

ON EQUILIBRIUM REFINEMENT FOR DISCONTINUOUS GAMES

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In moving from finite-action to infinite-action games, standard refinements of the Nash equilibrium concept cease to satisfy certain “natural” properties. For instance, perfect equilibria in compact, continuous games need not be admissible. This paper highlights additional properties of two standard refinement specifications that are not inherited by supersets of the set of finite games. The analysis reveals the following about the behavior of perfectness and strategic stability within a class of (possibly) discontinuous games: (1) Equilibria that assign positive probability to the interior of the set of strategies weakly dominated for some player can be chosen; (2) nonadmissible equilibria need not be ruled out when they are weakly dominated by admissible perfect equilibria; and (3) nonadmissible equilibria may be selected when admissible equilibria are ruled out.

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1. Introduction

The notion of *perfect equilibrium* was introduced by Selten (1975) as a refinement of the Nash equilibrium concept. Perfect equilibria are Nash equilibria that are immune to some slight trembles of the players’ actions. That is, Nash equilibria survive perfectness if they are good approximations of equilibrium behavior in some perturbed game in which the players make slight mistakes in the execution of their strategies. Kohlberg and Mertens’ (1986) *strategic stability* refines the set of perfect equilibria by requiring that equilibria be robust to *any* slight tremble of the players’ actions.

While Selten’s (1975) and Kohlberg and Mertens’ (1986) approaches to refinement are specific to games with finitely many actions, some authors have studied infinite-game extensions of the original equilibrium concepts (cf. Simon and Stinchcombe, 1995; Al-Najjar, 1995; Carbonell-Nicolau, 2011, 2011a, 2011b).

It is well known that for finite games, perfect equilibria are *admissible*, i.e., they put no mass on weakly dominated strategies. This property is no longer satisfied as one expands the universe of games under consideration. For infinite games, existence and admissibility are not generally compatible. In fact, there are (compact, continuous) games that have a *unique* Nash equilibrium in weakly dominated strategies (e.g., Simon and Stinchcombe, 1995, Example 2.1).

In an attempt to better understand the failure of admissibility in infinite games, this paper highlights additional properties of perfectness and stability for infinite normal-form games.

Some important remarks revolving around the admissibility of refined equilibrium points have been furnished elsewhere. For instance, for continuous games, perfect equilibria can be shown to satisfy a weakening of admissibility, termed *limit admissibility* in Simon and Stinchcombe (1995), and requiring that equilibria put mass only on the limits of weakly undominated strategies. However, it is possible to generate relatively simple examples of discontinuous games in which perfectness fails limit admissibility (Carbonell-Nicolau, 2011a).

Three additional observations are provided in this paper. We consider a class of (possibly) discontinuous games for which the existence of stable sets (and hence the existence of perfect equilibria) has been established elsewhere (Carbonell-Nicolau, 2011b), and point out the following:

1. Stability and perfectness fail limit admissibility.
2. Stability and perfectness need not rule out nonadmissible equilibria when they are weakly dominated by admissible perfect equilibria.
3. Stability and perfectness may select nonadmissible equilibria and, at the same time, rule out admissible equilibria.

2. Preliminaries

A *normal-form game* is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty action space for player i , and $u_i : X \rightarrow \mathbb{R}$, a bounded and Borel measurable map with domain $X := \times_{i=1}^N X_i$, denotes player i 's payoff function. When X_i is compact and metric for each $i \in \{1, \dots, N\}$, G is called a *compact metric game*.

The *mixed extension* of G is the game $\overline{G} = (M_i, U_i)_{i=1}^N$, where, for each i , M_i denotes the set of Borel probability measures on X_i , endowed with the weak* topology, and $U_i : M \rightarrow \mathbb{R}$ is defined by

$$U_i(\mu) := \int_X u_i d\mu,$$

where $M := \times_{i=1}^N M_i$.

Given a compact, metric game $G = (X_i, u_i)_{i=1}^N$, the set M , together with the Prokhorov metric on M , can be viewed as a metric space.^a The Prokhorov metric $\varrho : M^2 \rightarrow \mathbb{R}$ is defined as

$$\varrho(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon, \text{ for all } B\},$$

where

$$B^\varepsilon := \{x \in X : d(x, y) < \varepsilon \text{ for some } y \in B\},$$

and d denotes the metric associated with X .

Given a player i , the set $\times_{j \neq i} X_j$ (respectively, $\times_{j \neq i} M_j$) is denoted as X_{-i} (respectively, M_{-i}), and, given i , $x_i \in X_i$ (respectively, $\mu_i \in M_i$), and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in X_{-i}$ (respectively, $(\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_N) \in M_{-i}$), we slightly abuse notation and write (x_i, x_{-i}) (respectively, (μ_i, μ_{-i})) for (x_1, \dots, x_N) (respectively, (μ_1, \dots, μ_N)).

The formal definition of the solution concepts considered in this paper requires the following notation.

A measure μ_i in M_i is said to be *strictly positive* if $\mu_i(O) > 0$ for every nonempty open subset O of X_i .

For each i , let \widehat{M}_i be the set of all strictly positive members of M_i , and define $\widehat{M} := \times_i \widehat{M}_i$. For $\nu = (\nu_1, \dots, \nu_N) \in \widehat{M}$ and $\delta = (\delta_1, \dots, \delta_N) \in [0, 1]^N$, define

$$M_i(\delta_i \nu_i) := \{\mu_i \in M_i : \mu_i \geq \delta_i \nu_i\}$$

and $M(\delta \nu) := \times_i M_i(\delta_i \nu_i)$. The game

$$\overline{G}_{\delta \nu} = (M_i(\delta_i \nu_i), U_i|_{M(\delta \nu)})_{i=1}^N$$

is called a *Selten perturbation* of G . We often work with perturbations $\overline{G}_{\delta \nu}$ satisfying $\delta_1 = \dots = \delta_N$. When referring to these objects, we simply write $\overline{G}_{\delta \nu}$ with $\delta = \delta_1 = \dots = \delta_N$.

Definition 1. A strategy profile $x = (x_1, \dots, x_N) \in X$ is a *Nash equilibrium* of $G = (X_i, u_i)_{i=1}^N$ if for each i , $u_i(x) \geq u_i(y_i, x_{-i})$ for every $y_i \in X_i$.

In this paper we focus on two refinements of the Nash equilibrium concept: *Trembling-hand perfectness* and *stability*. We emphasize properties of these two solution concepts within a class of games whose members possess stable sets (and hence trembling-hand perfect equilibria). The relevant existence results, which have been proven elsewhere, are stated below.

The following notion of perfectness extends Selten’s (1975) concept to infinite games. Alternative equivalent formulations of perfectness can be found in Carbonell-Nicolau (2011a).

^aFor compact metric games the weak* topology on M coincides with the topology induced by the Prokhorov metric on M .

Definition 2. A strategy profile $\mu \in M$ is a *trembling-hand perfect (thp) equilibrium* of G if there are sequences (δ^n) , (ν^n) , and (μ^n) such that $(0, 1)^N \ni \delta^n \rightarrow 0$, $\nu^n \in \widehat{M}$, $\mu^n \rightarrow \mu$, and each μ^n is a Nash equilibrium of the perturbed game $\overline{G}_{\delta^n, \nu^n}$.

The notion of strategic stability, introduced in Kohlberg and Mertens (1986) for finite games, refines the set of trembling-hand perfect equilibria.

For $\emptyset \neq E \subseteq M$ and $\mu \in M$, define

$$\varrho(\mu, E) := \inf \{ \varrho(\mu, \nu) : \nu \in E \}.$$

For $\varepsilon > 0$ and $E \subseteq M$, a profile $\mu \in M$ is said to be ε -close to E if $\varrho(\mu, E) < \varepsilon$.

Given a game $G = (X_i, u_i)_{i=1}^N$, let \mathcal{S}_G be the family of all nonempty closed sets E of Nash equilibria of \overline{G} with the following property: for each $\varepsilon > 0$, there exists $\alpha > 0$ such that for each $\delta \in (0, 1)^N$ and every $\nu \in \widehat{M}$, the perturbed game $\overline{G}_{\delta, \nu}$ has a Nash equilibrium ε -close to E .

Definition 3. A set of strategy profiles in M is a *stable set* of G if it is a minimal element of the set \mathcal{S}_G ordered by set inclusion.

We highlight properties of stability and perfectness for members of a collection of games defined in terms of the following condition.

Condition (B). For each i and every $\varepsilon > 0$, there is a sequence (f_k) of Borel measurable maps $f_k : X_i \rightarrow X_i$ such that the following is satisfied:

- (a) For each $(x_i, x_{-i}) \in X_i \times X_{-i}$ and each k , there is a neighborhood $O_{x_{-i}}$ of x_{-i} for which $u_i(f_k(x_i), y_{-i}) > u_i(x_i, x_{-i}) - \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.
- (b) For each $(x_i, x_{-i}) \in X_i \times X_{-i}$, there exists a real number $K_{(x_i, x_{-i})}$ such that for each $k \geq K_{(x_i, x_{-i})}$, there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

Let \mathfrak{g} be the class of all compact, metric games $(X_i, u_i)_{i=1}^N$ satisfying Condition (B) and upper semicontinuity of the sum $\sum_i u_i$. Stable sets can be shown to exist within \mathfrak{g} .

Theorem 1 (Carbonell-Nicolau, 2010b, Theorem 1). *Suppose that G is a member of \mathfrak{g} . Then G has a stable set, and all stable sets of G contain only trembling-hand perfect equilibria, which are also Nash.*

3. Three Properties of Strategic Stability

Definition 4. A strategy $x_i \in X_i$ is *weakly dominated* for i if there exists a strategy $\mu_i \in M_i$ such that $U_i(\mu_i, x_{-i}) \geq U_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, with strict inequality for some x_{-i} .

Definition 5. A strategy profile $\mu \in M$ is *admissible* if $\mu_i(D_i) = 0$ for all i , where D_i denotes the set of strategies weakly dominated for i .

Definition 6. A strategy profile $\mu \in M$ is *limit admissible* if $\mu_i(O_i) = 0$ for all i , where O_i denotes the interior of the set of strategies weakly dominated for i .

We say that stability (respectively, trembling-hand perfectness) satisfies admissibility (respectively, limit admissibility) if it selects sets of admissible (respectively, limit admissible) strategy profiles.

It is well known that Kohlberg and Mertens' (1986) stability satisfies admissibility within the class of finite normal-form games (i.e., the class of normal-form games whose action spaces are finite). It is also well known that there are compact, continuous games that have a *unique* Nash equilibrium in weakly dominated strategies (e.g., Simon and Stinchcombe, 1995, Example 2.1). Consequently, stability (and hence trembling-hand perfectness) fails admissibility within \mathbf{g} . A natural question is whether stability satisfies limit admissibility in \mathbf{g} .

Simon and Stinchcombe (1995) show that perfect equilibria in metric, compact, and continuous games are limit admissible. Carbonell-Nicolau (2011) shows that perfectness need not select limit admissible profiles in a (strict) superset of \mathbf{g} . The following example illustrates that more is true: Members of stable sets in games belonging to \mathbf{g} need not be limit admissible.

Example 1. Consider the two-player game $G = ([0, 1], [0, 1], u_1, u_2)$, where

$$u_1(x_1, x_2) := \begin{cases} 0 & \text{if } x_1 \in [0, \frac{1}{2}) \text{ and } x_2 \in (0, 1], \\ -1 & \text{if } x_1 \in [0, \frac{1}{2}) \text{ and } x_2 = 0, \\ (2x_1 - 1)(2x_2 - 1) & \text{if } (x_1, x_2) \in [\frac{1}{2}, 1] \times [0, 1], \end{cases}$$

and

$$u_2(x_1, x_2) := \begin{cases} 1 & \text{if } x_1 \in [0, \frac{1}{2}] \text{ and } x_2 = 0, \\ -2x_1 + 2 & \text{if } x_1 \in (\frac{1}{2}, 1] \text{ and } x_2 = 0, \\ 2 & \text{if } x_1 \in [0, \frac{1}{2}] \text{ and } x_2 = \frac{1}{2}, \\ -4x_1 + 4 & \text{if } x_1 \in (\frac{1}{2}, 1] \text{ and } x_2 = \frac{1}{2}, \\ 6x_1 & \text{if } x_1 \in [0, \frac{1}{2}] \text{ and } x_2 = 1, \\ -6x_1 + 6 & \text{if } x_1 \in (\frac{1}{2}, 1] \text{ and } x_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to verify that G is a member of \mathbf{g} .

Observe that any $x_1 \in [0, \frac{1}{2})$ is weakly dominated for player 1 by $\frac{1}{2}$. Consequently, the strategy profile $(\rho_1, \rho_2) \in M$ is not limit admissible if $\rho_1([0, \frac{1}{2})) > 0$. We show that some (ρ_1, ρ_2) with $\rho_1([0, \frac{1}{2})) > 0$ is part of a stable set. Let E be a stable set (Theorem 1). We assume that

$$\rho \in E \Rightarrow \rho \left(\left[0, \frac{1}{2} \right) \times [0, 1] \right) = 0 \tag{1}$$

and derive a contradiction. Let (μ_1^n, μ_2^n) be a sequence of trembles in \widehat{M} with the following properties: (1) $\mu_1^n = (1 - \frac{1}{n}) 0 + \frac{1}{n} p$, where p denotes the Lebesgue measure over $[0, 1]$; and (2) $\mu_2^n = (1 - \frac{1}{n}) \frac{1}{4} + \frac{1}{n} p$. Let $\mu^n = (\mu_1^n, \mu_2^n)$ and $(0, 1) \ni \delta^n \searrow 0$. Let $\rho^n = (\rho_1^n, \rho_2^n)$ be a Nash equilibrium of the Selten perturbation $\overline{G}_{\delta^n \mu^n}$ (such an equilibrium exists (cf. Carbonell-Nicolau, 2010b)). Observe that because $\rho_i^n \in M_i(\delta^n \mu_i^n)$ for each i , we have $\rho_i^n = (1 - \delta^n) \sigma_i^n + \delta^n \mu_i^n$ for some $\sigma_i^n \in M_i$.

Clearly, $\sigma_2^n(\{\frac{1}{2}, 1\}) = 1$, since $u_2(x_1, \frac{1}{2}) \geq u_2(x_1, y_2)$ for $y_2 \in [0, 1] \setminus \{\frac{1}{2}, 1\}$ and $x_1 \in [0, 1]$, with strict inequality for $x_1 \in [0, 1] \setminus \{1\}$.

If $\sigma_2^n(\{\frac{1}{2}\}) = 1$ for infinitely many n , then there exists $\epsilon > 0$ such that $\sigma_1^n([0, \frac{1}{3}]) \geq \epsilon$ for infinitely many n , and this contradicts (1). To see that $\sigma_1^n([0, \frac{1}{3}]) \geq \epsilon$ for infinitely many n , note that if $\sigma_2^n(\{\frac{1}{2}\}) = 1$, since for each $\epsilon > 0$ there is a sufficiently large n for which

$$\begin{aligned} U_1 \left(x_1, (1 - \delta^n) \frac{1}{2} + \delta^n \mu_2^n \right) &= 0 \\ &> U_1 \left(y_1, (1 - \delta^n) \frac{1}{2} + \delta^n \mu_2^n \right) \\ &= \delta^n U_1(y_1, \mu_2^n) \end{aligned}$$

for each $x_1 \in [0, \frac{1}{2}]$ and every $y_1 \in (\frac{1}{2} + \epsilon, 1]$, we must have $\sigma_1^n([0, \frac{1}{2}]) \rightarrow 1$, and in this case $\frac{1}{2}$ is a best response to σ_1^n only if $\sigma_1^n([0, \frac{1}{3}])$ is bounded away from zero.

If $\sigma_2^n(\{1\}) > 0$ for infinitely many n , we distinguish two cases. Suppose first that $\sigma_2^n(\{1\}) \rightarrow 0$. By (1), we must have $\sigma_1^n([\frac{1}{2}, 1]) \rightarrow 1$. But if $\sigma_1^n([\frac{1}{2}, 1]) \rightarrow 1$, for each n player 2's best response to σ_1^n must be $\sigma_2^n(\{1\}) = 1$ unless $\sigma_1^n(\{1\}) = 1$ for infinitely many n , for

$$u_2 \left(x_1, \frac{1}{2} \right) < u_2(x_1, 1), \quad \text{for all } x_1 \in \left(\frac{1}{2} - \epsilon, 1 \right),$$

for some $\epsilon > 0$. If $\sigma_2^n(\{1\}) = 1$ for infinitely many n , the desired contradiction can be obtained as in the next paragraph. If $\sigma_1^n(\{1\}) = 1$ for infinitely many n , then for large n we have

$$\begin{aligned} U_2 \left((1 - \delta^n) 1 + \delta^n \mu_1^n, \frac{1}{2} \right) &= \delta^n U_2 \left(\mu_1^n, \frac{1}{2} \right) \\ &= \delta^n \left(1 - \frac{1}{n} \right) U_2 \left(0, \frac{1}{2} \right) + \delta^n \frac{1}{n} U_2 \left(p, \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned}
 &= \delta^n \left(1 - \frac{1}{n}\right) 2 + \delta^n \frac{1}{n} U_2 \left(p, \frac{1}{2}\right) \\
 &\approx \delta^n 2 \\
 &> U_2((1 - \delta^n) 1 + \delta^n \mu_1^n, 1) \\
 &= \delta^n U_2(\mu_1^n, 1) \\
 &= \delta^n \left(1 - \frac{1}{n}\right) U_2(0, 1) + \delta^n \frac{1}{n} U_2(p, 1) \\
 &= \delta^n \frac{1}{n} U_2(p, 1), \tag{2}
 \end{aligned}$$

so $\sigma_2^n(\{1\}) = 0$ for infinitely many n , and hence $\sigma_2^n(\{\frac{1}{2}\}) = 1$ for infinitely many n . One can now reason as in the previous paragraph.

It remains to consider the case when there exists $\varepsilon > 0$ such that $\sigma_2^n(\{1\}) \geq \varepsilon$ for infinitely many n . In this case, since $\sigma_2^n(\{\frac{1}{2}\} \cup \{1\}) = 1$, for large n we have

$$\begin{aligned}
 U_1(1, (1 - \delta^n) \sigma_2^n + \delta^n \mu_2^n) &\geq (1 - \delta^n) \varepsilon U_1(1, 1) + \delta^n U_1(1, \mu_2^n) \\
 &= (1 - \delta^n) \varepsilon + \delta^n U_1(1, \mu_2^n) \\
 &> U_1(x_1, (1 - \delta^n) \sigma_2^n + \delta^n \mu_2^n) \\
 &= 0,
 \end{aligned}$$

for all $x_1 \in [0, \frac{1}{2}]$, and so $\sigma_1^n((\frac{1}{2}, 1]) = 1$ for infinitely many n . For any such n , player 2's best response to σ_1^n must be $\sigma_2^n(\{1\}) = 1$ unless $\sigma_1^n(\{1\}) = 1$ for infinitely many n , for

$$u_2 \left(x_1, \frac{1}{2}\right) < u_2(x_1, 1), \quad \text{for all } x_1 \in \left(\frac{1}{2} - \varepsilon, 1\right),$$

for some $\varepsilon > 0$. If $\sigma_2^n(\{1\}) = 1$ for infinitely many n , it is clear that $\sigma_1^n(\{1\}) = 1$ for infinitely many n . If $\sigma_1^n(\{1\}) = 1$ for infinitely many n , then $\sigma_2^n(\{1\}) = 0$ for infinitely many n (this follows from (2)), thereby contradicting that there exists $\varepsilon > 0$ such that $\sigma_2^n(\{1\}) \geq \varepsilon$ for infinitely many n .

We have established the following:

Proposition 1. *For the class of games \mathfrak{g} , stability and trembling-hand perfectness fail limit admissibility.*

Our second observation is that stability and perfectness need not rule out nonadmissible equilibria when they are weakly dominated by admissible trembling-hand perfect equilibria. The following example illustrates this point.

Example 2. Define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) := \begin{cases} -1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let (f^n) be a sequence of maps $f^n : [0, 1] \rightarrow \mathbb{R}$ with the following properties:

- For each n , $f^n(x) = 0$ for all $x \in [0, 1 - \frac{1}{2n}] \cup [1 - \frac{1}{4n}, 1]$.
- For each n , $f^n(x) > 0$ for all $x \in (1 - \frac{1}{2n}, 1 - \frac{1}{4n})$.
- For each n , $f^n(0) = -1$ and f^n is continuous on $(0, 1]$.
- f^n converges uniformly to f .

Consider the two-player game

$$G = (X_1, X_2, u_1, u_2) = \left(\left[\frac{1}{2}, 1 \right] \cup \bigcup_{n=1}^{\infty} \left\{ \frac{1}{2} - \frac{1}{2n} \right\}, [0, 1], u_1, u_2 \right),$$

where

$$u_1(x_1, x_2) := \begin{cases} f^n(x_2) & \text{if } x_1 = \frac{1}{2} - \frac{1}{2n} \text{ and } n \in \mathbb{N}, \\ f(x_2) & \text{if } x_1 \in \left[\frac{1}{2}, 1 \right), \\ 0 & \text{if } x_1 = 1, \end{cases}$$

and

$$u_2(x_1, x_2) := \begin{cases} 1 & \text{if } x_2 = 0, \\ 2 & \text{if } x_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to verify that G is a member of \mathbf{g} . Moreover the action $\frac{1}{2}$ is weakly dominated by 1 for player 1, since

$$u_1\left(\frac{1}{2}, x_2\right) = \begin{cases} -1 < 0 = u_1(1, x_2) & \text{if } x_2 = 0, \\ 0 = u_1(1, x_2) & \text{if } x_2 \in (0, 1]. \end{cases}$$

Next, we show that $(1, 1)$ is a thp equilibrium. To see that $(1, 1)$ is Nash, note that $u_2(x_1, 1) > u_2(x_1, x_2)$ for all $x_2 \in [0, 1)$ and $u_1(x_1, 1) = 0$ for all $x_1 \in X_1$. To see that $(1, 1)$ is thp, note that because $u_2(x_1, 1) > u_2(x_1, x_2)$ for all $x_2 \in [0, 1)$, player 2's action 1 strongly dominates any other action in player 2's action space, so any thp equilibrium (μ_1, μ_2) of G must have $\text{supp}(\mu_2) = \{1\}$. In addition, letting $\nu_2 = \frac{1}{2}0 + \frac{1}{2}p_2$ for each n , where $p_2 \in \widehat{M}_2$ (i.e., ν_2^n assigns probability $\frac{1}{2}$ to the action 0 and randomizes according to p_2 with probability $\frac{1}{2}$), we have, for each $x_1 \in [0, 1)$,

$$\begin{aligned} U_1 \left(1, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2 \right) &= 0 \\ &> U_1 \left(x_1, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2 \right) \end{aligned}$$

$$\begin{aligned}
 &= \left(1 - \frac{1}{n}\right) u_1(x_1, 1) + \frac{1}{n} U_1(x_1, \nu_2) \\
 &= \frac{1}{n} U_1(x_1, \nu_2) \\
 &= \frac{1}{2n} u_1(x_1, 0) + \frac{1}{2n} U_1(x_1, p_2) \\
 &= \frac{1}{2n}(-1) + \frac{1}{2n} U_1(x_1, p_2) \\
 &\geq \frac{1}{2n}(-1).
 \end{aligned}$$

Therefore, player 1 best-responds to $(1 - \frac{1}{n})1 + \frac{1}{n}\nu_2$ by playing 1, so for given $\nu_1 \in \widehat{M}_1$, $((1 - \frac{1}{n})1 + \frac{1}{n}\nu_1, (1 - \frac{1}{n})1 + \frac{1}{n}\nu_2)$ is a Nash equilibrium of $\overline{G}_{n-1}(\nu_1, \nu_2)$, and since

$$\left(\left(1 - \frac{1}{n}\right) 1 + \frac{1}{n}\nu_1, \left(1 - \frac{1}{n}\right) 1 + \frac{1}{n}\nu_2 \right) \rightarrow (1, 1),$$

$(1, 1)$ is thp.

Let E be a stable set in G (Theorem 1). We claim that $(\frac{1}{2}, 1) \in E$. In fact, there exists $(\nu^n) = (\nu_1, \nu_2^n)$ with $\nu^n \in \widehat{M}$ for each n such that any sequence of Nash equilibria corresponding to the sequence of Selten perturbations $(\overline{G}_{n-1, \nu^n})$ has the form $((1 - \frac{1}{n})\mu_1^n + \frac{1}{n}\nu_1, (1 - \frac{1}{n})1 + \frac{1}{n}\nu_2^n)$ (for some sequence (μ_1^n) in M_1) and satisfies $\mu_1^n \rightarrow \frac{1}{2}$. To see this, define the sequence (a^n) in \mathbb{R} inductively as follows:

- $a^1 \in (\frac{1}{2}, \frac{3}{4})$; and
- for each $n > 1$, $a^n \in (1 - \frac{1}{4(n-1)}, 1 - \frac{1}{4n})$.

Let $(\nu^n) = (\nu_1, \nu_2^n)$ be a sequence with the following properties:

- $\nu_1 \in \widehat{M}_1$; and
- for each n , $\nu_2^n = (1 - \delta^n)a^n + \delta^n p$, where p represents the Lebesgue measure over $[0, 1]$, and where (δ^n) is a sequence in $(0, 1)$ satisfying the following:

$$\begin{aligned}
 n > 1 &\Rightarrow (1 - \delta^n) f^n(a^n) \\
 &> \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2m}, p \right), \quad \text{for all } m \in \{1, \dots, n - 1\}
 \end{aligned} \tag{3}$$

(note that the construction of (f^n) and (a^n) entails $f^n(a^n) > 0$ for each n).

For each n , the Selten perturbation $\overline{G}_{n-1, \nu^n}$ has a Nash equilibrium (cf. Carbonell-Nicolau, 2011b), and since $u_2(x_1, 1) > u_2(x_1, x_2)$ for all $x_2 \in [0, 1)$ (so that player 2's action 1 strongly dominates any other action in player 2's action space), any such equilibrium must be of the form $((1 - \frac{1}{n})\mu_1^n + \frac{1}{n}\nu_1, (1 - \frac{1}{n})1 + \frac{1}{n}\nu_2^n)$ (for some sequence (μ_1^n) in M_1).

It only remains to show that $\mu_1^n \rightarrow \frac{1}{2}$. Given $n > 1$, let

$$\left(\left(1 - \frac{1}{n} \right) \mu_1^n + \frac{1}{n} \nu_1, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2^n \right)$$

be a Nash equilibrium of $\overline{G}_{n-1\nu^n}$. Then μ_1^n must be a best response to $(1 - \frac{1}{n})1 + \frac{1}{n}\nu_2^n$ for player 1, and because

$$\begin{aligned} U_1 \left(x_1, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2^n \right) &= \frac{1}{n} U_1 (x_1, \nu_2^n) \\ &= 0 \\ &< U_1 \left(0, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2^n \right) \\ &= \frac{1}{n} U_1 (0, \nu_2^n) \\ &> 0, \quad \text{for all } x_1 \in \left[\frac{1}{2}, 1 \right], \end{aligned}$$

the support of μ_1^n must be contained in $X_1 \setminus [\frac{1}{2}, 1] = \bigcup_{m=1}^\infty \{ \frac{1}{2} - \frac{1}{2m} \}$. Now, since, for $m \in \{1, \dots, n-1\}$,

$$\begin{aligned} &U_1 \left(\frac{1}{2} - \frac{1}{2m}, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2^n \right) \\ &= \frac{1}{n} U_1 \left(\frac{1}{2} - \frac{1}{2m}, \nu_2^n \right) \\ &= \frac{1}{n} \left((1 - \delta^n) u_1 \left(\frac{1}{2} - \frac{1}{2m}, a^n \right) + \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2m}, p \right) \right) \\ &= \frac{1}{n} \left((1 - \delta^n) f^m(a^n) + \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2m}, p \right) \right) \\ &= \frac{1}{n} \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2m}, p \right) \\ &< \frac{1}{n} (1 - \delta^n) f^n(a^n) \\ &\leq \frac{1}{n} \left((1 - \delta^n) f^n(a^n) + \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2n}, p \right) \right) \\ &= \frac{1}{n} \left((1 - \delta^n) u_1 \left(\frac{1}{2} - \frac{1}{2n}, a^n \right) + \delta^n U_1 \left(\frac{1}{2} - \frac{1}{2n}, p \right) \right) \\ &= \frac{1}{n} U_1 \left(\frac{1}{2} - \frac{1}{2n}, \nu_2^n \right) \\ &= U_1 \left(\frac{1}{2} - \frac{1}{2n}, \left(1 - \frac{1}{n} \right) 1 + \frac{1}{n} \nu_2^n \right) \end{aligned}$$

(where the first inequality follows from (3)), it follows that the support of μ_1^n is contained in $\bigcup_{k=n}^\infty \{\frac{1}{2} - \frac{1}{2k}\}$. Consequently, because every member of the union $\bigcup_{k=n}^\infty \{\frac{1}{2} - \frac{1}{2k}\}$ is arbitrarily close to $\frac{1}{2}$ for sufficiently large n , we have $\mu_1^n \rightarrow \frac{1}{2}$, as we sought.

To sum up, in this example player 2 has a (strongly) dominant action, 1, and player 1 has a weakly dominated action, $\frac{1}{2}$. In addition, the equilibrium $(\frac{1}{2}, 1)$ is part of any stable set in G , and the thp equilibrium $(1, 1)$ has the property that 1 weakly dominates $\frac{1}{2}$ for player 1.

We say that a strategy profile $x \in X$ is weakly dominated by a strategy profile $\mu \in M$ if μ_i weakly dominates x_i for some i .

The conclusions from Example 2 are summarized in the following statement:

Proposition 2. *For the class of games \mathbf{g} , stability and trembling-hand perfectness need not rule out nonadmissible equilibria when they are weakly dominated by admissible trembling-hand perfect equilibria.*

Finally, we ask whether stability can choose nonadmissible equilibria and, at the same time, rule out admissible equilibria. The following example shows that the answer is in the affirmative.

Example 3. Consider the three-player game $G = (A, A, A, u_1, u_2, u_3)$, where the action space for all the players is

$$A := \left(\{1\} \times \left[0, \frac{1}{2}\right] \right) \cup \{2\},$$

and the payoffs for the cases where not all three players choose a member of $\{1\} \times [0, 1]$ are given in Fig. 1 (the second component of each element of $\{1\} \times [0, 1]$ is payoff irrelevant at profiles where some player chooses 2, so for these profiles a choice $(1, x_i)$ is simply represented as 1 in the figure; player 1 chooses a row, player 2 a column, and player 3 a matrix, and the first entry of each box corresponds to player 1's payoff, and so on). If every player chooses a member of $\{1\} \times [0, \frac{1}{2}]$, then player 3's payoff is zero, and, for $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, player i 's payoff is given by

$$u_i((1, x_i), (1, x_l)_{l \neq i}) := \begin{cases} x_i & \text{if } x_i \leq \frac{1}{2}x_j, \\ \frac{x_j(1-x_i)}{2-x_j} & \text{if } \frac{1}{2}x_j < x_i. \end{cases}$$

(Fig. 1 coincides with Fig. 2.2.1 of van Damme, 1987, p. 29, except that the first box of the first matrix is replaced by the infinite game given above.)

It is routine to verify that G belongs to \mathbf{g} (G is even continuous).

All thp equilibria of G are of the form $((1, 0), (1, 0), (1, x_3))$, where $x_3 \in [0, \frac{1}{2}]$. Moreover, the action $(1, 0)$ is weakly dominated by $(1, a)$, for any $a \in (0, \frac{1}{2}]$, for players 1 and 2. At the same time, $(2, (1, 0), (1, 0))$ is an undominated Nash equilibrium of G .

	1	2	
1		0,-1,0	
2	0,0,0	-1,-1,0	
	1		

	1	2
1	0,0,-1	-1,-1,-1
2	-1,0,-1	0,-1,-1
	1	2

Fig. 1. The payoffs of Example 3.

We summarize the conclusion from Example 3 in the following statement:

Proposition 3. *For the class of games \mathbf{g} , stability and trembling-hand perfectness may select nonadmissible equilibria and, at the same time, rule out admissible equilibria.*

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