcavity.

Second, as we have already mentioned (in footnote 4), better-reply security applied to the payoffs of the original game need not give better-reply security in the mixed extension.

Finally, we argue that standard strengthenings of better-reply security—payoff security (Reny, 1999) or uniform payoff security (Monteiro and Page, 2007)—along with upper semicontinuity of the sum of payoffs, need not imply the hypothesis of Proposition 1.

Definition 6. The game G is ***payoff secure*** if for each $\varepsilon > 0$, $x \in X$, and i, there exists $y_i \in X_i$ such that $u_i(y_i, O_{x_{-i}}) > u_i(x) - \varepsilon$ for some neighborhood $O_{x_{-i}}$ of x_{-i}.

Definition 7. Given $Y_i \subseteq X_i$ for each i, the game G is ***uniformly payoff secure over*** $\bigtimes_i Y_i$ if for each i, $\varepsilon > 0$, and $x_i \in Y_i$, there exists $y_i \in X_i$ such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that $u_i(y_i, O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.

The game G is ***uniformly payoff secure*** if it is uniformly payoff secure over X.

The following example demonstrates that payoff security or uniform payoff security (applied to the payoffs of the original game), along with upper semicontinuity of the sum of payoffs, need not give payoff security or better-reply security in $G_{(\delta,\mu)}$.[6] Consequently, stronger conditions are needed to apply Proposition 1.[7]

Example 2. Let (α^n) be a sequence in $(0,1)$ with $\alpha^n \nearrow 1$. Consider the two-player game $G = ([0,1],[0,1],u_1,u_2)$, where

$$u_1(x_1,x_2) := \begin{cases} 1 & \text{if } x_2 = \alpha^n,\ n = 1,2,\dots, \text{ or if } x_2 = 1, \\ x_1 & \text{elsewhere,} \end{cases}$$

[6] A similar example is given in Example 3 of Carbonell-Nicolau (2010a) to illustrate that payoff security or uniform payoff security, along with upper semicontinuity of the sum of payoffs, need not give payoff security or better-reply security in $\overline{G}_{\delta\mu}$. While Example 3 of Carbonell-Nicolau (2010a) would serve a similar purpose here, Example 2 (which features a concave game) illustrates that, even in concave games, Proposition 1 cannot be used to prove versions of Theorem 3 and Corollaries 1–2 in which generic entire payoff security is weakened to uniform payoff security (see Remark 6).

[7] An alternative is to generalize some aspect of the main theorem of Reny (1999) in such a way that the corresponding generalization of Proposition 1 warrants the obtention of useful, weaker conditions. We do not pursue this argument here.

and $u_2(x_1, x_2) := u_1(x_2, x_1)$.

The sum $\sum_i u_i$ is upper semicontinuous (in fact, each u_i is upper semicontinuous), and G is clearly uniformly payoff secure, for $u_i(1, x_{-i}) = 1$ for all $x_{-i} \in [0,1]$. However, $G_{(\delta,\mu)}$ fails payoff security whenever $\mu \in \widehat{M}$ and $\delta \in (0,1)$. To see this, fix $\mu \in \widehat{M}$ and $\delta \in (0,1)$. We need to show that there exist ε, i, and $(x_1, x_2) \in [0,1]^2$ such that for all $y_i \in [0,1]$ there is a point $y_{-i} \in [0,1]$ arbitrarily close to x_{-i} for which

$$U_i\big((1-\delta)y_i + \delta\mu_i, (1-\delta)y_{-i} + \delta\mu_{-i}\big) \leqslant U_i\big((1-\delta)x_i + \delta\mu_i, (1-\delta)x_{-i} + \delta\mu_{-i}\big) - \varepsilon.$$

Thus, it suffices to establish the following for $\varepsilon > 0$ sufficiently small: there is an n such that for each neighborhood $O_{(1-\delta)\alpha^n + \delta\mu_2}$ of $(1-\delta)\alpha^n + \delta\mu_2$ and every $y_1 \in [0,1]$,

$$U_1\big((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2\big) \leqslant U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big) - \varepsilon,$$

for some $(1-\delta)y_2 + \delta\mu_2 \in O_{(1-\delta)\alpha^n + \delta\mu_2}$.

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) := x$. Choose $\varepsilon > 0$ with the property that for large enough n

$$\delta(1-\delta) \int_{[0,1]} f \, \mathrm{d}\mu_1 \leqslant \delta(1-\delta)\alpha^n - \varepsilon. \tag{3}$$

Take any neighborhood $O_{(1-\delta)\alpha^n + \delta\mu_2}$ of $(1-\delta)\alpha^n + \delta\mu_2$ and any $y_1 \in [0,1]$. Clearly, we may pick some $y_2 \in (\alpha^n, \alpha^{n+1})$ sufficiently close to α^n to ensure that $(1-\delta)y_2 + \delta\mu_2 \in O_{(1-\delta)\alpha^n + \delta\mu_2}$.

By linearity of U_1 we have

$$\begin{aligned}
&U_1\big((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2\big) \\
&\quad = (1-\delta)^2 U_1(y_1, y_2) + (1-\delta)\delta U_1(y_1, \mu_2) + \delta(1-\delta)U_1(\mu_1, y_2) + \delta^2 U_1(\mu),
\end{aligned}$$

and the right-hand side is clearly less than or equal to $(1-\delta)^2 + (1-\delta)\delta + \delta(1-\delta)U_1(\mu_1, y_2) + \delta^2 U_1(\mu)$, so

$$U_1\big((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2\big) \leqslant (1-\delta)^2 + (1-\delta)\delta + \delta(1-\delta)U_1(\mu_1, y_2) + \delta^2 U_1(\mu). \tag{4}$$

On the other hand,

$$\begin{aligned}
&U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big) \\
&\quad = (1-\delta)^2 U_1\big(\alpha^n, \alpha^n\big) + (1-\delta)\delta U_1\big(\alpha^n, \mu_2\big) + \delta(1-\delta)U_1\big(\mu_1, \alpha^n\big) + \delta^2 U_1(\mu) \\
&\quad = (1-\delta)^2 + (1-\delta)\delta\alpha^n + \delta(1-\delta) + \delta^2 U_1(\mu),
\end{aligned}$$

where the first equality uses linearity of U_1 and the second equality is a consequence of the definition of u_1. Combining this equation with (4) gives

$$\begin{aligned}
&U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big) - U_1\big((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2\big) \\
&\quad = (1-\delta)\delta\alpha^n - \delta(1-\delta)U_1(\mu_1, y_2) \\
&\quad = (1-\delta)\delta\alpha^n - \delta(1-\delta) \int_{[0,1]} f \, \mathrm{d}\mu_1 \geqslant \varepsilon,
\end{aligned}$$

where the last inequality follows from (3). Consequently,

$$U_1\big((1-\delta)y_1 + \delta\mu_1, (1-\delta)y_2 + \delta\mu_2\big) \leqslant U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big) - \varepsilon,$$

and so $G_{(\delta,\mu)}$ is not payoff secure.

The perturbation $\overline{G}_{\delta\mu}$ also fails better-reply security. To see this, choose $\mu \in \widehat{M}$ and $\delta \in (0,1)$, and observe that

$$\big(\big(\alpha^n, \alpha^n\big), \big(\gamma_1^n, \gamma_2^n\big)\big),$$

where

$$\gamma_1^n = U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big),$$

and

$$\gamma_2^n = U_2\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big),$$

belongs to $\overline{\Gamma}_{G_{(\delta,\mu)}}$. Moreover, the strategy profile (α^n, α^n) is not a Nash equilibrium in $G_{(\delta,\mu)}$, for

$$U_1\big((1-\delta)1 + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big) > U_1\big((1-\delta)\alpha^n + \delta\mu_1, (1-\delta)\alpha^n + \delta\mu_2\big).$$

Reasoning as in the previous paragraph one can show that for large enough n there is no $(1-\delta)y_1 + \delta\mu_1$ for which $U_1((1-\delta)y_1 + \delta\mu_1, O_{(1-\delta)\alpha^n + \delta\mu_2}) > \gamma_1^n$ for some neighborhood $O_{(1-\delta)\alpha^n + \delta\mu_2}$ of $(1-\delta)\alpha^n + \delta\mu_2$, and similarly for player 2. It follows that $G_{(\delta,\mu)}$ is not better-reply secure.

The following condition appears in Carbonell-Nicolau (2010a).

Condition (A). There exists $(\mu_1, \ldots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i(f(x_i), O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for every $x_i \in Y_i$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f(x_i), z_{-i}) - u_i(x_i, z_{-i}) < \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.[8]

Remark 2. The following implications are immediate:

$$\begin{aligned} \text{continuity} \quad &\Rightarrow \quad \text{(A)} \\ &\Rightarrow \quad \text{uniform payoff security} \\ &\Rightarrow \quad \text{payoff security.} \end{aligned}$$

Lemma 1. *Suppose that a compact, metric game G satisfies Condition* (A)*. Then there exists $\mu \in \widehat{M}$ such that $G_{(\delta,\mu)}$ is payoff secure for every $\delta \in [0, 1)$.*

Remark 3. One can show that each one of items (a) and (b) in the statement of Condition (A) alone does not generally imply payoff security of $G_{(\delta,\mu)}$.

We now give an outline of the proof of Lemma 1. The details are relegated to Section 5.1. The main argument relies on the following intermediate result:

($\star$) Suppose that G is compact, metric, and satisfies Condition (A). Then there exists $(\mu_1, \ldots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a Borel measurable map $f : X_i \to X_i$ satisfying the following:
 ($\star$.1) For each $x_i \in X_i$ and every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} for which $U_i(f(x_i), O_{\sigma_{-i}}) > U_i(x_i, \sigma_{-i}) - \varepsilon$.
 ($\star$.2) For every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) - U_i(\mu_i, p_{-i}) < \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$, where $\mu_i^f \in M_i$ is defined by $\mu_i^f(B) := \mu_i(f^{-1}(B \cap f(X_i)))$.

This is Lemma 7 of Section 5.1. In general, uniform payoff security does not give item ($\star$.1) (in fact, while uniform payoff security implies the existence of a map f satisfying the properties in ($\star$.1), the map need not be Borel measurable) or item ($\star$.2).

Fix $\delta \in [0, 1)$, and let $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{M}$ be the measure given by ($\star$). We need to show that $G_{(\delta,\mu)}$ is payoff secure. Fix $\varepsilon > 0$, $x = (x_1, \ldots, x_N) \in X$, and i. We show that there exists $y_i \in X_i$ such that $u_i^{(\delta,\mu)}(y_i, O_{x_{-i}}) > u_i^{(\delta,\mu)}(x) - \varepsilon$ for some neighborhood $O_{x_{-i}}$ of x_{-i}. For notational convenience, let

$$\sigma = (\sigma_1, \ldots, \sigma_N) := \big((1-\delta)x_1 + \delta\mu_1, \ldots, (1-\delta)x_N + \delta\mu_N\big).$$

From ($\star$) we see that there is a Borel measurable map $f : X_i \to X_i$ satisfying the following:

(i) For every $y_i \in X_i$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} for which $U_i(f(y_i), O_{\sigma_{-i}}) > U_i(y_i, \sigma_{-i}) - \frac{\varepsilon}{4}$.
(ii) There is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) - U_i(\mu_i, p_{-i}) < \frac{\varepsilon}{2}$ for all $p_{-i} \in V_{\sigma_{-i}}$.

Define

$$p_i^f := (1-\delta)f(x_i) + \delta\mu_i \quad \text{and} \quad \upsilon_i^f := (1-\delta)f(x_i) + \delta\mu_i^f.$$

Using (i) one can show the following:

[8] The following generalization of Condition (A) leaves all of our results intact.

Condition (A′). There exists $(\mu_1, \ldots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:
(a) For each k, $x_i \in X_i$, and $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i(f_k(x_i), O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.
(b) For each $y_{-i} \in X_{-i}$, there is a subset Y_i of X_i with $\mu_i(Y_i) = 1$ such that for each $x_i \in Y_i$ and every sufficiently large k, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_k(x_i), z_{-i}) - u_i(x_i, z_{-i}) < \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

(a) There exists a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$\int_{X_i} U_i\big(f(\cdot), O_{\sigma_{-i}}\big)\,\mathrm{d}\sigma_i > \int_{X_i} U_i(\cdot, \sigma_{-i})\,\mathrm{d}\sigma_i - \tfrac{\varepsilon}{2}.$$

(This is Claim 1 in the proof of Lemma 1.)

In light of (a) and (ii), the proof of Lemma 1 can be completed as follows. Fact (ii), together with the definitions of p_i^f and υ_i^f, gives, for any p_{-i} in some neighborhood of σ_{-i},

$$\begin{aligned} U_i\big(p_i^f, p_{-i}\big) &= (1-\delta)U_i\big(f(x_i), p_{-i}\big) + \delta U_i(\mu_i, p_{-i}) \\ &> (1-\delta)U_i\big(f(x_i), p_{-i}\big) + \delta U_i\big(\mu_i^f, p_{-i}\big) - \tfrac{\varepsilon}{2} \\ &= U_i\big(\upsilon_i^f, p_{-i}\big) - \tfrac{\varepsilon}{2}. \end{aligned} \tag{5}$$

In addition, the definitions of σ_i and υ_i^f entail

$$\begin{aligned} U_i\big(\upsilon_i^f, p_{-i}\big) &= \int_{X_i} U_i(\cdot, p_{-i})\,\mathrm{d}\upsilon_i^f \\ &= (1-\delta)U_i\big(f(x_i), p_{-i}\big) + \delta \int_{X_i} U_i(\cdot, p_{-i})\,\mathrm{d}\mu_i^f \\ &= (1-\delta)U_i\big(f(x_i), p_{-i}\big) + \delta \int_{X_i} U_i\big(f_k(\cdot), p_{-i}\big)\,\mathrm{d}\mu_i = \int_{X_i} U_i\big(f(\cdot), p_{-i}\big)\,\mathrm{d}\sigma_i. \end{aligned} \tag{6}$$

Consequently, for every p_{-i} in some neighborhood of σ_{-i}, we have

$$U_i\big(p_i^f, p_{-i}\big) > U_i\big(\upsilon_i^f, p_{-i}\big) - \tfrac{\varepsilon}{2} = \int_{X_i} U_i\big(f(\cdot), p_{-i}\big)\,\mathrm{d}\sigma_i - \tfrac{\varepsilon}{2} > U_i(\sigma_i, \sigma_{-i}) - \varepsilon,$$

implying that $G_{(\delta,\mu)}$ is payoff secure. Here, the first inequality follows from (5), the second inequality is given by (a), and the equality is a consequence of (6).

Lemma 1, together with Proposition 1, implies that Condition (A), along with upper semicontinuity of $\sum_i u_i$, ensures the existence of a pure-strategy thp equilibrium of G, provided that $G_{(\delta,\mu)}$ is quasiconcave. This is shown below. We now ask if there are conditions on the payoff functions of G that give quasiconcavity of $G_{(\delta,\mu)}$. While quasiconcavity of G is too weak to ensure quasiconcavity of $G_{(\delta,\mu)}$ (recall the discussion following the proof of Proposition 1), concavity of G does imply quasiconcavity of $G_{(\delta,\mu)}$.

Lemma 2. *If G is concave, then $G_{(\delta,\mu)}$ is quasiconcave.*

Proof. Fix i, $x_{-i} \in X_{-i}$. Let $\{x_i, y_i\} \subseteq X_i$, and take $z_i = \alpha x_i + (1-\alpha)y_i \in \mathrm{co}\{x_i, y_i\}$. We wish to show that

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant \min_{w_i \in \{x_i, y_i\}} u_i^{(\delta,\mu)}(w_i, x_{-i}).$$

We have

$$\begin{aligned} u_i^{(\delta,\mu)}(z_i, x_{-i}) &= \sum_{k=0}^{N} \delta^k (1-\delta)^{N-k} \sum_{\substack{I\subseteq\{1,\ldots,N\} \\ :\#I=N-k}} U_i\big((z_i, x_{-i})_I, \mu_{-I}\big) \\ &= \sum_{k=0}^{N} \delta^k (1-\delta)^{N-k} \sum_{\substack{I\subseteq\{1,\ldots,N\} \\ :\#I=N-k}} U_i\big(\big(\alpha x_i + (1-\alpha)y_i, x_{-i}\big)_I, \mu_{-I}\big) \\ &\geqslant \sum_{k=0}^{N} \delta^k (1-\delta)^{N-k} \sum_{\substack{I\subseteq\{1,\ldots,N\} \\ :\#I=N-k}} \big(\alpha U_i\big((x_i, x_{-i})_I, \mu_{-I}\big) + (1-\alpha)U_i\big((y_i, x_{-i})_I, \mu_{-I}\big)\big) \\ &= \alpha u_i^{(\delta,\mu)}(x_i, x_{-i}) + (1-\alpha)u_i^{(\delta,\mu)}(y_i, x_{-i}) \\ &\geqslant \min_{w_i \in \{x_i, y_i\}} u_i^{(\delta,\mu)}(w_i, x_{-i}), \end{aligned}$$

where the first inequality follows from concavity of G. $\square$

Lemma 3. *Suppose that a compact, metric game G satisfies Condition* (A). *Suppose further that $\sum_i u_i$ is upper semicontinuous. Then $\overline{G}$ is better-reply secure, and there exists $\mu \in \widehat{M}$ such that $G_{(\delta,\mu)}$ is better-reply secure for every $\delta \in [0, 1)$.*

Proof. Since (A) implies uniform payoff security, $\overline{G}$ is payoff secure (Monteiro and Page, 2007, Theorem 1). By Proposition 3.2 of Reny (1999), if $G_{(\delta,\mu)}$ (respectively, $\overline{G}$) is payoff secure with $\sum_i u_i^{(\delta,\mu)}$ (respectively, $\sum_i U_i$) upper semicontinuous, then $G_{(\delta,\mu)}$ (respectively, $\overline{G}$) is better-reply secure. Hence, in light of Lemma 1, it suffices to show that if $\sum_i u_i$ is upper semicontinuous, then $\sum_i u_i^{(\delta,\mu)}$ (respectively, $\sum_i U_i$) is upper semicontinuous. But this follows from the fact that $\sum_i U_i$ (and hence $\sum_i u_i^{(\delta,\mu)}$) is upper semicontinuous if $\sum_i u_i$ is upper semicontinuous (Reny, 1999, Proposition 5.1). $\square$

Lemmata 1–3 can be combined with Proposition 1 to obtain our first main result.

Theorem 3. *Suppose that G is compact, metric, concave, and satisfies Condition* (A). *Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G has a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

In Section 4 we apply Theorem 3 to various economic games.

We now consider a strengthening of (A), in terms of two independent conditions, that is particularly useful when verifying the measurability of the map f from Condition (A) is cumbersome. The two proposed conditions do not require direct verification of the measurability of the map f, and, for concave games whose action spaces have a nonempty interior, one of the two conditions is redundant, and a simple corollary of Theorem 3 can be proven.

Let A_i be the set of all accumulation points of X_i (i.e., the set A_i of points $x_i \in X_i$ such that $(V \setminus \{x_i\}) \cap A_i \neq \emptyset$ for each neighborhood V of x_i). Since X_i is compact and metric, it can be written as a disjoint union $A_i \cup K_i$, where A_i is closed and dense in itself (i.e., with no isolated points) and K_i is countable.

Let $\widetilde{M}_i$ be the set of measures μ_i in M_i such that $\mu_i(\{x_i\}) = 0$ and $\mu_i(N_\epsilon(x_i)) > 0$ for each $x_i \in A_i$ and every $\epsilon > 0$, and $\mu_i(\{x_i\}) > 0$ for every $x_i \in K_i$. Define $\widetilde{M} := \bigtimes_i \widetilde{M}_i$.

Clearly, $\widetilde{M}_i$ is a subset of $\widehat{M}_i$. Moreover, $\widetilde{M}_i$ is nonempty. In fact, it is not difficult to show that $\widetilde{M}_i$ is dense in M_i for each i.

Definition 8. Given $Y_i \subseteq X_i$ for each i, we say that G is ***entirely payoff secure over*** $\bigtimes_i Y_i$ if for each i, $\varepsilon > 0$, and $x_i \in Y_i$, and for every neighborhood O of x_i, there exist $y_i \in O$ and a neighborhood O_{x_i} of x_i such that for every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i(y_i, O_{y_{-i}}) > u_i(O_{x_i}, y_{-i}) - \varepsilon$.

We say that G is ***entirely payoff secure*** if it is entirely payoff secure over X.

Definition 9. Given $Y_i \subseteq X_i$ for each i, we say that the game G is ***generically entirely payoff secure over*** $\bigtimes_i Y_i$ if there is, for each i, a set $Z_i \subseteq Y_i$ with $Y_i \setminus Z_i$ countable such that G is uniformly payoff secure over $\bigtimes_i (Y_i \setminus Z_i)$ and entirely payoff secure over $\bigtimes_i Z_i$.

A game G is ***generically entirely payoff secure*** if it is entirely payoff secure over $\bigtimes_i K_i$ and generically entirely payoff secure over $\bigtimes_i A_i$ (recall that $X_i = A_i \cup K_i$, where A_i is closed and dense in itself and K_i is countable).

Remark 4. The following implications are immediate:

$$
\begin{aligned}
\text{continuity} \quad &\Rightarrow \quad \text{entire payoff security}\\
&\Rightarrow \quad \text{generic entire payoff security}\\
&\Rightarrow \quad \text{uniform payoff security}\\
&\Rightarrow \quad \text{payoff security.}
\end{aligned}
$$

Definition 10. The game G is ***locally equi-upper semicontinuous*** if for each i, $x_{-i} \in X_{-i}$, and $x_i \in X_i$, and for each $\varepsilon > 0$, there exists a neighborhood O_{x_i} of x_i such that for every $y_i \in O_{x_i}$ there exists a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

Definition 11. The game G is ***generically locally equi-upper semicontinuous*** if there exists $(\mu_1, \ldots, \mu_N) \in \widetilde{M}$ such that for each i and $x_{-i} \in X_{-i}$, there exists $Y_i \subseteq X_i$ with $\mu_i(Y_i) = 1$ such that for each $x_i \in Y_i$ and $\varepsilon > 0$, there exists a neighborhood O_{x_i} of x_i such that for every $y_i \in O_{x_i}$ there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

As shown in Carbonell-Nicolau (2010a), generic entire payoff security and generic local equi-upper semicontinuity imply Condition (A).

Lemma 4. *Suppose that G is generically entirely payoff secure and generically locally equi-upper semicontinuous. Then G satisfies Condition* (A).

We furnish the proof of Lemma 4 in Section 5.2. Lemma 4, combined with Theorem 3, yields the following result.

Corollary 1 *(to Theorem 3). Suppose that G is compact, metric, concave, generically entirely payoff secure, and generically locally equi-upper semicontinuous. Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G has a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

When each X_i is a normed space with a nonempty interior, local equi-upper semicontinuity in the hypothesis of Corollary 1 can be dropped.[9] This flows from the following observations. Suppose that each X_i is a convex, compact subset of a normed vector space. If the interior of X_i, X_i°, is nonempty for each i, then, because X_i° is dense in X_i (e.g., Aliprantis and Border, 2006, Lemma 5.28), we must have $K_i \subseteq X_i^\circ$ (recall that the members of K_i are isolated points of X_i). In addition, if G is concave and each X_i has a nonempty interior, then each $u_i(\cdot, x_{-i})$ is continuous on X_i° (e.g., Aliprantis and Border, 2006, Theorem 5.43). Now, given $\mu = (\mu_1, \dots, \mu_N) \in \widetilde{M}$, define, for each i, $\sigma_i \in M_i$ by $\sigma_i(B_i) := \frac{\mu_i(B_i \cap X_i^\circ)}{\mu_i(X_i^\circ)}$. Clearly, $\sigma_i \in \widetilde{M}_i$ and $\sigma_i(X_i^\circ) = 1$ for each i. Hence, each $u_i(\cdot, x_{-i})$ is continuous on X_i°, and there exists $(\sigma_1, \dots, \sigma_N) \in \widetilde{M}$ such that $\sigma_i(X_i \setminus X_i^\circ) = 0$ for each i. We now show that this, together with concavity of G and boundedness of each u_i, implies that G is generically locally equi-upper semicontinuous. Observe that it suffices to prove that for each i and every $(x_i, x_{-i}) \in X_i^\circ \times X_{-i}$,

$$\limsup_{x_i^n \to x_i} \Big(\limsup_{x_{-i}^n \to x_{-i}} \big[u_i\big(x_i^n, x_{-i}^n\big) - u_i\big(x_i, x_{-i}^n\big) \big] \Big) \leqslant 0, \tag{7}$$

where the first lim sup is taken over all sequences $x_i^n \to x_i$, and similarly for the second lim sup.

To establish (7), we first show that for each i and every $y_i \in X_i^\circ$, there exist $K \in \mathbb{R}$ and $\delta > 0$ such that for every $y_{-i} \in X_{-i}$, $u_i(\cdot, y_{-i})$ is Lipschitz continuous on $N_\delta(y_i)$ with Lipschitz constant K.

Because u_i is bounded, there exists $\{m_i, M_i\} \subseteq \mathbb{R}$ such that $m_i \leqslant u_i(z) \leqslant M_i$ for all $z \in X$. Take

$$M > \max\{M_1 - m_1, \dots, M_N - m_N\}.$$

Now fix i and $y_i \in X_i^\circ$. Clearly, we may pick a sufficiently small $\varepsilon > 0$ such that $N_\varepsilon(y_i) \subseteq X_i^\circ$. Now let $K \in \mathbb{R}$ and $\delta \in (0, \varepsilon)$ satisfy $K > \frac{M}{\varepsilon - \delta}$. Fix $y_{-i} \in X_{-i}$, and take any pair $(z_i, w_i) \in N_\delta(y_i)^2$ with $z_i \neq w_i$. It suffices to show that

$$u_i(w_i, y_{-i}) - u_i(z_i, y_{-i}) \leqslant K \|z_i - w_i\|, \tag{8}$$

where $\|\cdot\|$ represents the norm associated with X_i. Let

$$a_i := z_i + \tfrac{\varepsilon - \delta}{\|z_i - w_i\|}(z_i - w_i).$$

The choice of a_i entails $a_i \in N_\varepsilon(y_i)$, and z_i can be expressed as a convex combination of a_i and w_i:

$$z_i = \big(\tfrac{\|z_i - w_i\|}{\varepsilon - \delta + \|z_i - w_i\|}\big) a_i + \big(\tfrac{\varepsilon - \delta}{\varepsilon - \delta + \|z_i - w_i\|}\big) w_i.$$

Consequently, by concavity of $u_i(\cdot, y_{-i})$ we have

$$u_i(z_i, y_{-i}) \geqslant \big(\tfrac{\|z_i - w_i\|}{\varepsilon - \delta + \|z_i - w_i\|}\big) u_i(a_i, y_{-i}) + \big(\tfrac{\varepsilon - \delta}{\varepsilon - \delta + \|z_i - w_i\|}\big) u_i(w_i, y_{-i}).$$

Rearranging terms, we have

$$\big(\tfrac{\|z_i - w_i\|}{\varepsilon - \delta + \|z_i - w_i\|}\big)\big(u_i(w_i, y_{-i}) - u_i(a_i, y_{-i})\big) \geqslant u_i(w_i, y_{-i}) - u_i(z_i, y_{-i}).$$

Hence, since

$$\big(\tfrac{\|z_i - w_i\|}{\varepsilon - \delta + \|z_i - w_i\|}\big)\big(u_i(w_i, y_{-i}) - u_i(a_i, y_{-i})\big) \leqslant \big(\tfrac{M}{\varepsilon - \delta}\big) \|z_i - w_i\| \leqslant K \|z_i - w_i\|,$$

(8) follows.

We now establish (7). Fix i and $(x_i, x_{-i}) \in X_i^\circ \times X_{-i}$. Because there exist $K \in \mathbb{R}$ and $\delta > 0$ such that

$$\big| u_i(y_i, y_{-i}) - u_i(z_i, y_{-i}) \big| \leqslant K \|y_i - z_i\|$$

for every $(y_i, z_i, y_{-i}) \in N_\delta(y_i)^2 \times X_{-i}$, given $\varepsilon > 0$, one may choose $J \in \mathbb{R}$ such that $\frac{1}{J} < \min\{\varepsilon, \delta\}$, and for such J we have, for $(y_i, y_{-i}) \in N_{\frac{1}{J}}(x_i) \times X_{-i}$,

$$u_i(y_i, y_{-i}) - u_i(x_i, y_{-i}) \leqslant K \|x_i - y_i\| \leqslant \tfrac{K}{J} < \varepsilon.$$

We have demonstrated that when each X_i is a normed space with a nonempty interior, local equi-upper semicontinuity can be dispensed with in the statement of Corollary 1. Corollary 2 summarizes this finding.

We call a normal-form game $G = (X_i, u_i)_{i=1}^N$ ***normed*** if X_i is a normed space for each i.

[9] The requirement that X_i be normed is not essential. See Remark 5.

Corollary 2 *(to Theorem 3). Suppose that G is compact, normed, concave, and generically entirely payoff secure, with $\sum_i u_i$ upper semicontinuous. Suppose further that the interior of each X_i is nonempty. Then G has a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

Remark 5. If X_i is a compact, metric vector space, then the topology on X_i is induced by a quasi-norm, i.e., a map $\|\cdot\|: X_i \to \mathbb{R}_+$ satisfying

- $\|x\| = 0 \Leftrightarrow x = 0$;
- $\|\alpha x\| = |\alpha|\|x\|$ if $(\alpha, x) \in \mathbb{R} \times X_i$; and
- for some $\kappa \geqslant 1$ and for every $(x, y) \in X_i^2$,

$$\|x + y\| \leqslant \kappa \big(\|x\| + \|y\|\big)$$

(e.g., Bourgin, 1943). Hence, since a straightforward modification of the proof of Corollary 2 can be furnished in terms of quasi-norms, it suffices that G be metric (rather than normed) in the statement of Corollary 2.

Remark 6. Proposition 1 cannot be used to prove versions of Theorem 3 and Corollaries 1–2 in which generic entire payoff security is weakened to uniform payoff security. In fact, the game G from Example 2 is compact, metric, concave, uniformly payoff secure, and locally equi-upper semicontinuous, with $\sum_i u_i$ upper semicontinuous, and yet the perturbations $G_{(\delta,\mu)}$ fail payoff security and better-reply security.

We now turn to games that fail to be concave. We first note that concavity cannot be weakened to quasiconcavity in Theorem 3 and Corollaries 1–2. To see this, consider the game G from Example 1. This game is compact, metric, quasiconcave, and entirely payoff secure, with $\sum_i u_i$ upper semicontinuous. Moreover, G is locally equi-upper semicontinuous. In fact, since G is continuous and compact, it is uniformly continuous (i.e., each u_i is uniformly continuous on X) by the Heine–Cantor theorem. Uniform continuity of G implies local equi-upper semicontinuity of G. Consequently, since G does not have a pure-strategy thp equilibrium, it is not possible to relax concavity in Theorem 3 without strengthening other aspects of the theorem's hypothesis. We first introduce a condition (Condition (LC) below) that imposes local concavity of each player's payoff (in own strategies) on certain neighborhoods of the player's domain of actions, along with a uniform bound on some payoff differences.

Definition 12. We say that $u_i(\cdot, x_{-i})$ satisfies **Condition (L)** over $Y_i \subseteq X_i$ relative to $\epsilon > 0$ if every $x_i \in Y_i$ has a neighborhood O such that

$$\frac{u_i(z_i,x_{-i})-u_i(y_i,x_{-i})}{u_i(y_i,y_{-i})-u_i(z_i,y_{-i})} \geqslant \epsilon$$

for all $\{z_i, y_i\} \subseteq O \cap Y_i$ with $u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$ with $u_i(y_i, y_{-i}) > u_i(z_i, y_{-i})$.

In some cases, this condition can be weakened. See Remark 7 below.

Given x_{-i} and small ϵ, Condition (L) can only be binding at points z_i for which there are nearby elements y_i such that $u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$ and $u_i(z_i, x_{-i})$ is sufficiently close to $u_i(y_i, x_{-i})$. This will typically happen, of course, at some points of continuity z_i of the map $u_i(\cdot, x_{-i})$. For such points, (L) requires that for elements y_{-i} in X_{-i} such that $u_i(y_i, y_{-i}) > u_i(z_i, y_{-i})$, if the difference $u_i(y_i, y_{-i}) - u_i(z_i, y_{-i})$ vanishes as y_i approaches z_i, it not vanish slower than $u_i(z_i, x_{-i}) - u_i(y_i, x_{-i})$ does. For instance, when Gâteaux derivatives exist, it suffices that the Gâteaux derivatives of the maps $u_i(\cdot, x_{-i})$ and $u_i(\cdot, y_{-i})$ at z_i be uniformly bounded away from $\pm\infty$, and that the Gâteaux derivative of the first map be uniformly bounded away from zero. Observe that Condition (L) will typically fail when the Gâteaux derivative of $u_i(\cdot, x_{-i})$ is zero. Thus, imposing (L) on the whole domain X_i would be overly restrictive. We shall only assume (L) over certain subdomains Y_i of X_i (for example, all of X_i except, for each i, small neighborhoods around the points of X_i at which the Gâteaux derivative of $u_i(\cdot, x_{-i})$ vanishes), and then require (local) concavity of each $u_i(\cdot, y_{-i})$ on $X_i \setminus Y_i$.

Definition 13. Given i, $x_{-i} \in X_{-i}$, and $\{x_i, y_i\} \subseteq X_i$, we say that $u_i(\cdot, x_{-i})$ ***is decreasing*** **(*increasing*)** ***on*** $\mathrm{co}\{x_i, y_i\}$ if for any pair $(\alpha, \beta) \in [0,1]^2$ with $\alpha \geqslant \beta$ $(\alpha \leqslant \beta)$ we have

$$u_i\big(\alpha x_i + (1-\alpha) y_i, x_{-i}\big) \geqslant u_i\big(\beta x_i + (1-\beta) y_i, x_{-i}\big).$$

Condition (LC). For each i, there exists $\epsilon_i > 0$ such that the following holds. Given $x_{-i} \in X_{-i}$ and $\{x_i, y_i\} \subseteq X_i$ such that $u_i(\cdot, x_{-i})$ is decreasing on $\mathrm{co}\{x_i, y_i\}$, there exists $z_i \in \mathrm{co}\{x_i, y_i\}$ such that (1) $u_i(\cdot, y_{-i})$ is concave on $\mathrm{co}\{x_i, z_i\}$ for every $y_{-i} \in X_{-i}$, and (2) $u_i(\cdot, x_{-i})$ satisfies Condition (L) over $\mathrm{co}\{z_i, y_i\}$ relative to ϵ_i.

Condition (LC) is often rather simple to verify. This is illustrated by the following example (but see also Examples 8–9 in Section 4).

Example 3. For the two-player game G from Example 1, Condition (LC) can be verified as follows. Fix i, say $i = 1$. It is easy to see that there exists $K \geqslant 1$ such that for each $w_1 \in [0,1]$, there is a sufficiently small neighborhood $N_\varepsilon(w_1)$ of w_1 such that

$$\tfrac{1}{K}|a_1 - b_1| \leqslant u_1(a_1, y_2) - u_1(b_1, y_2) \leqslant K|a_1 - b_1|$$

whenever $\{a_1, b_1\} \subseteq N_\varepsilon(w_1)$, $u_1(a_1, y_2) - u_1(b_1, y_2) > 0$, and $y_2 \in [0,1]$. Let $0 < \epsilon_1 < \frac{1}{K^2}$. Take $x_2 \in [0,1]$. It suffices to show that for any $\{x_1, y_1\} \subseteq [0,1]$, $u_1(\cdot, x_2)$ satisfies Condition (L) over $\operatorname{co}\{x_1, y_1\}$ relative to ϵ_1. Choose $\{x_1, y_1\} \subseteq [0,1]$ and $w_1 \in \operatorname{co}\{x_1, y_1\}$. Let $N_\varepsilon(w_1)$ be a neighborhood of w_1 with the above properties. Given $\{a_1, b_1\} \subseteq N_\varepsilon(w_1)$ with $u_1(a_1, x_2) - u_1(b_1, x_2) > 0$ and $y_2 \in [0,1]$ with $u_1(b_1, y_2) - u_1(a_1, y_2) > 0$, we have

$$\frac{u_1(a_1,x_2)-u_1(b_1,x_2)}{u_1(b_1,y_2)-u_1(a_1,y_2)} \geqslant \frac{\frac{1}{K}|a_1-b_1|}{u_1(b_1,y_2)-u_1(a_1,y_2)} \geqslant \frac{\frac{1}{K}|a_1-b_1|}{K|a_1-b_1|} = \frac{1}{K^2} > \epsilon_1,$$

and so $u_1(\cdot, x_2)$ satisfies Condition (L) over $\operatorname{co}\{x_1, y_1\}$ relative to ϵ_1.

It can be shown that Condition (LC) is strong enough to ensure that strictly quasiconcave games G have $G_{(\delta,\mu)}$ quasiconcave for every $(\delta, \mu) \in [0, \alpha] \times \widehat{M}$ and some $\alpha \in (0,1)$ (this is implied by Lemma 5 below). However, quasiconcave games that are not strictly quasiconcave need not have the same property. Indeed, the continuous, quasiconcave game G from Example 1 satisfies (LC) (Example 3) but fails strict quasiconcavity. Since this game lacks a pure-strategy thp equilibrium and $\overline{G}$ and $G_{(\delta,\mu)}$ are better-reply secure (even continuous), it follows from Proposition 1 that for each $\alpha > 0$ there exists $\delta \in [0, \alpha]$ such that $G_{(\delta,\mu)}$ is not quasiconcave. What is more, one cannot replace concavity in the statement of Theorem 3 by quasiconcavity and Condition (LC), for G is compact, metric, quasiconcave, entirely payoff secure, and locally equi-upper semicontinuous, with $\sum_i u_i$ upper semicontinuous, and in addition G satisfies Condition (LC).

Condition (LC) will be combined with the following strengthening of quasiconcavity.

Definition 14. The game G is said to be ***strongly quasiconcave*** if it is quasiconcave and for each i, $x_{-i} \in X_{-i}$, and $\{x_i, y_i\} \subseteq X_i$, $z_i \in \operatorname{co}\{x_i, y_i\}$ and $u_i(z_i, x_{-i}) = \min_{w_i \in \{x_i, y_i\}} u_i(w_i, x_{-i})$ imply $u_i(z_i, y_{-i}) \geqslant u_i(w_i, y_{-i})$ for all $y_{-i} \in X_{-i}$ and for some $w_i \in \{x_i, y_i\}$.

This condition can be weakened in some cases. See Remark 7.

Clearly, if G is strictly quasiconcave, then it is trivially strongly quasiconcave, for strict quasiconcavity avoids the condition

$$u_i(z_i, x_{-i}) = \min_{w_i \in \{x_i, y_i\}} u_i(w_i, x_{-i}).$$

The proof of the following lemma is relegated to Section 5.3.

Lemma 5. *Suppose that G is a strongly quasiconcave game satisfying* (LC). *Then, there exists $\alpha \in (0,1)$ such that $G_{(\delta,\mu)}$ is quasiconcave for every $(\delta, \mu) \in [0, \alpha] \times \widehat{M}$.*

Lemma 5 can be used, together with Lemma 3 and Proposition 1, to prove our second main result on the existence of pure-strategy thp equilibria. In fact, suppose that G is compact, metric, and strongly quasiconcave, with $\sum_i u_i$ upper semicontinuous. Suppose further that G satisfies (A) and (LC). Let $\mu \in \widehat{M}$ have the properties of Lemma 3. Lemma 5 gives $\alpha \in (0,1)$ such that $G_{(\delta,\mu)}$ is quasiconcave for every $0 \leqslant \delta \leqslant \alpha$. Hence, in light of Proposition 1, it suffices to show that $G_{(\delta,\mu)}$ is better-reply secure (for any $0 \leqslant \delta < 1$) and that $\overline{G}$ is better-reply secure. But this is guaranteed by Lemma 3.

We have obtained the following result.

Theorem 4. *Suppose that G is compact, metric, strongly quasiconcave, and satisfies* (A) *and* (LC). *Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G has a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

Remark 7. It is possible to state a slightly stronger version of Theorem 4 by combining Condition (A) with Definitions 12 and 14 as follows: Let $(\mu_1, \dots, \mu_N)$ be the measure given by Condition (A), and replace "for all $y_{-i} \in X_{-i}$" by "for μ_{-i}-almost all $y_{-i} \in X_{-i}$" in Definition 12, and "for all $y_{-i} \in X_{-i}$" by "for $y_{-i} = x_{-i}$ and for μ_{-i}-almost all $y_{-i} \in X_{-i}$" in Definition 14.

Remark 8. Let $f : (A, d_A) \to (B, d_B)$ be a function between metric spaces. If there exists $K \geqslant 1$ such that

$$\tfrac{1}{K} d_A(x, y) \leqslant d_B\big(f(x), f(y)\big) \leqslant K d_A(x, y), \quad \text{for all } (x, y) \in A^2,$$

f is said to be *bi-Lipschitz*. If for every $x \in A$, there is a neighborhood O_x of x such that $f|_{O_x}$ is bi-Lipschitz, then f is *locally bi-Lipschitz*.

Call a game $G = (X_i, u_i)_{i=1}^N$ *locally bi-Lipschitz* if each $u_i(\cdot, x_{-i})$ is locally bi-Lipschitz. One can show that if G is compact, metric, and locally bi-Lipschitz, then G is strongly quasiconcave and satisfies (LC). Consequently, in view of Theorem 4, any compact, metric, and locally bi-Lipschitz game satisfying upper semicontinuity of $\sum_i u_i$ and Condition (A) possesses a pure-strategy thp equilibrium.

Theorem 4, together with Lemma 4, gives the following corollary.

Corollary 3 *(to Theorem 4). Suppose that G is compact, metric, strongly quasiconcave, generically entirely payoff secure, generically locally equi-upper semicontinuous, and satisfies* (LC). *Suppose further that $\sum_i u_i$ is upper semicontinuous. Then G has a pure-strategy trembling-hand perfect equilibrium, and all trembling-hand perfect equilibria of G are Nash.*

In the next section, two economic games illustrate Theorem 4 and Corollary 3.

4. Applications

Example 4 *(Differentiated commodity Bertrand pricing game).* This example is taken from Simon and Stinchcombe (1995, Example 2.1). Consider the game $G = ([0, \frac{1}{2}], [0, \frac{1}{2}], u_1, u_2)$, where

$$u_i(x_i, x_j) := \begin{cases} x_i & \text{if } x_i \leqslant \frac{1}{2}x_j, \\ \frac{x_j(1-x_i)}{2-x_j} & \text{if } \frac{1}{2}x_j < x_i. \end{cases}$$

This game is a stylized version of a differentiated commodity Bertrand duopoly game in which each agent i's best response is always to undercut the other agent by a finite amount.[10]

It is readily seen that each $u_i(\cdot, x_{-i})$ is concave for each x_{-i} (so that G is concave) and G is continuous. It follows from Theorem 3 (or Corollary 2) that G has a pure-strategy thp equilibrium.

Example 5 *(Rank-order tournament).* Consider the two player rank-order tournament $G = ([0, E], [0, E], u_1, u_2)$ of Lazear and Rosen (1981). Here, E is a large positive real. Each player i chooses an investment level $e_i \in [0, E]$ and produces output

$$q_i = e_i + \varepsilon_i,$$

where ε_i is a random variable drawn out of a known distribution, with zero mean and variance σ^2, whose cdf is strictly increasing and twice differentiable (say, with support $\mathbb{R}$). The players' actions are simultaneous.

The rules of the game specify a fixed prize W_1 to the winner and a fixed prize W_2 to the loser. The winner of the contest is determined by the largest drawing of q. In case of a tie, the prizes are assigned according to an equiprobable distribution. Assume that both contestants have the same cost of investment $C(e)$, where C is convex and twice differentiable on $(0, E)$. Given a strategy profile (e_1, e_2), the payoffs to i are given by

$$u_i(e_1, e_2) := H(e_i - e_{-i})\big(W_1 - C(e_i)\big) + \big(1 - H(e_i - e_{-i})\big)\big(W_2 - C(e_i)\big),$$

where H is the cdf of $\varepsilon_{-i} - \varepsilon_i$. Here, $H(e_i - e_{-i})$ is the probability that i wins the contest. Thus, $u_i(e_1, e_2)$ represents i's expected wealth at (e_1, e_2).

Given i and e_{-i}, we have $u_i''(e_i, e_{-i}) \leqslant 0$ for all $e_i \in (0, E)$ if

$$(W_1 - W_2)H''(e_i - e_{-i}) \leqslant C''(e_i), \quad \text{for all } e_i \in (0, E). \tag{9}$$

Thus, this condition ensures concavity of $u_i(\cdot, e_{-i})$.[11] Given (9), Theorem 3 (or Corollary 2) gives a pure-strategy thp equilibrium.

Example 6 *(Rent-seeking).* Tullock (1980) considers the following rent-seeking game: $G = ([0, 1], [0, 1], u_1, u_2)$, where

$$u_i(x_i, x_{-i}) := \left(\frac{x_i}{x_i + x_{-i}}\right)R - x_i,$$

where $R > 0$ represents the rent value and x_ι denotes individual ι's expenditure. The game G is continuous and concave, and therefore Theorem 3 applies.

Example 7 *(Probabilistic voting).* In standard models of probabilistic voting (cf. Austen-Smith and Banks, 2005 and references therein), there are two candidates proposing platforms in a policy space X. For any pair of platforms $(a, b) \in X^2$, any voter $i \in N$ (N a finite set of voters), and any candidate $c \in \{A, B\}$, the candidates assess that individual i will vote for candidate

10 Similar games appear in Bester (1992).

11 Given convexity of C, it is clear that (9) is met if $W_1 = W_2$. Furthermore, it can be shown that (9) holds if σ^2 is large enough.

c with probability $p_i^c(u_i(a), u_i(b)) \in [0,1]$, where $u_i : X \to \mathbb{R}$.[12] Assuming that no voter abstains, we have $p_i^A(u_i(a), u_i(b)) + p_i^B(u_i(a), u_i(b)) = 1$ for each $i \in N$ and every $(a,b) \in X^2$. Suppose that for each $c \in \{A, B\}$ and every $i \in N$, $p_i^c(\cdot)$ is continuous and nondecreasing in i's utility from c's platform.

Elections are determined by plurality rule with ties broken by a fair lottery. The candidates simultaneously choose platforms to maximize their expected pluralities: given $(a,b) \in X^2$, candidate A's expected plurality is given by

$$\Pi(a,b) := \sum_{i \in N} \bigl(p_i^A\bigl(u_i(a), u_i(b)\bigr) - p_i^B\bigl(u_i(a), u_i(b)\bigr)\bigr),$$

and so B's expected plurality is $-\Pi(a,b)$.

An action profile $(a,b) \in X^2$ constitutes a pure-strategy Nash equilibrium of the associated normal-form game G if $\Pi(a,x) \geqslant \Pi(a,b) \geqslant \Pi(x,b)$, all $x \in X$.

If X is a compact, convex subset of $\mathbb{R}^n$, and $\Pi(x,y)$ is concave in x and convex in y, then G has a pure-strategy Nash equilibrium (Theorem 7.9 of Austen-Smith and Banks, 2005). By Theorem 3, these assumptions are also sufficient for the existence of a pure-strategy thp equilibrium.

Example 8 *(Timing game).* Consider a two-player game $G = ([0,1],[0,1],u_1,u_2)$, where

$$u_i(t_i, t_{-i}) := \begin{cases} 10 & \text{if } t_i < t_{-i}, \\ \frac{1}{t_i+1} & \text{if } t_i = t_{-i}, \\ -10 & \text{if } t_i > t_{-i}, \end{cases}$$

for $i \in \{1,2\}$.[13]

It is routine to check that G is quasiconcave. To see that G is strongly quasiconcave, observe that for each i, $t_{-i} \in [0,1]$, $\{t_i, s_i\} \subseteq [0,1]$, and $t_i \in \operatorname{co}\{t_i, s_i\}$, we have

$$u_i(t_i, \tau_{-i}) \geqslant u_i(\bar{t}_i, \tau_{-i}), \quad \text{for all } \tau_{-i} \in [0,1],$$

where $\bar{t}_i := \max\{t_i, s_i\}$.

To see that G satisfies (LC), fix $(t_i, t_{-i}) \in [0,1]^2$. Let $s_i \in [0,1]$ and suppose that $u_i(t_i, t_{-i}) > u_i(s_i, t_{-i})$. Then $t_i \leqslant t_{-i} \leqslant s_i$, with at least one inequality strict. Consequently, for every $s_{-i} \in [0,1]$, $u_i(s_i, s_{-i}) \not> u_i(t_i, s_{-i})$, and so Condition (LC) is trivially met.

We now show that G satisfies Condition (A). For each i, let μ_i be the Lebesgue measure over $[0,1]$. Fix i and $\varepsilon > 0$, and define, for each $k \in \mathbb{N}$, $f_k : [0,1] \to [0,1]$ by $f_k(t_i) := (1 - \frac{1}{k})t_i$. It is clear that each f_k is Borel measurable. We now verify that, for large k, f_k satisfies items (a) and (b) of Condition (A).

Given $t_i \in [0,1]$, k, and $t_{-i} \in [0,1]$, let $O_{t_{-i}}$ be a neighborhood of t_{-i} satisfying the following: if $t_{-i} > t_i$, then $\{t_i\} \cap O_{t_{-i}} = \emptyset$, and if $t_i = t_{-i} > 0$, then $\{(1 - \frac{1}{k})t_i\} \cap O_{t_{-i}} = \emptyset$. We have

$$u_i\bigl(f_k(t_i), O_{t_{-i}}\bigr) = u_i\bigl(\bigl(1 - \tfrac{1}{k}\bigr)t_i, O_{t_{-i}}\bigr) \begin{cases} = 10 > u_i(t_i, t_{-i}) - \varepsilon & \text{if } t_{-i} > t_i, \\ = 10 > u_i(t_i, t_{-i}) - \varepsilon & \text{if } t_{-i} = t_i, \\ > u_i(t_i, t_{-i}) - \varepsilon = -10 - \varepsilon & \text{if } t_{-i} < t_i. \end{cases}$$

Now fix $t_{-i} \in [0,1]$ and let $Y_i = [0,1] \setminus \{t_{-i}\}$. Clearly, $\mu_i(Y_i) = 1$. Pick any $t_i \in Y_i$. If $t_i < t_{-i}$, then, for each k, and for any neighborhood $O_{t_{-i}}$ of t_{-i} such that $\{t_i\} \cap O_{t_{-i}} = \emptyset$,

$$u_i\bigl(f_k(t_i), \tau_{-i}\bigr) - u_i(t_i, \tau_{-i}) = u_i\bigl(\bigl(1 - \tfrac{1}{k}\bigr)t_i, \tau_{-i}\bigr) - u_i(t_i, \tau_{-i}) = 10 - 10 < \varepsilon,$$

for all $\tau_{-i} \in O_{t_{-i}}$. If, on the other hand, $t_i > t_{-i}$, then, for large k, $f_k(t_i)$ is sufficiently close to t_i, and for any neighborhood $O_{t_{-i}}$ of t_{-i} such that $\{t_i\} \cap O_{t_{-i}} = \emptyset$,

$$u_i\bigl(f_k(t_i), \tau_{-i}\bigr) - u_i(t_i, \tau_{-i}) = u_i\bigl(\bigl(1 - \tfrac{1}{k}\bigr)t_i, \tau_{-i}\bigr) - u_i(t_i, \tau_{-i}) = -10 - (-10) < \varepsilon,$$

for all $\tau_{-i} \in O_{t_{-i}}$. This gives item (b) of Condition (A).

Finally, it is straightforward to verify that $\sum_i u_i$ is upper semicontinuous. By Theorem 4, therefore, G possesses a pure-strategy thp equilibrium.

Example 9 *(Cournot competition with discontinuous costs).* Consider a market for a single homogeneous good with inverse demand function $P : [0,1] \to \mathbb{R}$. Two firms compete in quantities and face cost function $C : [0,1] \to \mathbb{R}$. The associated game is $G = ([0,1],[0,1],u_1,u_2)$, where

$$u_i(Q_1, Q_2) := P(Q_1 + Q_2)Q_i - C(Q_i).^{14}$$

[12] The map u_i represents voter i's preferences over policies. This utility function does not completely describe a voter, for the model postulates that voters may have idiosyncratic biases in favor of some candidate. These biases are unknown to the candidates.

[13] This is a stylized version of games that have been used to model behavior in duels as well as in R&D and patent races (e.g. Karlin, 1959).

We let $P(Q) := 2 - Q$ and assume that the cost function takes the form

$$C(Q) := \begin{cases} Q^2 & \text{if } Q \in [0, \frac{1}{2}], \\ (2Q)^2 & \text{if } Q \in (\frac{1}{2}, 1]. \end{cases}$$

(See Baye and Morgan, 2002 for a discussion of economic phenomena leading to cost discontinuities.)

The game G is strictly quasiconcave, and hence strongly quasiconcave. We now argue that G satisfies (LC). The payoff to player i at (Q_i, Q_{-i}) is

$$u_i(Q_i, Q_{-i}) = \begin{cases} (2 - Q_1 - Q_2)Q_i - Q_i^2 & \text{if } Q_i \in [0, \frac{1}{2}], \\ (2 - Q_1 - Q_2)Q_i - 4Q_i^2 & \text{if } Q_i \in (\frac{1}{2}, 1]. \end{cases}$$

Given Q_{-i}, $u_i(\cdot, Q_{-i})$ is concave on any $\text{co}\{Q_i, q_i\}$ that does not intersect with $\{\frac{1}{2}\}$, so for such $\text{co}\{Q_i, q_i\}$ (LC) is met. Moreover, we have the following:

1. For any Q_{-i}, $u_i(\cdot, Q_{-i})$ is concave on $[0, \frac{1}{2}]$ and decreasing on $(\frac{1}{2}, 1]$.
2. The right derivative of $u_i(\cdot, Q_{-i})$ at any point in $(\frac{1}{2}, 1]$ is uniformly bounded away from zero and $-\infty$.

This means that if $u_i(\cdot, Q_{-i})$ is decreasing on $\text{co}\{Q_i, q_i\}$, where $Q_i < q_i$, and $\text{co}\{Q_i, q_i\}$ intersects with $\{\frac{1}{2}\}$ (so that $Q_i \leqslant \frac{1}{2} \leqslant q_i$, with at least one inequality strict), then (1) $u_i(\cdot, Q^*_{-i})$ is concave on $\text{co}\{Q_i, \frac{1}{2}\}$ for every Q^*_{-i}, and (2) $u_i(\cdot, Q_{-i})$ satisfies Condition (L) over $\text{co}\{\frac{1}{2}, q_i\}$ relative to some $\epsilon_i > 0$ that can be chosen independently of Q_i, q_i, and Q_{-i}.

We now argue that G is entirely payoff secure. Fix i, $\varepsilon > 0$, $Q_i \in [0, 1]$, and a neighborhood O of Q_i. The game G is entirely payoff secure if there exist $q_i \in O$ and a neighborhood O_{Q_i} of Q_i such that for every $Q_{-i} \in [0, 1]$, there is a neighborhood $O_{Q_{-i}}$ of Q_{-i} such that

$$u_i(q_i, O_{Q_{-i}}) > u_i(O_{Q_i}, Q_{-i}) - \varepsilon. \tag{10}$$

This is clearly true if $Q_i \neq \frac{1}{2}$, for u_i is continuous everywhere except at points (s_i, s_{-i}) with $s_i = \frac{1}{2}$. If $Q_i = \frac{1}{2}$, then one can choose $q_i < Q_i$ and a neighborhood O_{Q_i} of Q_i in such a way that for every $Q_{-i} \in [0, 1]$ there is a sufficiently small neighborhood $O_{Q_{-i}}$ of Q_{-i} with the property that

$$u_i(q_i, O_{Q_{-i}}) = (2 - q_i - O_{Q_{-i}})q_i - q_i^2 > (2 - s_i - Q_{-i})s_i - s_i^2 - \varepsilon,$$

for all $s_i \in O_{Q_i}$. Note that for such q_i and O_{Q_i}, (10) is satisfied.

Next, we show that G is generically locally equi-upper semicontinuous. Observe that given i and $(Q_i, Q_{-i}) \in ((0, 1) \setminus \{\frac{1}{2}\}) \times [0, 1]$, $u_i(\cdot, \cdot)$ is continuous on $[Q_i - \varepsilon, Q_i + \varepsilon] \times [Q_{-i} - \varepsilon, Q_{-i} + \varepsilon]$ for some $\varepsilon > 0$, and hence uniformly continuous on $[Q_i - \varepsilon, Q_i + \varepsilon] \times [Q_{-i} - \varepsilon, Q_{-i} + \varepsilon]$ (by the Heine–Cantor theorem). It is now easy to see that

$$\limsup_{Q_i^n \to Q_i} \Big(\limsup_{Q_{-i}^n \to Q_{-i}} \big[u_i\big(Q_i^n, Q_{-i}^n\big) - u_i\big(Q_i, Q_{-i}^n\big) \big] \Big) \leqslant 0, \tag{11}$$

where the first lim sup is taken over all sequences $Q_i^n \to Q_i$, and similarly for the second lim sup. Eq. (11) (with arbitrary Q_{-i} and generic Q_i) gives generic local equi-upper semicontinuity of G.

Finally, each u_i is upper semicontinuous, and so $\sum_i u_i$ is upper semicontinuous. Hence, Corollary 3 gives a pure-strategy thp equilibrium.

5. Proofs

5.1. Proof of Lemma 1

The proof Lemma 1 relies on a number of lemmata, which are stated before the main argument. Lemmata 6–7 have been proven elsewhere (cf. Carbonell-Nicolau, 2010a, Lemmata 4–5). Since these results are key to the development of the proof of Lemma 1, their proofs have been included here.

Given a metric space X and $Y \subseteq X$, $\mathbb{P}(Y)$ denotes the set of Borel probability measures on Y, and $\mathbb{P}_*(Y)$ is the subset of finitely supported measures in $\mathbb{P}(Y)$ that assign rational values to each Borel set.

Lemma 6. *Let X be a compact metric space. Suppose that $f : X \to \mathbb{R}$ is bounded and Borel measurable. For each $\mu \in \mathbb{P}(X)$ and every $\varepsilon > 0$, there exists $\nu^* \in \mathbb{P}_*(X) \cap N_\varepsilon(\mu)$ such that $|\int_X f\, \mathrm{d}\mu - \int_X f\, \mathrm{d}\nu^*| < \varepsilon$.*

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[14] Similar games can be found in Novshek (1985).

Proof. Let X be a compact metric space. Suppose that $f: X \to \mathbb{R}$ is bounded and Borel measurable. Fix $\mu \in \mathbb{P}(X)$ and $\varepsilon > 0$. Since f is bounded and Borel measurable and X is compact and metric, f is the uniform limit of a sequence (f_k) of Borel measurable simple functions (e.g., Aliprantis and Border, 2006, Theorem 4.38). Therefore,

$$\left|\int_X f\,\mathrm{d}\nu - \int_X f_k\,\mathrm{d}\nu\right| \leqslant \sup_{x\in X}\bigl|f(x) - f_k(x)\bigr|, \quad \text{for all } \nu \in \mathbb{P}(X),$$

and so it suffices to establish the lemma when f is a simple function.[15]

Suppose that f is simple with standard representation $f = \sum_k a_k \chi_{A_k}$. For each k, choose $x_k \in A_k$. Given $y \in X$, let δ_y stand for the Dirac measure on X with support $\{y\}$. Define the Borel probability measure ν on X as follows:

$$\nu(B) := \sum_k \mu(B \cap A_k)\delta_{x_k}(B \cap A_k).$$

Then ν has finite support, and it is clear that $\int_X f\,\mathrm{d}\nu = \int_X f\,\mathrm{d}\mu$. Furthermore, there is no loss of generality in assuming that $\nu \in N_\varepsilon(\mu)$, for if ν did not belong to $N_\varepsilon(\mu)$, one could successively refine the partition (A_k) of X and define the corresponding finite measures analogously, thereby generating a sequence (ν^n) of finitely supported measures with limit point μ such that $\int_X f\,\mathrm{d}\nu^n = \int_X f\,\mathrm{d}\mu$ for each n.

We conclude that there is a finitely supported measure $\nu \in N_\varepsilon(\mu)$ satisfying $|\int_X f\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\nu| < \varepsilon$. It is now easy to see that there is a measure ν^* in $\mathbb{P}_*(X)$ with the same support as ν such that $|\int_X f\,\mathrm{d}\nu^* - \int_X f\,\mathrm{d}\mu| < \varepsilon$.[16] $\square$

Lemma 7. *Suppose that G is compact, metric, and satisfies Condition* (A). *Then there exists $(\mu_1, \dots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a map $f: X_i \to X_i$ such that the following is satisfied*:

(i) *For each $x_i \in X_i$ and every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} for which $U_i(f(x_i), O_{\sigma_{-i}}) > U_i(x_i, \sigma_{-i}) - \varepsilon$.*

(ii) *For every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) - U_i(\mu_i, p_{-i}) < \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$, where $\mu_i^f \in M_i$ is defined by $\mu_i^f(B) := \mu_i(f^{-1}(B \cap f(X_i)))$.*

Proof. Since G satisfies (A), there exists $(\mu_1, \dots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is a map $f_{(i,\varepsilon)}: X_i \to X_i$ such that the following is satisfied:

(a) For each $x_i \in X_i$ and every $y_{-i} \in X_{-i}$, there is a neighborhood $O_{y_{-i}}$ of y_{-i} for which $u_i(f_{(i,\varepsilon)}(x_i), O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon$.

(b) For every $y_{-i} \in X_{-i}$, there is a subset $Y_{(i,\varepsilon,y_{-i})}$ of X_i with $\mu_i(Y_{(i,\varepsilon,y_{-i})}) = 1$ such that for each $x_i \in Y_{(i,\varepsilon,y_{-i})}$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that $u_i(f_{(i,\varepsilon)}(x_i), z_{-i}) - u_i(x_i, z_{-i}) < \varepsilon$ for all $z_{-i} \in V_{y_{-i}}$.

By (b), we have, given (i, ε) and $y_{-i} \in X_{-i}$,

$$\int_{X_i} \limsup_{y_{-i}^m \to y_{-i}} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i = \int_{Y_{(i,\varepsilon,y_{-i})}} \limsup_{y_{-i}^m \to y_{-i}} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i < \varepsilon. \tag{12}$$

In addition,

$$\limsup_{y_{-i}^m \to y_{-i}} \int_{X_i} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i \leqslant \int_{X_i} \limsup_{y_{-i}^m \to y_{-i}} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i. \tag{13}$$

In fact, the negation of this inequality implies that there is a sequence (y_{-i}^m) in X_{-i} such that

$$\limsup_m \int_{X_i} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i > \int_{X_i} \limsup_m \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y_{-i}^m\bigr) - u_i\bigl(\cdot, y_{-i}^m\bigr)\bigr]\,\mathrm{d}\mu_i,$$

thereby contradicting the reverse Fatou lemma. From (12) and (13) we see that

[15] We thank an anonymous referee for suggesting this proof.

[16] We sketch an alternative proof of the fact that there is a finitely supported measure $\nu \in N_\varepsilon(\mu)$ satisfying $|\int_X f\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\nu| < \varepsilon$. We thank an anonymous referee for suggesting the argument.

Lusin's theorem gives a compact subset Y of X such that $\mu(X \setminus Y)2M < \frac{\varepsilon}{4}$ and $f|_Y$ is continuous, where M is large enough to ensure that $|f(x)| < M$ for all $x \in X$. Now, since the set of finitely supported members of $\mathbb{P}(Y)$ is dense in $\mathbb{P}(Y)$, there exists a finitely supported measure p in $\mathbb{P}(Y) \cap N_\varepsilon(\mu|_Y)$ such that $|\int_Y f\,\mathrm{d}\mu - \int_Y f\,\mathrm{d}p| < \frac{\varepsilon}{2}$. Hence, because there exists a finitely supported probability measure ρ on $X \setminus Y$ such that $\rho \in N_\varepsilon(\mu|_{X\setminus Y})$ and $\rho(X \setminus Y)2M < \frac{\varepsilon}{4}$, letting $\nu(B) := p(B \cap Y) + \rho(B \cap (X \setminus Y))$ gives $\nu \in N_\varepsilon(\mu)$ and $|\int_X f\,\mathrm{d}\mu - \int_X f\,\mathrm{d}\nu| < \varepsilon$.

$$\limsup_{y^m_{-i}\to y_{-i}} \int_{X_i} \bigl[u_i\bigl(f_{(i,\varepsilon)}(\cdot), y^m_{-i}\bigr) - u_i\bigl(\cdot, y^m_{-i}\bigr)\bigr]\,\mathrm{d}\mu_i = \limsup_{y^m_{-i}\to y_{-i}} \bigl[U_i\bigl(\mu_{f_{(i,\varepsilon)}}, y^m_{-i}\bigr) - U_i\bigl(\mu_i, y^m_{-i}\bigr)\bigr] < \varepsilon,$$

where

$$\mu_{f_{(i,\varepsilon)}}(B) := \mu_i\bigl(f^{-1}_{(i,\varepsilon)}\bigl(B \cap f_{(i,\varepsilon)}(X_i)\bigr)\bigr).$$

We conclude that for each (i,ε) and every $y_{-i} \in X_{-i}$, there is a neighborhood $V_{y_{-i}}$ of y_{-i} such that

$$U_i(\mu_{f_{(i,\varepsilon)}}, z_{-i}) - U_i(\mu_i, z_{-i}) < \varepsilon, \quad \text{for all } z_{-i} \in V_{y_{-i}}. \tag{14}$$

Now define $\phi_{f_{(i,\varepsilon)}} : X_{-i} \to \mathbb{R}$ and $\overline{\phi}_{f_{(i,\varepsilon)}} : X_{-i} \to \mathbb{R}$ by

$$\phi_{f_{(i,\varepsilon)}}(y_{-i}) := U_i(\mu_{f_{(i,\varepsilon)}}, y_{-i}) - U_i(\mu_i, y_{-i})$$

and

$$\overline{\phi}_{f_{(i,\varepsilon)}}(y_{-i}) := \limsup_{y^n_{-i}\to y_{-i}} \bigl[U_i\bigl(\mu_{f_{(i,\varepsilon)}}, y^n_{-i}\bigr) - U_i\bigl(\mu_i, y^n_{-i}\bigr)\bigr].$$

Since the map $\overline{\phi}_{f_{(i,\varepsilon)}}$ is upper semicontinuous, so is the map $p_{-i} \mapsto \int_{X_{-i}} \overline{\phi}_{f_{(i,\varepsilon)}}\,\mathrm{d}p_{-i}$ defined on M_{-i} (cf. Aliprantis and Border, 2006, Theorem 15.5).

Fix i, $\varepsilon > 0$, and $\sigma_{-i} \in M_{-i}$. Given $\eta > 0$, because the map $p_{-i} \mapsto \int_{X_{-i}} \overline{\phi}_{f_{(i,\eta)}}\,\mathrm{d}p_{-i}$ defined on M_{-i} is upper semicontinuous, there exists a neighborhood $V^{\eta}_{\sigma_{-i}}$ of σ_{-i} such that

$$\int_{X_{-i}} \overline{\phi}_{f_{(i,\eta)}}\,\mathrm{d}p_{-i} < \int_{X_{-i}} \overline{\phi}_{f_{(i,\eta)}}\,\mathrm{d}\sigma_{-i} + \tfrac{\varepsilon}{2}, \quad \text{for all } p_{-i} \in V^{\eta}_{\sigma_{-i}}.$$

Hence, since $\phi_{f_{(i,\eta)}} \leqslant \overline{\phi}_{f_{(i,\eta)}}$,

$$\int_{X_{-i}} \phi_{f_{(i,\eta)}}\,\mathrm{d}p_{-i} < \int_{X_{-i}} \overline{\phi}_{f_{(i,\eta)}}\,\mathrm{d}\sigma_{-i} + \tfrac{\varepsilon}{2}, \quad \text{for all } p_{-i} \in V^{\eta}_{\sigma_{-i}}. \tag{15}$$

Now, from (14) we see that

$$\overline{\phi}_{f_{(i,\eta)}}(y_{-i}) \underset{\eta\to 0}{\longrightarrow} 0, \quad \text{for every } y_{-i} \in X_{-i},$$

so the dominated convergence theorem gives $\lim_{n\to\infty} \int_{X_{-i}} \overline{\phi}_{f_{(i,\frac{1}{n})}}\,\mathrm{d}\sigma_{-i} = 0$. Therefore, in light of (15), for any sufficiently large n, there exists a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_{f_{(i,\frac{1}{n})}}, p_{-i}) - U_i(\mu_i, p_{-i}) = \int_{X_{-i}} \phi_{f_{(i,\frac{1}{n})}}\,\mathrm{d}p_{-i} < \int_{X_{-i}} \overline{\phi}_{f_{(i,\frac{1}{n})}}\,\mathrm{d}\sigma_{-i} + \tfrac{\varepsilon}{2} < \varepsilon,$$

for all $p_{-i} \in V_{\sigma_{-i}}$.

We have seen that there exists $(\mu_1, \dots, \mu_N) \in \widehat{M}$ such that for each i and every $\varepsilon > 0$ there is, for any sufficiently large n, a Borel measurable map $f_{(i,\frac{1}{n})} : X_i \to X_i$ such that the following is satisfied: for every $\sigma_{-i} \in M_{-i}$, there is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_{f_{(i,\frac{1}{n})}}, p_{-i}) - U_i(\mu_i, p_{-i}) < \varepsilon$ for all $p_{-i} \in V_{\sigma_{-i}}$. This establishes (ii).

To see that the map $f_{(i,\frac{1}{n})}$ (given any large enough n) also satisfies (i), take $x_i \in X_i$ and $\sigma_{-i} \in M_{-i}$. Define $\xi_{f_{(i,\frac{1}{n})}} : X_{-i} \to \mathbb{R}$ by

$$\xi_{f_{(i,\frac{1}{n})}}(x_{-i}) := \liminf_{x^m_{-i}\to x_{-i}} u_i\bigl(f_{(i,\frac{1}{n})}(x_i), x^m_{-i}\bigr),$$

where the lim inf is taken over all sequences $x^m_{-i} \to x_{-i}$. Since the map $\xi_{f_{(i,\frac{1}{n})}}$ is lower semicontinuous, so is the map $p_{-i} \mapsto \int_{X_{-i}} \xi_{f_{(i,\frac{1}{n})}}\,\mathrm{d}p_{-i}$ defined on M_{-i} (cf. Aliprantis and Border, 2006, Theorem 15.5). Consequently,

$$\int_{X_{-i}} \xi_{f_{(i,\frac{1}{n})}}\,\mathrm{d}\nu_{-i} \geqslant \int_{X_{-i}} \xi_{f_{(i,\frac{1}{n})}}\,\mathrm{d}\sigma_{-i} - \tfrac{\varepsilon}{2}$$

for every ν_{-i} in some neighborhood $O_{\sigma_{-i}}$ of σ_{-i}. Hence, since

$$u_i\bigl(f_{(i,\frac{1}{n})}(x_i), y_{-i}\bigr) \geqslant \xi_{f_{(i,\frac{1}{n})}}(y_{-i}) \geqslant u_i(x_i, y_{-i}) - \tfrac{1}{n}, \quad \text{for all } y_{-i} \in X_{-i},$$

we obtain, for every $\nu_{-i} \in O_{\sigma_{-i}}$, and for any large enough n,

$$U_i\big(f_{(i,\frac{1}{n})}(x_i), \nu_{-i}\big) \geqslant \int_{X_{-i}} \xi_{f_{(i,\frac{1}{n})}} \, \mathrm{d}\nu_{-i} \geqslant \int_{X_{-i}} \xi_{f_{(i,\frac{1}{n})}} \, \mathrm{d}\sigma_{-i} - \tfrac{\varepsilon}{2} \geqslant U_i(x_i, \sigma_{-i}) - \varepsilon,$$

as desired. □

Lemma 1. *Suppose that a compact, metric game G satisfies Condition* (A)*. Then there exists $\mu \in \widehat{M}$ such that $G_{(\delta,\mu)}$ is payoff secure for every $\delta \in [0, 1)$.*

Proof. Fix $\delta \in [0, 1)$, and let $\mu = (\mu_1, \ldots, \mu_N) \in \widehat{M}$ be the measure given by Condition (A). We fix $\varepsilon > 0$, $x = (x_1, \ldots, x_N) \in X$, and i, and show that there exists $y_i \in X_i$ such that $u_i^{(\delta,\mu)}(y_i, O_{x_{-i}}) > u_i^{(\delta,\mu)}(x) - \varepsilon$ for some neighborhood $O_{x_{-i}}$ of x_{-i}. To shorten notation, let

$$\sigma = (\sigma_1, \ldots, \sigma_N) := \big((1-\delta)x_1 + \delta\mu_1, \ldots, (1-\delta)x_N + \delta\mu_N\big).$$

Lemma 7 gives a Borel measurable map $f : X_i \to X_i$ satisfying the following:

(i) For each $y_i \in X_i$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that $U_i(f(y_i), O_{\sigma_{-i}}) > U_i(y_i, \sigma_{-i}) - \frac{\varepsilon}{4}$.
(ii) There is a neighborhood $V_{\sigma_{-i}}$ of σ_{-i} such that $U_i(\mu_i^f, p_{-i}) - U_i(\mu_i, p_{-i}) < \frac{\varepsilon}{2}$ for all $p_{-i} \in V_{\sigma_{-i}}$, where $\mu_i^f \in M_i$ is defined by $\mu_i^f(B) := \mu_i(f^{-1}(B \cap f(X_i)))$.

Claim 1. *There exists a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that*

$$\int_{X_i} U_i\big(f(\cdot), O_{\sigma_{-i}}\big) \, \mathrm{d}\sigma_i > \int_{X_i} U_i(\cdot, \sigma_{-i}) \, \mathrm{d}\sigma_i - \tfrac{\varepsilon}{2}.$$

Proof. By (i), for every $y_i \in X_i$, there is a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i\big(f(y_i), O_{\sigma_{-i}}\big) > U_i(y_i, \sigma_{-i}) - \tfrac{\varepsilon}{4}.$$

For each $n \in \mathbb{N}$, define

$$X_i^n := \bigcup_{\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i})} \big\{y_i \in X_i \colon U_i\big(f(y_i), \nu_{-i}\big) < U_i(y_i, \sigma_{-i}) - \tfrac{\varepsilon}{4}\big\}.$$

Each X_i^n is Borel measurable. In fact, Lemma 6 gives

$$X_i^n = \bigcup_{\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i}) \cap \mathbb{P}_*(X_{-i})} X_i(\nu_{-i}), \tag{16}$$

where $X_i(\nu_{-i}) := \{y_i \in X_i \colon U_i(f(y_i), \nu_{-i}) < U_i(y_i, \sigma_{-i}) - \frac{\varepsilon}{4}\}$. Now, since u_i and f are Borel measurable, for each $\nu_{-i} \in N_{\frac{1}{n}}(\sigma_{-i})$ the set $X_i(\nu_{-i})$ is Borel measurable. Therefore, each X_i^n is (by (16)) a countable union of Borel sets, and hence a Borel set itself.

Now observe that we have $\bigcap_n X_i^n = \emptyset$ and $X_i^1 \supseteq X_i^2 \supseteq \cdots$. Consequently, for any large enough n,

$$\sigma_i\big(X_i^n\big) \sup_{(\nu,\rho) \in M^2} \big[U_i(\nu) - U_i(\rho)\big] < \tfrac{\varepsilon}{4}.$$

Hence, for any sufficiently large n,

$$\begin{aligned}
\int_{X_i} U_i\big(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i})\big) \, \mathrm{d}\sigma_i &= \int_{X_i \setminus X_i^n} U_i\big(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i})\big) \, \mathrm{d}\sigma_i + \int_{X_i^n} U_i\big(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i})\big) \, \mathrm{d}\sigma_i \\
&> \int_{X_i \setminus X_i^n} U_i(\cdot, \sigma_{-i}) \, \mathrm{d}\sigma_i + \tfrac{\varepsilon}{4} + \int_{X_i^n} U_i\big(f(\cdot), N_{\frac{1}{n}}(\sigma_{-i})\big) \, \mathrm{d}\sigma_i \\
&> U_i(\sigma_i, \sigma_{-i}) - \tfrac{\varepsilon}{2},
\end{aligned}$$

as desired. □

Define

$$p_i^f := (1-\delta)f(x_i) + \delta\mu_i \quad \text{and} \quad \upsilon_i^f := (1-\delta)f(x_i) + \delta\mu_i^f,$$

where, recall, $\mu_i^f \in M_i$ is defined by $\mu_i^f(B) := \mu_i(f^{-1}(B \cap f(X_i)))$.

By (ii), there exists a neighborhood $O_{\sigma_{-i}}$ of σ_{-i} such that

$$U_i(\mu_i, p_{-i}) > U_i(\mu_i^f, p_{-i}) - \tfrac{\varepsilon}{2}, \quad \text{for all } p_{-i} \in O_{\sigma_{-i}}.$$

This, together with the definitions of p_i^f and υ_i^f, gives, for any p_{-i} in some neighborhood of σ_{-i},

$$\begin{aligned} U_i(p_i^f, p_{-i}) &= (1-\delta)U_i(f(x_i), p_{-i}) + \delta U_i(\mu_i, p_{-i}) \\ &> (1-\delta)U_i(f(x_i), p_{-i}) + \delta U_i(\mu_i^f, p_{-i}) - \tfrac{\varepsilon}{2} \\ &= U_i(\upsilon_i^f, p_{-i}) - \tfrac{\varepsilon}{2}. \end{aligned} \tag{17}$$

In addition, the definitions of σ_i and υ_i^f entail

$$\begin{aligned} U_i(\upsilon_i^f, p_{-i}) &= \int_{X_i} U_i(\cdot, p_{-i})\, \mathrm{d}\upsilon_i^f = (1-\delta)U_i(f(x_i), p_{-i}) + \delta \int_{X_i} U_i(\cdot, p_{-i})\, \mathrm{d}\mu_i^f \\ &= (1-\delta)U_i(f(x_i), p_{-i}) + \delta \int_{X_i} U_i(f(\cdot), p_{-i})\, \mathrm{d}\mu_i = \int_{X_i} U_i(f(\cdot), p_{-i})\, \mathrm{d}\sigma_i. \end{aligned} \tag{18}$$

Consequently, for every p_{-i} in some neighborhood of σ_{-i} we have

$$U_i(p_i^f, p_{-i}) > U_i(\upsilon_i^f, p_{-i}) - \tfrac{\varepsilon}{2} = \int_{X_i} U_i(f(\cdot), p_{-i})\, \mathrm{d}\sigma_i - \tfrac{\varepsilon}{2} > U_i(\sigma_i, \sigma_{-i}) - \varepsilon.$$

Here, the first inequality follows from (17), the second inequality is given by Claim 1, and the equality is a consequence of (18). We conclude that $G_{(\delta,\mu)}$ is payoff secure. □

5.2. Proof of Lemma 4

Lemma 4. *Suppose that G is generically entirely payoff secure and generically locally equi-upper semicontinuous. Then G satisfies Condition* (A)*.*

Proof. Suppose that G is generically entirely payoff secure and generically locally equi-upper semicontinuous. Let $(\mu_1, \ldots, \mu_N)$ be the measure from $\widetilde{M}$ provided by the generic local equi-upper semicontinuity of G. Fix i and $\varepsilon > 0$. By the generic entire payoff security of G, for each $x_i \in X_i$ and $k \in \mathbb{N}$ there exist $g_k(x_i) \in X_i$ and $\beta_k(x_i) > 0$ such that for every $y_{-i} \in X_{-i}$, there exists a neighborhood $O_{y_{-i}}$ of y_{-i} such that

$$\begin{cases} u_i(g_k(x_i), O_{y_{-i}}) > u_i(x_i, y_{-i}) - \varepsilon & \text{if } x_i \in K_i \cup C_i, \\ g_k(x_i) \in N_{\frac{1}{k}}(x_i) \text{ and } u_i(g_k(x_i), O_{y_{-i}}) > u_i(N_{\beta_k(x_i)}(x_i), y_{-i}) - \varepsilon & \text{if } x_i \in A_i \setminus C_i, \end{cases}$$

where C_i is a countable subset of A_i. Moreover, since G is entirely payoff secure over $\bigtimes_j K_j$, we may take $g_k(x_i) = x_i$ for $x_i \in K_i$, and there is no loss of generality in assuming that $\beta_k(x_i) < \frac{1}{k}$.

Now, because $A_i \setminus C_i \subseteq X_i$ and X_i is compact and metric, $A_i \setminus C_i$ is separable. Hence, there is a countable subset $\{x_i^1, x_i^2, \ldots\}$ of $A_i \setminus C_i$ such that

$$\bigcup_{\ell} \bigl(N_{\beta_k(x_i^\ell)}(x_i^\ell) \cap (A_i \setminus C_i)\bigr) = \bigcup_{x_i \in A_i \setminus C_i} \bigl(N_{\beta_k(x_i)}(x_i) \cap (A_i \setminus C_i)\bigr).$$

Now define

$$V_{k1} := N_{\beta_k(x_i^1)}(x_i^1) \cap (A_i \setminus C_i),$$

$$V_{kl} := \bigl(N_{\beta_k(x_i^l)}(x_i^l) \cap (A_i \setminus C_i)\bigr) \setminus \left(\bigcup_{m=1}^{l-1} V_m\right), \quad l \in \{2, 3, \ldots\},$$

and $f_k : X_i \to X_i$ by

$$f_k(x_i) := \begin{cases} g_k(x_i^\ell) & \text{if } x_i \in V_{k\ell}, \\ g_k(x_i) & \text{if } x_i \in K_i \cup C_i. \end{cases}$$

Clearly, $f_k(A_i \setminus C_i) = \{f_k(x_i^1), f_k(x_i^2), \ldots\}$. Moreover, $f_k(x_i) \in N_{\frac{2}{k}}(x_i)$ for $x_i \in A_i \setminus C_i$. In fact, for $x_i \in A_i \setminus C_i$, we have $x_i \in V_{k\ell}$ for some $V_{k\ell}$, and so

$$d_i\big(x_i, f_k(x_i)\big) = d_i\big(x_i, f_k\big(x_i^\ell\big)\big) \leqslant d_i\big(x_i, x_i^\ell\big) + d_i\big(x_i^\ell, f_k\big(x_i^\ell\big)\big) < \beta_k\big(x_i^\ell\big) + \tfrac{1}{k} < \tfrac{2}{k},$$

where d_i denotes the metric associated with X_i.

We now show that each f_k is Borel measurable. Since $f_k(A_i \setminus C_i) = \{f_k(x_i^1), f_k(x_i^2), \ldots\}$ and $X_i \setminus (A_i \setminus C_i) = K_i \cup C_i$ is countable, $f_k(X_i) \subseteq f_k(A_i \setminus C_i) \cup f_k(C_i \cup K_i)$ is a countable set. Pick a Borel set B in $f_k(X_i)$, and let $\{a_i^1, a_i^2, \ldots\} \cup \{b_i^1, b_i^2, \ldots\}$ be an enumeration of B. There is no loss of generality in assuming that $\{a_i^1, a_i^2, \ldots\} \subseteq f_k(A_i \setminus C_i)$ and $\{b_i^1, b_i^2, \ldots\} \subseteq f_k(C_i \cup K_i)$. The set $f_k^{-1}(\{b_i^1, b_i^2, \ldots\})$ is clearly countable, and each $f_k^{-1}(\{a_i^l\})$ equals $V_{k\ell}$ for some ℓ. Therefore, the set $f_k^{-1}(B)$ is Borel measurable. Since B was arbitrary, f_k is Borel measurable.

Now fix $x_i \in X_i$, k, and $y_{-i} \in X_{-i}$. Suppose that $x_i \in A_i \setminus C_i$. Then $x_i \in V_{k\ell}$ for some ℓ, and therefore there is a neighborhood $O_{y_{-i}}$ of y_{-i} such that

$$u_i\big(f_k(x_i), O_{y_{-i}}\big) = u_i\big(g_k\big(x_i^\ell\big), O_{y_{-i}}\big) > u_i(x_i, y_{-i}) - \varepsilon.$$

We have obtained a sequence (f_k) of Borel measurable maps satisfying item (a) of Condition (A). It remains to show that (f_k) satisfies item (b). This flows from the following observations. The construction of f_k entails $f_k(x_i) \to x_i$ (as $k \to \infty$) for all $x_i \in A_i \setminus C_i$, and since $X_i \setminus (A_i \setminus C_i) = K_i \cup C_i$ is countable, $f_k(x_i) = x_i$ for $x_i \in K_i$, and $\mu_i \in \widetilde{M}_i$, it follows that $f_k(x_i) \to x_i$ μ_i-almost everywhere in X_i. This, together with generic local equi-upper semicontinuity of G, implies that there exists k such that f_k satisfies item (b) of Condition (A). □

5.3. *Proof of Lemma 5*

Lemma 5. *Suppose that G is a strongly quasiconcave game satisfying* (LC)*. Then, for some $\alpha \in (0, 1)$, $G_{(\delta,\mu)}$ is quasiconcave for every $(\delta, \mu) \in [0, \alpha] \times \widehat{M}$.*

Proof. Suppose that G is a strongly quasiconcave game satisfying (LC). Fix $\mu \in \widehat{M}$. The power set of $\{1, \ldots, N\}$ is denoted as $2^{\{1,\ldots,N\}}$, and we write $\#2^{\{1,\ldots,N\}}$ for the cardinality of $2^{\{1,\ldots,N\}}$. Choose $\delta > 0$ small enough to ensure that

$$\left((1-\delta)^N \min_i \epsilon_i - \#2^{\{1,\ldots,N\}} \sum_{k=1}^N \delta^k (1-\delta)^{N-k}\right) K > 0, \tag{19}$$

where

$$K := \max_i \sup_{(\mu,\nu) \in M^2} \big[U_i(\mu) - U_i(\nu)\big],$$

and each ϵ_i is given by Condition (LC).

Fix i, $x_{-i} \in X_{-i}$, $\{x_i, y_i\} \subseteq X_i$, and $z_i \in \operatorname{co}\{x_i, y_i\}$. We wish to show that

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant \min_{s_i \in \{x_i, y_i\}} u_i^{(\delta,\mu)}(s_i, x_{-i}). \tag{20}$$

It is straightforward to show that quasiconcavity of $u_i(\cdot, x_{-i})$ implies that there exists $q_i \in \operatorname{co}\{x_i, y_i\}$ such that $u_i(\cdot, x_{-i})$ is increasing on $\operatorname{co}\{x_i, q_i\}$ and decreasing on $\operatorname{co}\{q_i, y_i\}$. (The proof of this fact is omitted in the interest of brevity.)

It is convenient to first consider the case when $q_i \in \{x_i, y_i\}$. Say $q_i = y_i$ (a similar argument applies when $q_i = x_i$). Because $u_i(\cdot, x_{-i})$ is increasing on $\operatorname{co}\{x_i, y_i\}$, $u_i(\cdot, x_{-i})$ is decreasing on $\operatorname{co}\{y_i, x_i\}$, and so Condition (LC) gives $w_i \in \operatorname{co}\{y_i, x_i\}$ such that

(a) $u_i(\cdot, y_{-i})$ is concave on $\operatorname{co}\{y_i, w_i\}$ for every $y_{-i} \in X_{-i}$, and
(b) every $a_i \in \operatorname{co}\{w_i, x_i\}$ has a neighborhood O_{a_i} such that

$$\tfrac{u_i(c_i, x_{-i}) - u_i(b_i, x_{-i})}{u_i(b_i, y_{-i}) - u_i(c_i, y_{-i})} \geqslant \epsilon_i,$$

for all $\{c_i, b_i\} \subseteq O_{a_i} \cap \operatorname{co}\{w_i, x_i\}$ with $u_i(c_i, x_{-i}) > u_i(b_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$ with $u_i(b_i, y_{-i}) > u_i(c_i, y_{-i})$.

Because $u_i(\cdot, y_{-i})$ is concave on $\operatorname{co}\{y_i, w_i\}$ for every $y_{-i} \in X_{-i}$, one can show (by means of an argument similar to that of the proof of Lemma 2) that $u_i^{(\delta,\mu)}(\cdot, x_{-i})$ is concave on $\operatorname{co}\{y_i, w_i\}$. Therefore, if $z_i \in \operatorname{co}\{y_i, w_i\}$, then

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant \min_{s_i \in \{y_i, w_i\}} u_i^{(\delta,\mu)}(s_i, x_{-i}),$$

and if $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(y_i, x_{-i})$, (20) follows, as we sought.

We now consider the case when $z_i \in \mathrm{co}\{y_i, w_i\}$ and

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(w_i, x_{-i}) \tag{21}$$

(the remaining case, i.e., the case when $z_i \notin \mathrm{co}\{y_i, w_i\}$ (so that $z_i \in \mathrm{co}\{w_i, x_i\}$) can be dealt with similarly). Consider the neighborhoods O_{a_i} given in (b). Since

$$\{O_{a_i} \cap \mathrm{co}\{w_i, x_i\}\}_{a_i \in \mathrm{co}\{w_i, x_i\}}$$

is a cover for $\mathrm{co}\{w_i, x_i\}$ and $\mathrm{co}\{w_i, x_i\}$ is closed in X_i and hence compact, there exists a finite subcover

$$\{O_{\alpha_i^1} \cap \mathrm{co}\{w_i, x_i\}, \dots, O_{\alpha_i^k} \cap \mathrm{co}\{w_i, x_i\}\}.$$

Without loss of generality, we may set $w_i = \alpha_i^1$ and $x_i = \alpha_i^k$. Also there is no loss of generality in assuming that there is a collection $\{b_i^{12}, \dots, b_i^{k-1k}\}$ with $b_i^{\ell\ell+1} \in O_{\alpha_i^\ell} \cap O_{\alpha_i^{\ell+1}} \cap \mathrm{co}\{w_i, x_i\}$, for all $\ell \in \{1, \dots, k-1\}$, and $\alpha_i^2 \in \mathrm{co}\{b_i^{12}, b_i^{23}\}, \dots, \alpha_i^{k-1} \in \mathrm{co}\{b_i^{k-2k-1}, b_i^{k-1k}\}$.[17]

We show that either

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(b_i^{12}, x_{-i}) \tag{22}$$

holds or (20) holds. Because $u_i(\cdot, x_{-i})$ is decreasing on $\mathrm{co}\{y_i, x_i\}$ and we are assuming that $z_i \in \mathrm{co}\{y_i, \alpha_i^1\} = \mathrm{co}\{y_i, w_i\}$, we have

$$u_i(y_i, x_{-i}) \geqslant u_i(\alpha_i^1, x_{-i}) \geqslant u_i(b_i^{12}, x_{-i}).$$

If $u_i(\alpha_i^1, x_{-i}) = u_i(b_i^{12}, x_{-i})$, strong quasiconcavity gives either $u_i(\alpha_i^1, y_{-i}) \geqslant u_i(y_i, y_{-i})$ for all $y_{-i} \in X_{-i}$ or $u_i(\alpha_i^1, y_{-i}) \geqslant u_i(b_i^{12}, y_{-i})$ for all $y_{-i} \in X_{-i}$. In the first case, we have $u_i^{(\delta,\mu)}(\alpha_i^1, x_{-i}) \geqslant u_i^{(\delta,\mu)}(y_i, x_{-i})$. This, together with (21), gives (20). In the second case, we have $u_i^{(\delta,\mu)}(\alpha_i^1, x_{-i}) \geqslant u_i^{(\delta,\mu)}(b_i^{12}, x_{-i})$, which, combined with (21), yields (22). We have seen that if $u_i(\alpha_i^1, x_{-i}) = u_i(b_i^{12}, x_{-i})$ either the proof is complete or (22) holds.

We now show that the inequality $u_i(\alpha_i^1, x_{-i}) > u_i(b_i^{12}, x_{-i})$ implies (22). Assume $u_i(\alpha_i^1, x_{-i}) > u_i(b_i^{12}, x_{-i})$. Since $\alpha_i^1 = w_i$, condition (b) gives

$$u_i(c_i, x_{-i}) - u_i(b_i, x_{-i}) \geqslant \epsilon_i\big(u_i(b_i, y_{-i}) - u_i(c_i, y_{-i})\big) \tag{23}$$

for all $\{c_i, b_i\} \subseteq O_{\alpha_i^1} \cap \mathrm{co}\{w_i, x_i\}$ with $u_i(c_i, x_{-i}) > u_i(b_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$ with $u_i(b_i, y_{-i}) > u_i(c_i, y_{-i})$. But for any $y_{-i} \in X_{-i}$ such that $u_i(b_i, y_{-i}) \leqslant u_i(c_i, y_{-i})$, (23) is trivially true, so we have

$$u_i(c_i, x_{-i}) - u_i(b_i, x_{-i}) \geqslant \epsilon_i\big(u_i(b_i, y_{-i}) - u_i(c_i, y_{-i})\big)$$

for all $\{c_i, b_i\} \subseteq O_{\alpha_i^1} \cap \mathrm{co}\{w_i, x_i\}$ with $u_i(c_i, x_{-i}) > u_i(b_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$. Hence, because $\{\alpha_i^1, b_i^{12}\} \subseteq O_{\alpha_i^1} \cap \mathrm{co}\{w_i, x_i\}$ and $u_i(\alpha_i^1, x_{-i}) > u_i(b_i^{12}, x_{-i})$, we have

$$u_i(\alpha_i^1, x_{-i}) - u_i(b_i^{12}, x_{-i}) \geqslant \epsilon_i\big(u_i(b_i^{12}, y_{-i}) - u_i(\alpha_i^1, y_{-i})\big) \tag{24}$$

for all $y_{-i} \in X_{-i}$. Eq. (24) yields

$$u_i(\alpha_i^1, x_{-i}) - u_i(b_i^{12}, x_{-i}) \geqslant \epsilon_i \max_{I \subseteq \{1,\dots,N\}} \big(U_i\big((b_i^{12}, x_{-i})_I, \mu_{-I}\big) - U_i\big((\alpha_i^1, x_{-i})_I, \mu_{-I}\big)\big)$$

(if $i \notin I$, then $U_i((b_i^{12}, x_{-i})_I, \mu_{-I}) - U_i((\alpha_i^1, x_{-i})_I, \mu_{-I}) = 0$, and the inequality follows from the fact that $u_i(\alpha_i^1, x_{-i}) \geqslant u_i(b_i^{12}, x_{-i})$). We may therefore write

[17] To see this, let the elements $\alpha_i^1, \dots, \alpha_i^k$ be labeled so that $\alpha_i^\ell = \lambda_\ell w_i + (1-\lambda_\ell)x_i$ for each ℓ and $\lambda_1 \geqslant \dots \geqslant \lambda_k$. Let

$$\gamma_1 := \inf_{\substack{\gamma \in [0,1] \\ :\gamma\alpha_i^1 + (1-\gamma)x_i \in O_{\alpha_i^1}}} \gamma,$$

and add the set $O_{d_1} \cap \mathrm{co}\{\alpha_i^1, x_i\}$, where $d_i := \gamma_1 \alpha_i^1 + (1-\gamma_1)x_i$, to the finite subcover

$$\{O_{\alpha_i^1} \cap \mathrm{co}\{w_i, x_i\}, \dots, O_{\alpha_i^k} \cap \mathrm{co}\{w_i, x_i\}\}.$$

It is easy to see that $O_{\alpha_i^1} \cap O_{d_1} \cap \mathrm{co}\{w_i, x_i\} \neq \emptyset$. Therefore, after the addition of the set O_{d_1} and an appropriate relabeling of the sets in the new subcover, there exists a point b_i^{12} with the desired properties. Finitely many iterations of this argument generate the collection $\{b_i^{12}, \dots, b_i^{k-1k}\}$.

$$\begin{aligned}
&u_i^{(\delta,\mu)}\big(\alpha_i^1, x_{-i}\big) - u_i^{(\delta,\mu)}\big(b_i^{12}, x_{-i}\big)\\
&\quad= (1-\delta)^N\big(u_i\big(\alpha_i^1, x_{-i}\big) - u_i\big(b_i^{12}, x_{-i}\big)\big)\\
&\qquad+\sum_{k=1}^{N}\delta^k(1-\delta)^{N-k}\sum_{\substack{I\subseteq\{1,\ldots,N\}\\ :\#I=N-k}}\big(U_i\big(\big(\alpha_i^1, x_{-i}\big)_I, \mu_{-I}\big) - U_i\big(\big(b_i^{12}, x_{-i}\big)_I, \mu_{-I}\big)\big)\\
&\quad\geqslant (1-\delta)^N\epsilon_i \max_{I\subseteq\{1,\ldots,N\}}\big(U_i\big(\big(b_i^{12}, x_{-i}\big)_I, \mu_{-I}\big) - U_i\big(\big(\alpha_i^1, x_{-i}\big)_I, \mu_{-I}\big)\big)\\
&\qquad+\sum_{k=1}^{N}\delta^k(1-\delta)^{N-k}\sum_{\substack{I\subseteq\{1,\ldots,N\}\\ :\#I=N-k}}\big(U_i\big(\big(\alpha_i^1, x_{-i}\big)_I, \mu_{-I}\big) - U_i\big(\big(b_i^{12}, x_{-i}\big)_I, \mu_{-I}\big)\big)\\
&\quad\geqslant (1-\delta)^N\epsilon_i \max_{I\subseteq\{1,\ldots,N\}}\big(U_i\big(\big(b_i^{12}, x_{-i}\big)_I, \mu_{-I}\big) - U_i\big(\big(\alpha_i^1, x_{-i}\big)_I, \mu_{-I}\big)\big)\\
&\qquad-\sum_{k=1}^{N}\delta^k(1-\delta)^{N-k}\#2^{\{1,\ldots,N\}}\max_{I\subseteq\{1,\ldots,N\}}\big(U_i\big(\big(b_i^{12}, x_{-i}\big)_I, \mu_{-I}\big) - U_i\big(\big(\alpha_i^1, x_{-i}\big)_I, \mu_{-I}\big)\big)\\
&\quad\geqslant 0,
\end{aligned}$$

where the last inequality uses (19). Hence, $u_i^{(\delta,\mu)}(\alpha_i^1, x_{-i}) \geqslant u_i^{(\delta,\mu)}(b_i^{12}, x_{-i})$. Combining this inequality with (21) we see that (22) holds, as we sought.

Repeating the argument above, with b_i^{12} playing the role of α_i^1 and α_i^2 playing the role of b_i^{12}, gives either $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(y_i, x_{-i})$ or $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(\alpha_i^2, x_{-i})$. If $\alpha_i^2 = \alpha_i^k = x_i$, then $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(x_i, x_{-i})$, which implies (20). So if $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(y_i, x_{-i})$ or $\alpha_i^2 = \alpha_i^k = x_i$, the proof is complete. If $\alpha_i^2 \neq \alpha_i^k$, then finitely many iterations of the above argument give (20).

We now turn to the case when $q_i = z_i$. In this case Condition (LC) gives $w_i^1 \in \mathrm{co}\{x_i, z_i\}$ and $w_i^2 \in \mathrm{co}\{z_i, y_i\}$ satisfying the following:

(i) For every $y_{-i} \in X_{-i}$, $u_i(\cdot, y_{-i})$ is concave on $\mathrm{co}\{w_i^1, z_i\}$ and on $\mathrm{co}\{z_i, w_i^2\}$.

(ii) Every $a_i \in \mathrm{co}\{w_i^1, x_i\}$ has a neighborhood O_{a_i} such that

$$\tfrac{u_i(c_i,x_{-i})-u_i(b_i,x_{-i})}{u_i(b_i,y_{-i})-u_i(c_i,y_{-i})} \geqslant \epsilon_i,$$

for all $c_i, b_i \in O_{a_i} \cap \mathrm{co}\{w_i^1, x_i\}$ with $u_i(c_i, x_{-i}) > u_i(b_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$ with $u_i(b_i, y_{-i}) > u_i(c_i, y_{-i})$.

(iii) Every $a_i \in \mathrm{co}\{w_i^2, y_i\}$ has a neighborhood V_{a_i} such that

$$\tfrac{u_i(c_i,x_{-i})-u_i(b_i,x_{-i})}{u_i(b_i,y_{-i})-u_i(c_i,y_{-i})} \geqslant \epsilon_i,$$

for all $c_i, b_i \in V_{a_i} \cap \mathrm{co}\{w_i^2, y_i\}$ with $u_i(c_i, x_{-i}) > u_i(b_i, x_{-i})$ and for all $y_{-i} \in X_{-i}$ with $u_i(b_i, y_{-i}) > u_i(c_i, y_{-i})$.

Item (i) implies that $u_i(\cdot, y_{-i})$ is concave on $\mathrm{co}\{w_i^1, w_i^2\}$ for every $y_{-i} \in X_{-i}$. Consequently, by an argument analogous to that of the proof of Lemma 2, $u_i^{(\delta,\mu)}(\cdot, x_{-i})$ is concave on $\mathrm{co}\{w_i^1, w_i^2\}$, so either $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(w_i^1, x_{-i})$ or $u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}(w_i^2, x_{-i})$, and there is no loss of generality in assuming that

$$u_i^{(\delta,\mu)}(z_i, x_{-i}) \geqslant u_i^{(\delta,\mu)}\big(w_i^1, x_{-i}\big). \tag{25}$$

Define

$$\gamma_1 := \sup_{\substack{\gamma\in[0,1]\\ :u_i(\gamma x_i+(1-\gamma)z_i,x_{-i})=u_i(z_i,x_{-i})}} \gamma, \qquad \gamma_2 := \sup_{\substack{\gamma\in[0,1]\\ :u_i(\gamma y_i+(1-\gamma)z_i,x_{-i})=u_i(z_i,x_{-i})}} \gamma,$$

$e_i^1 := \gamma_1 x_i + (1-\gamma_1)z_i$, and $e_i^2 := \gamma_2 y_i + (1-\gamma_2)z_i$.

Reasoning as before, it is possible to obtain, using item (ii), a finite subcover

$$\big\{O_{\alpha_i^1} \cap \mathrm{co}\big\{w_i^1, x_i\big\}, \ldots, O_{\alpha_i^k} \cap \mathrm{co}\big\{w_i^1, x_i\big\}\big\}$$

of

$$\big\{O_{a_i} \cap \mathrm{co}\big\{w_i^1, x_i\big\}\big\}_{a_i\in\mathrm{co}\{w_i^1,x_i\}},$$

and a collection $\{b_i^{12},\dots,b_i^{k-1k}\}$ with $b_i^{\ell\ell+1}\in O_{\alpha_i^\ell}\cap O_{\alpha_i^{\ell+1}}\cap\mathrm{co}\{w_i^1,x_i\}$, for all $\ell\in\{1,\dots,k-1\}$, and $\alpha_i^2\in\mathrm{co}\{b_i^{12},b_i^{23}\}$, $\dots,\alpha_i^{k-1}\in\mathrm{co}\{b_i^{k-2k-1},b_i^{k-1k}\}$, and there is no loss of generality in assuming that $w_i^1=\alpha_i^1$, $e_i^1=\alpha_i^\ell$ for some ℓ, and $x_i=\alpha_i^k$.

Similarly, item (iii) gives a finite subcover

$$\left\{V_{\beta_i^1}\cap\mathrm{co}\left\{w_i^2,y_i\right\},\dots,V_{\beta_i^l}\cap\mathrm{co}\left\{w_i^2,y_i\right\}\right\}$$

of

$$\left\{V_{a_i}\cap\mathrm{co}\left\{w_i^2,y_i\right\}\right\}_{a_i\in\mathrm{co}\{w_i^2,y_i\}},$$

and a collection $\{c_i^{12},\dots,c_i^{l-1l}\}$ with $c_i^{\ell\ell+1}\in V_{\alpha_i^\ell}\cap V_{\alpha_i^{\ell+1}}\cap\mathrm{co}\{w_i^2,y_i\}$, for all $\ell\in\{1,\dots,l-1\}$, and $\beta_i^2\in\mathrm{co}\{c_i^{12},c_i^{23}\},\dots,\beta_i^{l-1}\in\mathrm{co}\{c_i^{l-2l-1},c_i^{l-1l}\}$, and there is no loss of generality in assuming that $w_i^2=\beta_i^1$, $e_i^2=\beta_i^\ell$ for some ℓ, and $y_i=\beta_i^l$.

Now one can use the argument from the case when $q_i=y_i$, with e_i^2 replacing y_i (observe that the definition of e_i^2 entails $z_i\in\mathrm{co}\{x_i,e_i^2\}$ and that $u_i(\cdot,x_{-i})$ is increasing on $\mathrm{co}\{x_i,e_i^2\}$), to conclude that either $u_i^{(\delta,\mu)}(\alpha_i^1,x_{-i})\geqslant u_i^{(\delta,\mu)}(e_i^2,x_{-i})$ or $u_i^{(\delta,\mu)}(\alpha_i^1,x_{-i})\geqslant u_i^{(\delta,\mu)}(b_i^{12},x_{-i})$. This, together with (25), gives either

$$u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}\left(e_i^2,x_{-i}\right) \tag{26}$$

or

$$u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}\left(b_i^{12},x_{-i}\right). \tag{27}$$

In the second case, a new iteration of the same argument, with b_i^{12} playing the role of α_i^1 and α_i^2 playing the role of b_i^{12}, gives either $u_i^{(\delta,\mu)}(b_i^{12},x_{-i})\geqslant u_i^{(\delta,\mu)}(\alpha_i^2,x_{-i})$ or $u_i^{(\delta,\mu)}(b_i^{12},x_{-i})\geqslant u_i^{(\delta,\mu)}(e_i^2,x_{-i})$. This, together with (27), gives either $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(\alpha_i^2,x_{-i})$ or $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(e_i^2,x_{-i})$. Observe that if $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(\alpha_i^2,x_{-i})$ and $\alpha_i^2=x_i$, the proof is complete. If $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(\alpha_i^2,x_{-i})$ and $\alpha_i^2\neq x_i$, the argument can be applied one more time. If $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(e_i^2,x_{-i})$, we are in the first case, which is considered next.

In the first case, we consider two (exhaustive) subcases: $e_i^2\in\mathrm{co}\{z_i,w_i^2\}$ and $e_i^2\in\mathrm{co}\{w_i^2,y_i\}$. Suppose first that $e_i^2\in\mathrm{co}\{z_i,w_i^2\}$. In this case, we have

$$u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}\left(w_i^2,x_{-i}\right). \tag{28}$$

In fact, it is easy to see that this inequality is implied by (25)–(26), together with the fact that $u_i^{(\delta,\mu)}(\cdot,x_{-i})$ is concave on $\mathrm{co}\{w_i^1,w_i^2\}$. Now one can use the argument from the case when $q_i=y_i$, with e_i^1 replacing y_i (observe that the definition of e_i^1 entails $z_i\in\mathrm{co}\{y_i,e_i^1\}$ and that $u_i(\cdot,x_{-i})$ is increasing on $\mathrm{co}\{y_i,e_i^1\}$), to conclude that

$$u_i^{(\delta,\mu)}\left(\beta_i^1,x_{-i}\right)\geqslant u_i^{(\delta,\mu)}\left(c_i^{12},x_{-i}\right). \tag{29}$$

Indeed, consider the case when $\beta_i^1\neq c_i^{12}$ (otherwise the inequality is trivial), and note that because $u_i(\cdot,x_{-i})$ is decreasing on $\mathrm{co}\{e_i^1,y_i\}$, we have

$$u_i\left(e_i^1,x_{-i}\right)\geqslant u_i\left(\beta_i^1,x_{-i}\right)\geqslant u_i\left(c_i^{12},x_{-i}\right).$$

If $u_i(\beta_i^1,x_{-i})=u_i(c_i^{12},x_{-i})$, then, because $e_i^2\in\mathrm{co}\{z_i,w_i^2\}$ and $\beta_i^1=w_i^2\neq c_i^{12}$, strong quasiconcavity gives $u_i(\beta_i^1,y_{-i})\geqslant u_i(c_i^{12},y_{-i})$ for all $y_{-i}\in X_{-i}$, which implies (29). If, on the other hand, $u_i(\beta_i^1,x_{-i})>u_i(c_i^{12},x_{-i})$, then the argument for the case when $q_i=y_i$ and $u_i(\alpha_i^1,x_{-i})>u_i(b_i^{12},x_{-i})$, with β_i^1 and c_i^{12} playing the roles of α_i^1 and b_i^{12} respectively, gives (29). Combining (28) and (29) yields $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(c_i^{12},x_{-i})$, and if $c_i^{12}=y_i$, the proof is complete.

Now consider the case when $e_i^2\in\mathrm{co}\{w_i^2,y_i\}$. Recall that $e_i^2=\beta_i^\ell$ for some ℓ. If $\beta_i^\ell=\beta_i^l=y_i$, by (26) there is nothing to prove. Otherwise, one can reason as before to show that $u_i^{(\delta,\mu)}(e_i^2,x_{-i})\geqslant u_i^{(\delta,\mu)}(c_i^{\ell\ell+1},x_{-i})$, which, combined with (26) gives $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(c_i^{\ell\ell+1},x_{-i})$, and if $c_i^{\ell\ell+1}=y_i$ the proof is complete.

The preceding argument can be repeated a finite number of times to obtain the desired inequality, (20).

Finally, the case when $q_i\neq z_i$ can be handled as follows. Say $z_i\in\mathrm{co}\{x_i,q_i\}$. Then the argument for the case when $q_i\in\{x_i,y_i\}$ can be used (with q_i playing the role of y_i) to show that $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant\min_{s_i\in\{x_i,q_i\}}u_i^{(\delta,\mu)}(s_i,x_{-i})$. If $u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(x_i,x_{-i})$, the proof is complete. If, on the other hand,

$$u_i^{(\delta,\mu)}(z_i,x_{-i})\geqslant u_i^{(\delta,\mu)}(q_i,x_{-i}), \tag{30}$$

the argument for the case when $q_i=z_i$ gives

$$u_i^{(\delta,\mu)}(q_i,x_{-i})\geqslant\min_{s_i\in\{x_i,y_i\}}u_i^{(\delta,\mu)}(s_i,x_{-i}),$$

which, combined with (30), gives (20). □

References

Aliprantis, C.D., Border, K.C., 2006. Infinite Dimensional Analysis. Springer, Berlin.
Austen-Smith, D., Banks, J.S., 2005. Positive Political Theory II: Strategy and Structure. University of Michigan Press.
Barelli, P., Soza, I., 2010. On the existence of Nash equilibria in discontinuous and qualitative games. Mimeo.
Baye, M.R., Morgan, J., 2002. Winner-take-all price competition. Econ. Theory 19, 271–282.
Berge, C., 1957. Théorie Générale des Jeux à n Personnes. Mémorial des Sciences Mathématiques, Fascicule 138. Gauthier–Villars, Paris.
Bester, H., 1992. Bertrand equilibrium in a differentiated duopoly. Int. Econ. Rev. 33, 433–448.
Bourgin, D.G., 1943. Linear topological spaces. Amer. J. Math. 65, 637–659.
Carbonell-Nicolau, O., 2010a. The existence of perfect equilibrium in discontinuous games. Mimeo.
Carbonell-Nicolau, O., 2010b. Perfect and limit admissible perfect equilibria in discontinuous games.
Carmona, G., 2010. A unifying approach to the existence of Nash equilibrium in discontinuous games. Mimeo.
Debreu, G., 1952. A social equilibrium existence theorem. Proc. Nat. Acad. Sci. 38, 886–893.
Fan, K., 1952. Fixed-point and minimax theorems in locally convex topological linear spaces. Proc. Nat. Acad. Sci. 38, 121–126.
Friedman, J.W., 1977. Oligopoly and the Theory of Games. North-Holland, Amsterdam.
Glicksberg, I.L., 1952. A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. Proc. Amer. Math. Soc. 3, 170–174.
Karlin, S., 1959. Mathematical Methods and Theory in Games, Programming and Economics. Addison–Wesley, Reading, MA.
Lazear, E.P., Rosen, S., 1981. Rank-order tournaments as optimum labor contracts. J. Polit. Economy 89, 841–864.
McLennan, A., Monteiro, P.K., Tourky, R., 2009. Games with discontinuous payoffs: a strengthening of Reny's existence theorem. Mimeo.
Monteiro, P.K., Page, F.H., 2007. Uniform payoff security and Nash equilibrium in compact games. J. Econ. Theory 134, 566–575.
Novshek, W., 1985. On the existence of Cournot equilibrium. Rev. Econ. Stud. 52, 85–98.
Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica 67, 1029–1056.
Selten, R., 1975. Reexamination of the perfectness concept for equilibrium points in extensive games. Int. J. Game Theory 4, 25–55.
Simon, L.K., Stinchcombe, M.B., 1995. Equilibrium refinement for infinite normal-form games. Econometrica 63, 1421–1443.
Tullock, G., 1980. Efficient rent-seeking. In: Buchanan, J.M., Tollison, R.D., Tullock, G. (Eds.), Toward a Theory of the Rent-Seeking Society. Texas A&M Press, College Station.
van Damme, E.E.C., 2002. Stability and Perfection of Nash Equilibria. Springer-Verlag, Berlin.