Average, Total, and Concentration Hierarchical Indices

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Abstract

Hierarchy is pervasive, yet existing measures obscure whether it represents a per-capita condition, an aggregate volume, or a concentration of authority. This paper develops an axiomatic framework for measuring hierarchy on directed acyclic graphs. A central contribution is clarifying the tension between novel vertical extension axioms of hierarchy and normalized separability. Resolving these axiomatic tensions, we characterize three mutually exclusive index families. Average bilateral indices quantify per-capita hierarchy and remain invariant under replication. Total bilateral indices measure aggregate structural volume and scale linearly. Concentration indices capture authority density and decline under replication. Ultimately, this framework demonstrates that the choice of a measurement lens is fundamentally normative: the underlying separability condition inevitably dictates an index's scaling behavior and its capacity to satisfy the extension axioms.

Keywords: hierarchy measurement, hierarchical index, directed acyclic graph, additive separability, organizational structure.
JEL classifications: D23, L22.

1 Introduction

Hierarchy shapes behavior across animal and human societies, regulating resource access, power dynamics, and social coordination. Whether formalized in organizational charts,

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embedded in informal networks, or constructed through status norms, hierarchical structures permeate nearly every domain of human interaction. High-status actors often exhibit greater confidence and influence, while those lower in the order face constraints that shape conformity, competition, and mobility.

These structures generate complex outcomes. Hierarchies can enhance stability, coordination, and efficiency in uncertain environments (Simon, 1962, 1973; Holland, 1998; Lawrence and Lorsch, 1967; Thompson, 1967; Lincoln and Kalleberg, 1985), and informal networks often bolster organizational resilience, profitability, and crisis management (Krackhardt and Stern, 1988; Sarkar et al., 2010; Culen, 2017). Yet the same architectures can enable power abuse, entrench inequality, and stifle innovation (Anderson, 1972; Fix, 2017, 2018, 2019, 2021; Wright, 2024). Hierarchy is thus not a passive backdrop but an active mechanism that channels social and economic outcomes.

Precise measurement is therefore essential. While prior axiomatic work has established ordinal frameworks via incomplete pre-orders, existing cardinal indices lack formal structural grounding. This leaves a clear gap for measures that yield complete rankings and are characterized by structural axioms. To address this, we develop a new class of hierarchical indices grounded in a significantly expanded axiomatic framework. Building on the ordinal foundations of Carbonell-Nicolau (2025a,b)—which relied on anonymity, replication invariance, and subordination removal—we introduce novel replication principles and vertical extension axioms. Whereas the prior framework leaves many structurally distinct hierarchies incomparable, particularly under depth transformations, our expanded axiomatization resolves this limitation, yielding complete cardinal rankings that capture structural dimensions systematically overlooked by conventional measures.

A central conceptual contribution is the recognition that measuring hierarchy requires a fundamental normative choice about how the concept should scale with organizational size. This choice manifests in three mutually incompatible visions: hierarchy as an extensive cumulative volume, an intensive per-capita average, or a concentrated global density. Each vision is anchored by a distinct replication principle dictating whether cloning an organization strictly increases, preserves, or dilutes the measured hierarchy. To ensure mathematical consistency, we formalize three corresponding separability structures that differ exclusively in population normalization: absolute additivity (no scaling), linear averaging (division by population size), and quadratic density scaling (division by the square of population size). Selecting a conceptual paradigm simultaneously fixes the index's replication behavior and the separability structure required to sustain it. No single index can satisfy more than one pathway.

Beyond replication, a complete measurement framework must account for vertical growth and structural transformation. We therefore introduce axioms formalizing how appending management layers—above or below an existing structure—affects the overall measurement. Strict (unrestricted) extension principles require that any valid vertical attachment increases the hierarchy score, while relaxed variants accommodate partial linkages provided the appended structure is itself sufficiently hierarchical.

Our analysis begins with two impossibility results. We prove that no regular hierarchical index can simultaneously satisfy per-capita or concentration-based scaling alongside unrestricted vertical extension. To resolve this tension, we characterize three complementary families of indices. The first identifies intensive, per-capita measures that remain invariant under replication but satisfy only relaxed extension conditions. The second identifies extensive, additive measures that scale linearly with population and satisfy unrestricted extension. The third identifies concentration measures that penalize structural fragmentation and strictly decrease under replication. Each family is uniquely characterized by its normalization rule and a tailored population-growth condition ensuring meaningful asymptotic behavior.

Applying this framework to established indices reveals systematic axiom violations, underscoring the need for structural validation. The global reaching centrality index (Mones et al., 2012), widely used in network science, satisfies anonymity but its reliance on a global reference point destroys separability and generates non-monotonic behavior, violating core extension and subordination removal axioms. Krackhardt's (1994) four dimensions of informal organization prove either trivial or axiomatically incompatible in our framework. Other measures face similar limitations: indices by Trusina et al. (2004), Corominas-Murtra et al. (2013), and Luo and Magee (2011) are calibrated to detect cycles, collapsing to constant values on acyclic graphs. Finally, the algorithmic approach of Czégel and Palla (2015) depends on an exogenous decay parameter; this sensitivity can violate extension axioms unless optimally tuned, limiting its robustness for structural comparison.

The remainder of the paper proceeds as follows. Section 2 formalizes hierarchies as directed acyclic graphs and introduces the structural primitives used throughout the analysis. Section 3 develops the expanded axiomatic framework, detailing the replication principles and vertical extension axioms that ground our measurement approach. Section 4 presents the core theoretical results, establishing the impossibility of combining normalized separability with unrestricted extension and providing complete characterizations for the average, total, and concentration index families. Section 5 benchmarks prominent hierarchy indices against our axiomatic body, demonstrating their systematic failure to satisfy core structural axioms. Section 6 concludes by discussing the implications of the normative trichotomy for organizational measurement and outlining directions for empirical research and the exploration of alternative functional specifications.

2 Hierarchies

A *hierarchy* is a *directed acyclic graph* (DAG) $h = (V(h), E(h))$, where $V(h)$ denotes the set of individuals and $E(h)$ denotes the set of directed edges. An edge $(j, i) \in E(h)$ indicates that j is a (direct) supervisor of i. Acyclicity ensures that following edges in their direction never returns to the starting node; equivalently, there are no directed cycles.

For $v \in V(h)$, let

$$\mathrm{Anc}_h(v) = \{u \in V(h) : u \rightsquigarrow v \text{ via a directed path of length } \geq 1\}$$

and

$$\mathrm{Desc}_h(v) = \{w \in V(h) : v \rightsquigarrow w \text{ via a directed path of length } \geq 1\}$$

denote the sets of (direct or indirect) supervisors and subordinates of v, respectively. We define the *depth* of v as

$$s_h(v) = |\mathrm{Anc}_h(v)|,$$

i.e., the total number of (direct or indirect) supervisors of v. Symmetrically, we define the *span* of v as

$$\sigma_h(v) = |\mathrm{Desc}_h(v)|,$$

the total number of (direct or indirect) subordinates of v.

This structure induces a unidirectional (downward) flow of authority. Individuals may have direct supervisors (immediate predecessors) and indirect supervisors (supervisors of their supervisors), producing transitive chains of command. The DAG assumption guarantees that the set of reporting lines is well-defined (acyclic), though not necessarily unique when multiple supervisors exist.

Because the graph is acyclic, an individual's depth is strictly greater than the depth of any of their supervisors: for any directed edge $j \to i$ in h, we have

$$\mathrm{Anc}_h(j) \subseteq \mathrm{Anc}_h(i) \quad \text{and} \quad j \in \mathrm{Anc}_h(i) \setminus \mathrm{Anc}_h(j),$$

hence $s_h(j) < s_h(i)$. Thus, if an individual i has k total supervisors, each of those supervisors has strictly fewer than k supervisors.

A node with no supervisors (in-degree 0) is a *root node*; a node with no subordinates (out-degree 0) is a *leaf node*. The *size* of a hierarchy is the total number of nodes, $|V(h)|$.

Let $\mathscr{H}$ denote the class of all (finite) hierarchies. For $n \geq 1$, write $\mathscr{H}_n = \{ h \in \mathscr{H} : |V(h)| = n \}$, so that

$$\mathscr{H} = \bigcup_{n \geq 1} \mathscr{H}_n.$$

A *composite hierarchy* $(h_1, \ldots, h_k)$, with $k \geq 1$, is an ordered k-tuple of pairwise node-disjoint hierarchies; we identify it with their disjoint union $h_1 \cup \cdots \cup h_k$.

3 Axiomatic framework

As established in Section 2, let $\mathscr{H}$ denote the universal class of finite hierarchies (DAGs). The approach in Carbonell-Nicolau (2025a,b) formalizes hierarchy via an *ordinal* pre-order on $\mathscr{H}$. In this paper, our primary object is a *cardinal hierarchical index*, a real-valued mapping $I : \mathscr{H} \to \mathbb{R}$ that induces a complete pre-order by numerical ranking: $h' \succcurlyeq h \Leftrightarrow I(h') \geq I(h)$. The symmetric part ($\sim$) corresponds to equality ($I(h') = I(h)$), and the asymmetric part ($\succ$) corresponds to strict inequality ($I(h') > I(h)$).

A hierarchy h' is a *relabeling* (isomorphic copy) of h if there exists a bijection $\phi : V(h) \to V(h')$ such that, for all $j, i \in V(h)$,

$$(j, i) \in E(h) \Leftrightarrow (\phi(j), \phi(i)) \in E(h').$$

Thus, h and h' share the same authority structure and differ only in node labels.

Anonymity (A). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A if $I(h') = I(h)$ whenever h' is a relabeling of h.

A *replication* of $h \in \mathscr{H}$ is the disjoint union of $k \geq 1$ identical copies of h. If hierarchy is conceived as an average, per-capita property of an organization, scaling up the population by simply cloning its structure should leave the overall measurement unchanged.

Replication Principle (RP). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies RP if $I(h') = I(h)$ whenever h' is a replication of h.

Subordination removal. Let $P_h(j) = \{ p \in V(h) : (p, j) \in E(h) \}$ denote the set of *immediate* supervisors of j.

Definition 1. Let $h \in \mathscr{H}$. We say h' is obtained from h by *removing* a subordination relation $(j, i) \in E(h)$ if:

1. Delete (j, i).
2. If $P_h(j) = \varnothing$ (i.e., j is a root), add no further edges.
3. If $P_h(j) \neq \varnothing$, then for each $p \in P_h(j)$:
   - add the edge (p, i) *unless* there already exists in h a directed path from p to i that does not traverse j;
   - otherwise, make no change.
4. All other supervisory relations in h remain unchanged.

The resulting h' is again a DAG: each new edge (p, i) points from a superior of j to a subordinate of j, so no directed cycle can be created.

An edge $(j, i) \in E(h)$ is *redundant* if there exists a directed path from j to i distinct from (j, i) itself.

Subordination Removal (SR). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies SR if $I(h) \geq I(h')$ whenever h' is obtained from h by removing a subordination relation. Moreover, $I(h) > I(h')$ whenever the removed edge is non-redundant in h.

Remark 1 (From pre-orders to indices)**.** The axioms A, RP, and SR originate in the ordinal framework of Carbonell-Nicolau (2025a,b), where they characterize the topological pre-order generated by successive subordination removals. Here we impose them directly on a cardinal index I.

3.1 Replication principles and structural scaling

We examine how a hierarchical index I behaves when a hierarchy h is *replicated* into a disjoint composite of $k \geq 1$ identical copies. Formally, h^k denotes the disjoint union of k pairwise node-disjoint hierarchies, each isomorphic to h.

Intuition yields conflicting, antagonistic answers regarding whether for $k > 1$, the composite h^k is equally, more, or less hierarchical than the isolated h. This tension reflects that "hierarchy" conflates at least three distinct structural notions: per-capita shape, aggregate mass, and global density. To disentangle these, we formalize three pairwise incompatible variants of the replication principle.

3.1.1 Hierarchy as a structural average (the intensive view)

The baseline Replication Principle (RP) introduced earlier belongs to the intensive measurement paradigm. If we conceptualize hierarchy as the expected, day-to-day structural experience of a randomly selected node, cloning the society does not alter this average local environment. We re-designate this condition as the Intensive Replication Principle.

Intensive Replication Principle (IRP). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies IRP if for any hierarchy h and any integer $k \geq 1$, $I(h^k) = I(h)$.

The intuitive appeal of IRP lies in treating hierarchy as a scale-independent condition rather than a cumulative volume. Because replication scales aggregate structure and population proportionally, the per-capita average remains invariant.

3.1.2 Hierarchy as an aggregate volume (the extensive view)

In contrast to the intensive perspective, one might view hierarchy not as a per-capita shape, but as a total volume of administrative overhead, systemic friction, or total relational control. From this macroeconomic perspective, 1,000 isolated small businesses contain vastly more aggregate "managerial substance" than a single small business.

Extensive Replication Principle (ERP). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies ERP if for any hierarchy h with at least one subordination relation (i.e., $E(h) \neq \emptyset$) and any integer $k > 1$, $I(h^k) > I(h)$.

The ERP captures the intuition that hierarchical relations are additive units of structure. While IRP treats hierarchy as a scale-invariant per-capita property, ERP conceptualizes it as a cumulative structural volume.

3.1.3 Hierarchy as global centralization (the concentration view)

Finally, intuition frequently suggests a third, antagonistic perspective: that 1,000 disconnected copies of a two-person team is severely *less* hierarchical than a single two-person team. This intuition views hierarchy as a measure of structural concentration or network density.

Concentration Replication Principle (CRP). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies CRP if for any hierarchy h with at least one subordination relation (i.e., $E(h) \neq \emptyset$) and any integer $k > 1$, $I(h^k) < I(h)$.

While CRP and ERP track opposite inequalities, both are logically grounded in their implicit normalization baselines. In a system of N nodes, the maximum possible number of supervisory relationships grows quadratically with population size. When a hierarchy is replicated k times ($k > 1$) without introducing cross-links, the actual number of reporting links increases linearly (by a factor of k), while the theoretical capacity for relationships expands quadratically (by a factor of k^2). This structural mismatch inevitably dilutes network density. CRP formalizes the intuition that a large population fragmented into isolated chains of command is less concentrated relative to its maximum possible connectivity.

No single index can satisfy more than one of IRP, ERP, and CRP. Thus, choosing a replication principle explicitly fixes whether an index measures structural averages, structural accumulation, or structural density.

Analogues with scaling principles in socioeconomic measurement are discussed in Carbonell-Nicolau (2026, §3.1.3).

3.2 Upward Extension

Conventions. Throughout this and the following two subsections, terms such as "leaf," "root," and "maximum depth" refer to the corresponding notions in the *pre-extension* hierarchies, i.e., before any cross-edges are added. Furthermore, when taking the disjoint union of two hierarchies to form a compound, we preserve all original nodes and edges with no nodes identified or merged.[1] In each compound defined below, the first argument acts as the upper component and the second as the lower component. Newly added cross-edges are exclusively directed from nodes in the upper component to nodes in the lower component. Because the components are disjoint and no cross-edges point back, directed cycles cannot be created, preserving acyclicity.

We now introduce an axiom concerning hierarchical extensions that place one hierarchy *above* another by linking the leaf nodes of the upper hierarchy to the root nodes of the lower hierarchy.

Appended compound (set-valued operator). For $h_o, h \in \mathscr{H}$, let $h_o \oplus h$ denote the *set* of hierarchies obtained by:

1. taking the disjoint union of h_o and h, and

[1] If the node sets are not disjoint, we replace the hierarchies with isomorphic copies that have disjoint node labels before taking the union. This standard convention makes the disjoint union well-defined; when I satisfies Anonymity (A), the choice of relabeling is immaterial for evaluation.

2. adding edges from each leaf node of h_o to one or more root nodes of h so that every leaf in h_o acquires at least one outgoing edge to a root in h.

Only such leaf-to-root edges are added; no other cross-edges are permitted.

Remark 2. Since every finite DAG has at least one leaf and at least one root (by finiteness, an unextendable directed path exists; its start is a root and its end a leaf), $h_o \oplus h \neq \emptyset$ for all $h_o, h \in \mathscr{H}$ (e.g., one can always pick an arbitrary root r of h and add an edge from every leaf of h_o to r). The operator $\oplus$ is *set-valued* and *order-dependent*: in general $h_o \oplus h \neq h \oplus h_o$, and there can be multiple valid compounds in $h_o \oplus h$ depending on how leaves are attached to roots. Note that while every leaf of h_o must connect to h (ensuring the upper structure is fully connected by cross-edges), we intentionally do not require every root of h to receive a connection. The operator formalizes placing a structure above h, which does not strictly require integrating every independent component of the base hierarchy.

Upward Extension (UE). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies UE if for all $h_o, h \in \mathscr{H}$ and every $h^* \in h_o \oplus h$, we have $I(h^*) > I(h)$.

Intuitively, appending h beneath h_o strictly increases hierarchical depth, so any resulting compound hierarchy must be evaluated as strictly more hierarchical than h alone. UE intentionally imposes unrestricted monotonicity: even placing a flat hierarchy above h must strictly increase the index, because the axiom formalizes "depth creation" rather than the "quality" of the appended module.

Figures illustrating the operator $h_o \oplus h$, the completeness requirement (every leaf of h_o attaches), and the distinction between UE and SR are provided in Carbonell-Nicolau (2026, Figures 1–6).

3.3 Downward Extension

We now introduce an axiom for hierarchical extensions that place one hierarchy *below* another by linking (via new edges) the root nodes of the lower hierarchy to leaf nodes of the upper hierarchy that have maximum depth.

Appended compound (set-valued operator). For $h, h_o \in \mathscr{H}$, let $h \boxplus h_o$ denote the *set* of hierarchies obtained by:

1. taking the disjoint union of h and h_o, and

2. for *each* root node r of h_o, adding one or more edges from leaf nodes of h of *maximum depth* to r.

Only such maximum-depth-leaf-to-root edges are added; no other cross-edges are permitted.

Remark 3. Note that $h \boxplus h_o \neq \varnothing$ for all $h, h_o \in \mathscr{H}$. The operator $\boxplus$ is *set-valued* and *order-dependent*: in general $h \boxplus h_o \neq h_o \boxplus h$. Note that maximum depth is evaluated globally across all nodes in h, rather than componentwise. Furthermore, in contrast to the top-coverage requirement of $\oplus$, while every root of h_o must be connected to h, we do not require every maximum-depth leaf of h to send a connection.

Downward Extension (DE). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies DE if for all $h, h_o \in \mathscr{H}$ and every $h^* \in h \boxplus h_o$, we have $I(h^*) > I(h)$.

Intuitively, appending h_o *below* h strictly deepens the hierarchy at its bottom boundary; any compound $h^* \in h \boxplus h_o$ must therefore be evaluated as strictly more hierarchical than h alone. Like UE, DE captures depth creation rather than module quality, and thus demands a strict increase even if the lower appendage h_o is entirely flat.

Figures illustrating the operator $h \boxplus h_o$, the maximum-depth attachment condition, and the relation between DE, UE, and SR are provided in Carbonell-Nicolau (2026, Figures 7–9).

3.4 Additional extension axioms

We introduce two additional axioms—Upward Extension* and Downward Extension*—that complement UE and DE by allowing *partial* attachments, compensated by a requirement that the appended hierarchy be (weakly) more hierarchical than the base.

Partial-append compound (set-valued operator). For $h, h_o \in \mathscr{H}$, let $h \uplus h_o$ denote the *set* of hierarchies obtained by:

1. taking the disjoint union of h and h_o, and
2. adding one or more edges from *at least one* leaf of h to *at least one* root of h_o.

Only leaf-to-root cross-edges are allowed; no other cross-edges are permitted. Note that attachments may be *partial*: not every leaf of h need connect, and not every root of h_o need be attached.

Remark 4. We have $h \uplus h_o \neq \varnothing$ for all $h, h_o \in \mathscr{H}$. Moreover, the operator $\uplus$ is *set-valued* and *order-dependent*, and

$$h_o \oplus h \;\subseteq\; h_o \uplus h \quad \text{and} \quad h \boxplus h_o \;\subseteq\; h \uplus h_o,$$

with strict inclusions whenever the completeness conditions of $\oplus$ or $\boxplus$ fail (e.g., incomplete leaf coverage or non-maximum-depth attachments).

An illustration of partial attachment under $\uplus$ and its role in UE* and DE* is provided in Carbonell-Nicolau (2026, Figure 10).

Upward Extension* (UE*). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies UE* if for all $h_o, h \in \mathscr{H}$ and every $h^* \in h_o \uplus h$, $I(h_o) \geq I(h)$ implies $I(h^*) > I(h)$.

Downward Extension* (DE*). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies DE* if for all $h, h_o \in \mathscr{H}$ and every $h^* \in h \uplus h_o$, $I(h_o) \geq I(h)$ implies $I(h^*) > I(h)$.

Conditional monotonicity. UE* and DE* are conditional, self-referential axioms: they restrict I's behavior only when I itself ranks the appendage weakly above the base (i.e., $I(h_o) \geq I(h)$). The dominance premise is uniform: it always compares the appended hierarchy h_o to the base h, independent of whether h_o is appended above (via UE*) or below (via DE*).

4 Hierarchical indices

We now examine hierarchical indices and their relationship to the axioms introduced in the preceding section.

Our analysis yields three distinct families of hierarchical indices, corresponding to three competing paradigms for what "hierarchy" should measure: the *average* (intensive) family, the *total* (extensive) family, and the *concentration* (density) family. We provide an axiomatic characterization of each class, highlighting the trade-offs implied by the separability and scaling axioms.

Hierarchical indices offer computational advantages through simple formulae, avoiding the complexities and incompleteness of pre-orders. This makes them readily applicable to real-world datasets. However, by inducing a complete order on hierarchies, these indices necessarily make definitive comparisons between any two hierarchical structures—even in ambiguous cases where the structural axioms alone would not single out a more hierarchical structure.

4.1 Classical separability: AS, ADD, and QS

Separability in classical measurement theory requires that an aggregate index be decomposable into a sum of localized evaluations. Here we adapt this idea to hierarchies by requiring that global hierarchy be computed from an additive aggregation of node-level environment scores. The three axioms below share the same additive numerator but differ only in normalization: ADD measures total structural mass, AS measures a per-capita average, and QS measures a quadratic (density) normalization. Additive Separability and Additivity are standard in inequality and poverty measurement, while quadratic normalizations underlie classical concentration measures.

To translate separability from scalar distributions (e.g., income) to networks, we isolate the structural information relevant to an individual's position. For any hierarchy $h \in \mathscr{H}$ and node $v \in V(h)$, define the *hierarchical environment* $N(v, h)$ as the subgraph induced by v and all nodes reachable from or to v via directed paths (i.e., all direct and indirect supervisors and subordinates of v). Let $\mathscr{N}$ denote the space of all such valid environments.

Additive Separability (AS). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies AS if there exists an environment evaluation function $f : \mathscr{N} \to \mathbb{R}$ such that for any hierarchy $h \in \mathscr{H}$,

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} f\big(N(v, h)\big).$$

AS evaluates hierarchy as the population average of individual structural scores. Restricting f to the local environment $N(v, h)$ enforces structural independence, as in the FGT poverty indices (Foster et al., 1984). By confining the impact of localized changes to their respective environments, this formulation guarantees subgroup consistency. Consequently, the overall index strictly increases whenever local evaluations weakly improve with at least one strict gain. Finally, AS strengthens IRP: replicating h k times multiplies both $\sum_{v \in V(h)} f(N(v, h))$ and $|V(h)|$ by k, leaving the average unchanged.

Additivity (ADD). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies ADD if there exists an environment evaluation function $f : \mathcal{N} \to \mathbb{R}$ such that for any hierarchy $h \in \mathcal{H}$,

$$I(h) = \sum_{v \in V(h)} f\big(N(v, h)\big).$$

In the inequality/poverty literature, strict additivity means the aggregate is the sum of individual evaluations (e.g., Shorrocks, 1980); ADD is its network analog. Here an individual's "endowment" is the localized position $N(v, h)$, and ADD treats hierarchy as an *extensive* (cumulative) volume rather than an *intensive* per-capita average. It implies component additivity $I(h_1 \cup h_2) = I(h_1) + I(h_2)$ for node-disjoint h_1, h_2, while the node-by-node summation ensures that structural changes within a single connected organization only affect the evaluations of the specific branches involved. Because replication doubles the number of evaluated positions, any ADD index guarantees ERP. Moreover, lacking a population denominator, ADD avoids dilution: appending a large flat hierarchy cannot depress the score, and under the natural normalization $I(h) = 0$ for a single isolated node (cf. INN below), disconnected flat components contribute nothing. This extensive-versus-intensive contrast mirrors the distinction between the absolute *total poverty gap* and normalized FGT measures, and in population ethics between Classical (Total) and Average Utilitarianism (Blackorby et al., 2005).

A third perspective views hierarchy as *global concentration* (density) relative to the population's dyadic capacity. To capture this, we replace linear/absent normalization with a quadratic denominator.

Quadratic Separability (QS). A hierarchical index $I : \mathcal{H} \to \mathbb{R}$ satisfies QS if there exists an environment evaluation function $f : \mathcal{N} \to \mathbb{R}$ such that for any hierarchy $h \in \mathcal{H}$,

$$I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} f\big(N(v, h)\big).$$

While AS and ADD use linear and absent population scaling, QS imposes a quadratic penalty. This parallels dyadic normalization in concentration/inequality measurement (e.g., the Gini coefficient), and is natural here because hierarchy is built from ordered boss–subordinate pairs. Under QS, independent components are down-weighted by squared population shares, with the "missing" weight interpretable as absent cross-component dyads. Consequently, whenever $I(h) > 0$ for non-flat h, QS implies CRP: replicating h by k scales the numerator linearly but the denominator by k^2, strictly diluting density. Thus the denominator exponent fixes the lens: $|V(h)|^0$ yields cumulative mass (ERP), $|V(h)|^1$ yields the per-capita average (IRP), and $|V(h)|^2$ yields global concentration (CRP).

Remark 5 (scaling exponents and higher-order structures)**.** The exponent p in $|V(h)|^p$ selects the relational order used for normalization. Under QS, $p = 2$ is grounded in the quadratic growth of potential directed pairs, so it bounds a genuine *dyadic density*. Higher integers normalize higher-order structures (e.g., $p = 3$ benchmarks triads such as $i \to j \to k$), while fractional exponents can reflect empirical scaling regimes. For instance, densification power laws (Leskovec et al., 2005) motivate denominators like $|V(h)|^{1.5}$, benchmarking density against observed super-linear but sub-quadratic edge growth. Such regimes may reflect cognitive constraints (Dunbar, 1992) or span-of-control limits (Williamson, 1967).

4.2 Impossibility

We first observe that the normalized separability axioms (AS and QS) are incompatible with the strict structural extension axioms (UE and DE).

Throughout this section, we restrict our attention to *regular* separable indices. A separable index (i.e., one satisfying AS, ADD, or QS, and thus admitting a local evaluation function f) is *regular* if it is strictly non-negative ($I(h) \geq 0$ for all h, and $I(h) > 0$ for complex non-flat structures) and its environment evaluation function does not asymptotically saturate (for any $K > 0$, there exists an environment N such that $f(N) > K$). Finally, a regular separable index exhibits monotonically bounded marginal returns: expanding a local environment cannot decrease its hierarchical evaluation, and its growth is strictly bounded. Formally, there exists a finite constant $M > 0$ such that for any hierarchical environments $N \subseteq N'$ (where $\subseteq$ denotes induced-subgraph inclusion), the change in evaluation satisfies

$$0 \leq f(N') - f(N) \leq M(|V(N')| - |V(N)|).$$

Theorem 1. *There exists no regular hierarchical index satisfying A that simultaneously satisfies either AS or QS and either UE or DE.*

Proof. Suppose, for contradiction, that there exists a regular hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfying A, either AS or QS, and also UE. Because I satisfies AS or QS, there exist $f : \mathscr{N} \to \mathbb{R}$ and $p \in \{1, 2\}$ such that for every $h \in \mathscr{H}$,

$$I(h) = \frac{1}{|V(h)|^p} \sum_{v \in V(h)} f\big(N(v, h)\big). \tag{1}$$

Let $c_2 = f(N_2)$ be the evaluation of a 2-node directed chain.

Step 1 (construct a baseline hierarchy h with $I(h) = C$ large enough). *Case $p = 1$ (AS).* Because f does not saturate, pick an environment N^* with $f(N^*) > c_2 + M$. Let $K = f(N^*)$, so $K > c_2 + M$. Form h_1 by appending a flat hierarchy of k nodes upward to N^*. Then for each of the k new nodes u, we have $N^* \subseteq N(u, h_1)$. By monotonicity, $f(N(u, h_1)) \geq f(N^*) = K$. Hence

$$\sum_{v \in V(h_1)} f(N(v, h_1)) \geq kK,$$

so

$$I(h_1) = \frac{1}{|V(h_1)|} \sum_{v \in V(h_1)} f(N(v, h_1)) \geq \frac{kK}{|V(h_1)|}.$$

As k increases, the fraction of nodes among $V(h_1)$ with evaluation at least K tends to 1, so we can choose k large enough so that

$$I(h_1) \geq c_2 + M + \delta$$

for some fixed $\delta > 0$.

Write $s = |V(N^*)|$, so $|V(h_1)| = m = k + s$. For each of the k new nodes u, the environment $N(u, h_1)$ is obtained from N^* by adding the single node u as a new ancestor. By bounded marginal returns,

$$f(N(u, h_1)) \leq f(N^*) + M = K + M.$$

For each node $w \in V(N^*)$, the construction adds at most k new ancestors to w, so again by bounded marginal returns,

$$f(N(w, h_1)) \leq f(N(w, N^*)) + Mk.$$

Let

$$A^* = \sum_{w \in V(N^*)} f(N(w, N^*)),$$

which is a finite constant independent of k. Summing the above inequalities yields

$$\sum_{v \in V(h_1)} f(N(v, h_1)) \leq k(K + M) + A^* + sMk.$$

Dividing by $m = k + s$ gives

$$I(h_1) \leq \frac{k(K + M + sM)}{k + s} + \frac{A^*}{k + s} \leq (K + M + sM) + A^* = \overline{B},$$

where $\overline{B}$ is a constant independent of k (hence independent of m).

Case $p = 2$ *(QS)*. Choose any complex non-flat hierarchy h_1. Regularity implies $I(h_1) > 0$.

Now let h be the disjoint union of h_1 and one isolated node r_2. Write $m = |V(h_1)|$, so $|V(h)| = m + 1$. Define $C = I(h)$.

If $p = 2$, then $C > 0$ because h is complex and regularity gives strict positivity.

If $p = 1$, then using (1) and letting $c_1 = f(N_1)$ (the one-node environment),

$$C = I(h) = \frac{mI(h_1) + c_1}{m + 1}.$$

Hence

$$C \geq I(h_1) - \frac{I(h_1) - c_1}{m + 1}.$$

By the uniform bound $I(h_1) \leq \overline{B}$, we have

$$\frac{I(h_1) - c_1}{m + 1} \leq \frac{\overline{B} - c_1}{m + 1}.$$

Since $m = k + s$ can be made arbitrarily large by increasing k, we may choose k (and thus m) large enough so that

$$\frac{\overline{B} - c_1}{m + 1} \leq \frac{\delta}{2},$$

which implies

$$\frac{I(h_1) - c_1}{m + 1} \leq \frac{\delta}{2}.$$

Then

$$C \geq c_2 + M + \frac{\delta}{2} > c_2 + M.$$

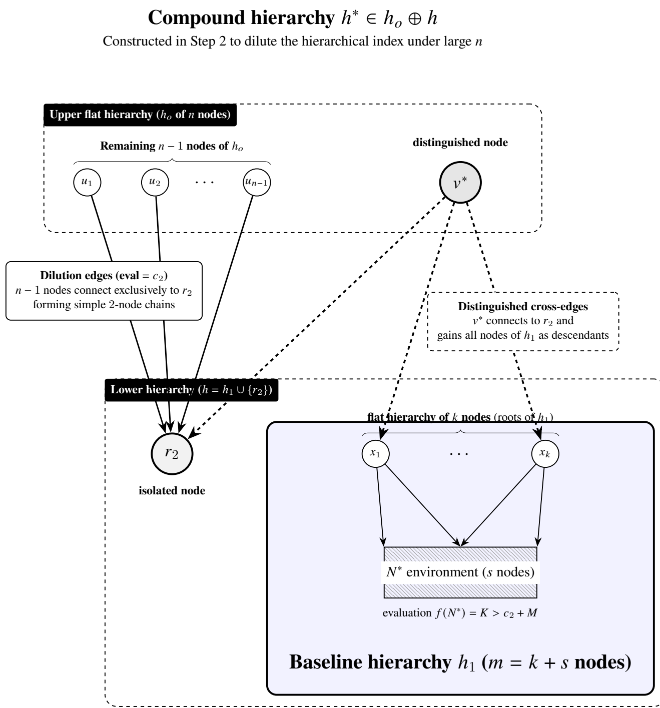

Figure 1: Compound hierarchy h^*.

Step 2 (append a large flat hierarchy above h in a way that dilutes the index). Let h_o be a completely flat hierarchy of n nodes. Form a compound $h^* \in h_o \oplus h$ as follows: choose exactly one node $v^* \in V(h_o)$ and connect v^* to all roots of h_1 and to r_2. Connect each of the remaining $n-1$ nodes of h_o exclusively to r_2 (Figure 1).

Let

$$S(h) = \sum_{v \in V(h)} f(N(v,h)),$$

which is a constant independent of n.

We now bound

$$S(h^*) = \sum_{v \in V(h^*)} f(N(v,h^*))$$

from above by an affine function of n with an explicit constant term.

(i) Nodes in h_1. Each of the m nodes of h_1 gains exactly one new ancestor (v^*). By bounded marginal returns,

$$f(N(v,h^*)) - f(N(v,h)) \le M \quad \text{for each } v \in V(h_1).$$

Therefore the total increase contributed by nodes in h_1 is at most mM.

(ii) The node r_2. The node r_2 gains exactly n new ancestors (all nodes of h_o). Thus, by bounded marginal returns,

$$f(N(r_2, h^*)) - f(N(r_2, h)) \leq Mn.$$

(iii) The $n-1$ *nodes in* $h_o \setminus \{v^*\}$. Each such node becomes the root of a simple 2-node chain pointing to r_2. By anonymity, their local environment evaluations equal c_2. Hence their total contribution is exactly $(n-1)c_2$.

(iv) The single node v^*. The node v^* has an environment of size $m+2$ (itself, r_2, and all nodes of h_1). This environment does not depend on n. Let

$$B = f(N(v^*, h^*)),$$

which is a finite constant independent of n.

Combining (i)–(iv) yields

$$S(h^*) \leq S(h) + mM + Mn + (n-1)c_2 + B = (c_2 + M)n + \underbrace{\big(S(h) + mM + B - c_2\big)}_{K_0},$$

where K_0 is a constant independent of n.

Therefore, using (1) and $|V(h^*)| = n + m + 1$,

$$I(h^*) = \frac{S(h^*)}{(n+m+1)^p} \leq \frac{(c_2 + M)n + K_0}{(n+m+1)^p}. \tag{2}$$

Step 3 (take n large and contradict UE). *Case* $p = 1$ *(AS)*. From (2),

$$I(h^*) \leq (c_2 + M)\frac{n}{n+m+1} + \frac{K_0}{n+m+1}.$$

Letting $n \longrightarrow \infty$ gives

$$\limsup_{n \to \infty} I(h^*) \leq c_2 + M.$$

But we built h so that $I(h) = C > c_2 + M$. Hence there exists n_0 such that for all $n \geq n_0$, $I(h^*) < C = I(h)$, contradicting UE (which requires $I(h^*) > I(h)$ for every $h^* \in h_o \oplus h$).

Case $p = 2$ (QS). From (2),

$$I(h^*) \le \frac{(c_2 + M)n + K_0}{(n + m + 1)^2} \le \frac{c_2 + M}{n + m + 1} + \frac{K_0}{(n + m + 1)^2}.$$

Letting $n \to \infty$ gives $\lim_{n\to\infty} I(h^*) = 0$. But we built h so that $I(h) = C > 0$. Thus for large n, $I(h^*) < C = I(h)$, again contradicting UE.

Step 4 (the same contradiction for DE). The argument for DE is symmetric. Let h be the disjoint union of a massive branch h_1 and a single isolated leaf l_2. Append a flat hierarchy of n new nodes *below* by forming $h^* \in h \boxplus h_{\text{flat}}$ so that every new node attaches exclusively to l_2. Then the same bounded-marginal-returns calculation shows that the numerator $S(h^*)$ is at most $(c_2 + M)n + K_0'$ for some constant K_0', while the denominator is $(n + m + 1)^p$. Hence $I(h^*)$ falls below $I(h)$ for large n, contradicting DE.

This completes the contradiction and proves the theorem. ∎

We further establish that while the population-averaging of AS permits compliance with relaxed extension axioms, the quadratic normalization of QS fundamentally precludes even partial structural extensions.

Theorem 2. *There exists no regular hierarchical index satisfying A and QS that simultaneously satisfies the partial extension axioms UE* or DE*.*

Proof. Suppose, by way of contradiction, that there exists a regular hierarchical index $I : \mathcal{H} \to \mathbb{R}$ that satisfies A, QS, and UE*. (The proof for DE* is symmetric.)

Because I satisfies QS, there exists an environment evaluation function $f : \mathcal{N} \to \mathbb{R}$ such that for every $h \in \mathcal{H}$,

$$I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} f\big(N(v, h)\big).$$

Let $c_1 = f(N_1)$ be the evaluation of the one-node environment, and let $c_2 = f(N_2)$ be the evaluation of a 2-node directed chain.

Construct a baseline hierarchy $h \in \mathcal{H}$ as the disjoint union of a complex (non-flat) hierarchy h_{core} and a single isolated node v_{flat}. Let $N = |V(h)|$. Define the aggregate evaluation (numerator)

$$S(h) = \sum_{v \in V(h)} f\big(N(v, h)\big),$$

so that by QS we have

$$I(h) = \frac{S(h)}{N^2}.$$

Because h is complex, regularity implies $I(h) = C > 0$.

Let h_o be an exact isomorphic copy of h. By A, $I(h_o) = I(h) = C$, so the weak dominance premise in UE* is satisfied.

Form a compound $h^* \in h_o \uplus h$ by adding exactly one directed edge from the flat node in h_o to the flat node in h, and adding no other cross-edge.

Claim A. *The local environments (and hence the evaluations) of the $2N - 2$ nodes belonging to the two core components remain unchanged when passing from the disjoint union $h_o \cup h$ to h^*.*

Proof of Claim A. Indeed, in $h_o \cup h$ the two flat nodes are isolated and have no edges to any core node. The single added edge connects the two flat nodes to each other but still creates no directed path from any core node to either flat node, nor from either flat node to any core node. Hence for every core node v in either copy, its set of ancestors and descendants is identical in $h_o \cup h$ and in h^*, so $N(v, h^*) = N(v, h_o \cup h)$ and thus $f(N(v, h^*)) = f(N(v, h_o \cup h))$. □

Claim B. *The only two nodes whose environments change are the two flat nodes, and each of them moves from environment N_1 to environment N_2.*

Proof of Claim B. Before the linkage, each flat node is isolated, so its environment is N_1. After adding the single edge between them, the induced environment of each flat node is exactly the 2-node chain N_2 (each flat node can reach or be reached by the other). Therefore, each flat node's evaluation changes from c_1 to c_2, so the combined change in the numerator is

$$\big(c_2 - c_1\big) + \big(c_2 - c_1\big) = 2(c_2 - c_1).$$

Moreover, by bounded marginal returns (the Lipschitz bound in regularity), adding one node to an environment changes its evaluation by at most M, hence $0 \le c_2 - c_1 \le M$, and therefore $0 \le 2(c_2 - c_1) \le 2M$. □

Combining Claim A and Claim B yields an exact expression for the numerator of h^*. Let $S(h_o \cup h)$ denote the numerator of the disjoint union $h_o \cup h$. Since h_o is isomorphic to h, we have $S(h_o \cup h) = S(h_o) + S(h) = 2S(h) = 2N^2C$. After the linkage, the only change is $2(c_2 - c_1)$, hence

$$S(h^*) = 2S(h) + 2(c_2 - c_1) = 2N^2C + 2(c_2 - c_1).$$

Since $|V(h^*)| = |V(h_o \cup h)| = 2N$, the QS formula gives

$$I(h^*) = \frac{S(h^*)}{(2N)^2} = \frac{2N^2C + 2(c_2 - c_1)}{4N^2} = \frac{C}{2} + \frac{c_2 - c_1}{2N^2}.$$

Using $0 \leq c_2 - c_1 \leq M$, we obtain the explicit bound

$$I(h^*) \leq \frac{C}{2} + \frac{M}{2N^2}.$$

Because $C > 0$, choose N large enough so that

$$\frac{M}{2N^2} < \frac{C}{2}.$$

For such an N we have

$$I(h^*) \leq \frac{C}{2} + \frac{M}{2N^2} < \frac{C}{2} + \frac{C}{2} = C = I(h).$$

Thus $I(h^*) < I(h)$, which contradicts the strict increase conclusion required by UE*.

The proof for DE* is symmetric: take the same two copies and add a single admissible cross-edge witnessing $h^* \in h \uplus h_o$ in the direction relevant for DE*; the same numerator computation yields $I(h^*) < I(h)$ for sufficiently large N, contradicting DE*. ■

Intuition (Theorem 1, Theorem 2). Theorem 1 relies on a *dilution* mechanism. AS and QS divide total structural scores by population size, whereas UE/DE demand that any valid vertical attachment strictly increases the index. By appending a massive, completely flat layer connected only through a single bottleneck node, the population (denominator) explodes while the aggregate structural score (numerator) barely changes. This severe dilution forces the index down, contradicting the extension axioms.

Theorem 2 uses a *fragmentation* argument specific to QS. The conditional axioms UE*/DE* block the "flat-layer" trick by requiring the appended structure to be sufficiently hierarchical. The proof circumvents this by linking two identical copies of a complex hierarchy via a single, minimal cross-edge. Doubling the population quadruples the quadratic denominator of QS. However, the single connecting edge alters the local environments of only two nodes. This negligible numerator gain is overwhelmed by the denominator's fourfold expansion, causing overall structural density to plummet and violating the axioms.

4.3 A class of bilateral average indices

Theorem 1 shows that no regular index can satisfy both localized averaging (AS) and unrestricted strict vertical monotonicity (UE/DE); hence one must relax one of these demands.

To proceed, we impose a normative invariance to internal routing: in many applications only reachability matters, not the particular intermediate chains. Because AS (and its analogues) decompose I into local contributions, this neutrality can be stated directly at the level of f.

Condition 1 (Environment Topology Neutrality, ETN)**.** *Let $I : \mathscr{H} \to \mathbb{R}$ satisfy one of AS, ADD, or QS with witness $f : \mathscr{N} \to \mathbb{R}$. We say that I satisfies ETN if for any environment $N \in \mathscr{N}$ centered at v, $f(N)$ is invariant to any rewiring of edges in N that preserves v's sets of direct and indirect supervisors and direct and indirect subordinates.*

Thus ETN makes a node's local score depend only on *who* can reach v and *who* v can reach, not on how those superiors/subordinates are internally connected (including bypass edges or direct vs. indirect reporting), which is nontrivial on DAGs. Combined with the population-average structure of AS and Anonymity (A), ETN forces a reduction to depth–span evaluations.

Lemma 1. *A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A, AS, and ETN if and only if there exists a function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for every $h \in \mathscr{H}$*

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

Proof. **Sufficiency (⇐).** Suppose

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g(s_h(v), \sigma_h(v))$$

for some function g. Define the local environment evaluation as $f(N(v,h)) = g(s_h(v), \sigma_h(v))$. Because the cardinalities $s_h(v)$ and $\sigma_h(v)$ are strictly invariant under graph isomorphism,

I is unchanged by any relabeling of the nodes, ensuring A holds. The additive functional form directly satisfies AS. Furthermore, because g depends exclusively on the sizes of the ancestor and descendant sets, any internal restructuring of edges that preserves those sets leaves the local evaluation unchanged, satisfying ETN.

Necessity (⇒). Assume I satisfies A, AS, and ETN.

Step 1. Symmetrization. By AS, there exists a local evaluation function $f : \mathscr{N} \to \mathbb{R}$ such that for any hierarchy $h \in \mathscr{H}$,

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} f(N(v,h)).$$

Although f itself is not assumed to be label-invariant, the global index I is. Let Π denote the set of all node-label permutations (relabelings) of the hierarchy h. By A, $I(h)$ is invariant under any $\pi \in \Pi$. Averaging the AS representation over all such permutations yields:

$$I(h) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} I(\pi(h)) = \frac{1}{|V(h)|} \sum_{v \in V(h)} \underbrace{\left[\frac{1}{|\Pi|} \sum_{\pi \in \Pi} f(\pi(N(v,h))) \right]}_{\bar{f}(N(v,h))}. \tag{3}$$

The bracketed term defines a symmetrized local evaluation function $\bar{f}$. Note that for any relabeling π, we have $\pi(N(v,h)) = N(\pi(v), \pi(h))$; thus the distinguished center node is transported along with the relabeling.

Claim 3. *The symmetrized evaluation $\bar{f}(N(v,h))$ depends strictly on the rooted isomorphism class (isomorphisms mapping the center to the center) of $N(v,h)$.*

Proof of Claim 3. To formally verify that the inner average in (3) depends only on the local environment, consider how a global permutation $\pi \in \Pi$ acts upon the environment's nodes $V(N(v,h))$. Let T be the set of all valid label injections $\tau : V(N(v,h)) \to V(h)$. We can partition the set of all global permutations Π into disjoint subsets $\Pi_\tau = \{\pi \in \Pi \mid \pi|_{V(N(v,h))} = \tau\}$, grouping together all permutations that apply the exact same labels to the local environment.

Because the function f evaluates only the isolated environment, its output depends solely on τ and is entirely blind to how π maps the remaining nodes $V(h) \setminus V(N(v,h))$. Thus, for any $\pi \in \Pi_\tau$, we have $f(\pi(N(v,h))) = f(\tau(N(v,h)))$.

The number of global permutations in each subset Π_τ is exactly $|\Pi_\tau| = (|V(h)| - |V(N(v,h))|)!$, representing all possible ways to distribute the leftover labels to the nodes outside the environment. We can therefore expand and factor the sum explicitly:

$$\begin{aligned}
\frac{1}{|V(h)|!}\sum_{\pi\in\Pi} f(\pi(N(v,h))) &= \frac{1}{|V(h)|!}\sum_{\tau\in T}\sum_{\pi\in\Pi_\tau} f(\pi(N(v,h))) \\
&= \frac{1}{|V(h)|!}\sum_{\tau\in T}\sum_{\pi\in\Pi_\tau} f(\tau(N(v,h))) \\
&= \frac{1}{|V(h)|!}\sum_{\tau\in T} |\Pi_\tau| f(\tau(N(v,h))) \\
&= \frac{(|V(h)| - |V(N(v,h))|)!}{|V(h)|!}\sum_{\tau\in T} f(\tau(N(v,h))).
\end{aligned}$$

Because the fraction on the right is the exact reciprocal of the total number of valid injections $|T|$, the ambient hierarchy merely contributes a combinatorial multiplier that is perfectly neutralized. The operation mathematically reduces to an exact, uniform arithmetic mean over all possible label injections of the isolated environment. If two environments are rooted isomorphic, summing over all possible injections from $V(h)$ will generate the exact same set of identically-evaluated terms. By integrating out the specific original node identities in this manner, $\bar{f}$ yields the exact same value for any two environments sharing the same topology, making it a well-defined function evaluated strictly on rooted isomorphism classes. □

We thus have

$$I(h) = \frac{1}{|V(h)|}\sum_{v\in V(h)} \bar{f}(N(v,h)).$$

Step 2. Canonical reduction via ETN. We first explicitly verify that because ETN holds for the witness f, it is inherited by the symmetrized witness $\bar{f}$. Let N and N' be two environments centered at v that differ only by an ETN-admissible rewiring. By the definition of ETN, any relabeling π preserves the cardinalities of the ancestor and descendant sets, meaning the relabeled environments $\pi(N)$ and $\pi(N')$ also differ only by an ETN-admissible rewiring. Since the original witness f satisfies ETN, it follows that $f(\pi(N)) = f(\pi(N'))$ for every possible relabeling $\pi \in \Pi$. Summing these identical evaluations over all $\pi \in \Pi$ and dividing by $|\Pi|$ guarantees exactly $\bar{f}(N) = \bar{f}(N')$.

Fix any node $v \in V(h)$. Notice that ETN restricts the evaluation based *only* on the absolute sets of ancestors and descendants of the centered node v. It is strong enough to allow us to completely ignore and eliminate all other internal topology (i.e., edges strictly among ancestors, strictly among descendants, or directly from ancestors to descendants). We restructure $N(v, h)$ into a canonical "double-star" environment, denoted $D_{s,\sigma}$, which consists of exactly $s = s_h(v)$ independent roots pointing directly to v, and v pointing directly to exactly $\sigma = \sigma_h(v)$ independent leaves, with no other internal edges.

Such a rewiring always exists: delete all internal edges among ancestors, among descendants, and between ancestors and descendants; then add directed edges $u \to v$ for each ancestor u, and $v \to w$ for each descendant w. The resulting environment is acyclic and preserves exactly the absolute sets of ancestors and descendants of v. By ETN, this structural rewiring yields:

$$\bar{f}(N(v, h)) = \bar{f}(D_{s_h(v),\sigma_h(v)}).$$

Step 3. Defining the universal function. Because $\bar{f}$ is strictly invariant on rooted isomorphism classes (from Claim 1) and the double-star $D_{s,\sigma}$ is completely characterized up to rooted isomorphism by the cardinalities s and σ, the evaluation $\bar{f}(D_{s,\sigma})$ depends strictly on these two integers. We can therefore define a universal function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ as:

$$g(s, \sigma) = \bar{f}(D_{s,\sigma}).$$

Substituting this directly into the symmetrized AS representation from Step 1 gives:

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

This completes the proof. ∎

4.3.1 Intensive Population Growth (IPG)

Definition 2. A hierarchical index $I : \mathcal{H} \to \mathbb{R}$ is a *bilateral average index* if there exists $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for all $h \in \mathcal{H}$,

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

We impose a regularity condition ensuring that macroscopic structural change leaves a non-vanishing per-capita trace. Let I be as in Definition 2. For a sequence of compounds $h'_k \in h_k \uplus h_k$, identify each $v \in V(h'_k)$ with its baseline counterpart $v_0 \in V(h_k)$, and write

$$\Delta s_k(v) = s_{h'_k}(v) - s_{h_k}(v_0), \qquad \Delta\sigma_k(v) = \sigma_{h'_k}(v) - \sigma_{h_k}(v_0).$$

Call $(h'_k \in h_k \uplus h_k)$ *macroscopic* if some $\rho > 0$ and k_0 satisfy, for all $k \geq k_0$,

$$\left|\{v \in V(h'_k) : \Delta s_k(v) > 0 \text{ or } \Delta\sigma_k(v) > 0\}\right| \geq \rho|V(h'_k)|.$$

Define discrete marginals

$$\Delta_s g(s,\sigma) = g(s+1,\sigma) - g(s,\sigma), \qquad \Delta_\sigma g(s,\sigma) = g(s,\sigma+1) - g(s,\sigma),$$

and node-level evaluation gains

$$\Delta g_k(v) = g\big(s_{h'_k}(v), \sigma_{h'_k}(v)\big) - g\big(s_{h_k}(v_0), \sigma_{h_k}(v_0)\big), \qquad \Delta g_k^+(v) = \max\{\Delta g_k(v), 0\}.$$

Let $\Delta g^+_{(1)} \geq \Delta g^+_{(2)} \geq \cdots$ be the order statistics of $\{\Delta g_k^+(v) : v \in V(h'_k)\}$, and for $c \geq 1$ set

$$M_k(c) = \frac{1}{|V(h'_k)|}\sum_{i=1}^{c} \Delta g^+_{(i)}.$$

Condition 2 (IPG: Intensive Population Growth)**.** *A bilateral average index I satisfies* IPG *if for any macroscopic sequence $(h'_k \in h_k \uplus h_k)$ with $|V(h_k)| \to \infty$ and any fixed $c \geq 1$,*

$$\liminf_{k\to\infty}\Big[I(h'_k) - I(h_k) - M_k(c)\Big] > 0.$$

Remark 6. IPG requires that the per-capita gain from a macroscopic shift remains strictly positive after removing the contribution of any fixed c outliers (captured by $M_k(c)$). In particular, bounded discrete marginals imply that $M_k(c) \to 0$ as $|V(h'_k)| \to \infty$. Separately, the strict inequality ensures that g responds persistently to widespread increases in depth or span (see Proposition 1 and Theorem 3).

Proposition 1. *If a bilateral average index satisfies* IPG*, its local evaluation function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ must be strictly increasing in both arguments and, for any fixed value of one coordinate, have discrete marginals strictly bounded away from zero in the other argument.*

Proof. Suppose I is a bilateral average index characterized by $g(s, \sigma)$. We first prove that g must be strictly increasing in each argument by constructing macroscopic compound sequences that isolate a unit increase in a single coordinate.

Step 1 (strict increase in span). Assume, toward a contradiction, that g is not strictly increasing in span. Then there exist $s, \sigma \in \mathbb{Z}_+$ such that $g(s, \sigma+1) \leq g(s, \sigma)$. Fix an integer $q \geq 3$. For each large integer m, construct a hierarchy h_m with $|V(h_m)| = qm$ containing exactly $K_m = m$ *targets* each with depth s and span σ, as follows.

Construct s shared ancestors (without loss, s distinct roots, each with an edge to every target) and a shared collection of σ leaves. If $\sigma = 0$, omit these leaves. Each target points to all σ shared leaves. If $\sigma \geq 1$, choose a distinguished leaf b_1. Arrange the remaining wiring so that the set of nodes that can reach b_1 is exactly the set of targets, the s shared ancestors, and a support set R_m satisfying $|R_m| \leq \bar{c}$ for some fixed $\bar{c}$ independent of m.[2] If $\sigma = 0$, set $R_m = \varnothing$ (Figure 2). Finally, designate at least one remaining node of h_m to be isolated (having no incoming or outgoing edges); this is possible for all sufficiently large m because the number of remaining nodes is $qm - m = (q-1)m$, which grows to infinity, exceeding the number of required non-target nodes.

Now form a compound $h'_m \in h_m \uplus h_m$, where $\uplus$ is the compounding operator in IPG. Choose $z^{\downarrow}$ to be an isolated node in the lower copy. If $\sigma \geq 1$, let $b_1^{\uparrow}$ be the copy of b_1 in the upper copy and add the single cross-edge $b_1^{\uparrow} \to z^{\downarrow}$. If $\sigma = 0$, add cross-edges from each upper-copy target to $z^{\downarrow}$. In either case, every upper-copy target gains $z^{\downarrow}$ as a new descendant, so its span increases by exactly 1 and its depth is unchanged.

Because $z^{\downarrow}$ has no outgoing edges, the only new reachability relations are paths ending at $z^{\downarrow}$. Hence, a node's span increases if and only if it could reach the linkage tail(s) before the link: $b_1^{\uparrow}$ when $\sigma \geq 1$, and the targets themselves when $\sigma = 0$. By construction, the nodes with $\Delta\sigma > 0$ are exactly the targets, the s shared ancestors, the support set R_m, and the linkage tail $b_1^{\uparrow}$ when $\sigma \geq 1$. Define the residual set

$$\widetilde{R}_m = (\text{both copies of } R_m) \cup \{z^{\downarrow}\} \cup (\{b_1^{\uparrow}\} \text{ if } \sigma \geq 1) \cup (\text{the } s \text{ shared ancestors in the upper copy}).$$

[2] The proof still holds if we set $R_m = \varnothing$ (using only direct edges to form the required depth and span). We include bounded support sets in the formal construction to illustrate that the limit infimum bounding argument is robust to arbitrary internal bridging architectures, provided the number of such bridging nodes remains independent of m.

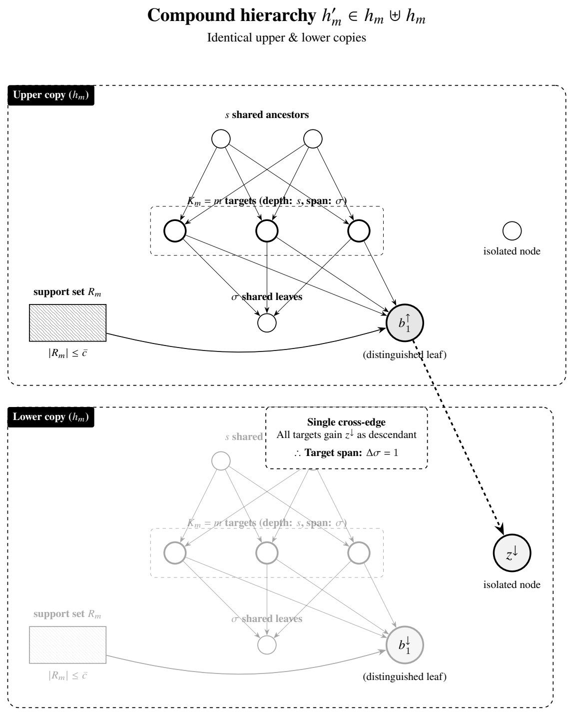

Figure 2: Compound hierarchy h'_m.

Then

$$|\widetilde{R}_m| \leq s + 2\bar{c} + 2 = c_0,$$

and c_0 is a finite constant independent of m. Moreover, every node outside the targets and $\widetilde{R}_m$ has $\Delta s = \Delta\sigma = 0$ and hence $\Delta g_m(v) = 0$.

Since $\frac{K_m}{|V(h'_m)|} = \frac{m}{2qm} = \frac{1}{2q} > 0$, the ratio of targets to total nodes is exactly constant for all m. Hence the sequence is macroscopic (with macroscopic constant $1/(2q)$).

For each target v we have $\Delta g_m(v) = g(s, \sigma + 1) - g(s, \sigma) \leq 0$. Thus every node with $\Delta g_m(v) > 0$ lies in $\widetilde{R}_m$, so there are at most c_0 nodes with strictly positive gains. Let $\Delta g_m^+(v) := \max\{\Delta g_m(v), 0\}$ and let $\Delta g_m^+(1) \geq \Delta g_m^+(2) \geq \cdots$ be the order statistics. Because at most c_0 of the $\Delta g_m^+(v)$ are nonzero, the total positive gain equals $\sum_{i=1}^{c_0} \Delta g_m^+(i)$, and by definition of $M_m(c)$ in IPG we have

$$\frac{1}{|V(h'_m)|} \sum_{v \in V(h'_m)} \Delta g_m^+(v) = \frac{1}{|V(h'_m)|} \sum_{i=1}^{c_0} \Delta g_m^+(i) \leq M_m(c_0).$$

Therefore,

$$I(h'_m) - I(h_m) - M_m(c_0) = \frac{1}{|V(h'_m)|} \sum_{v \in V(h'_m)} \Delta g_m(v) - M_m(c_0) \le \frac{1}{|V(h'_m)|} \sum_{v \notin \widetilde{R}_m} \Delta g_m(v) \le 0,$$

since nodes outside $\widetilde{R}_m$ are either targets (nonpositive gains) or unchanged nodes (zero gain). Because the sequence is macroscopic, taking $\liminf_{m\to\infty}$ strictly contradicts IPG. Hence $g(s, \sigma + 1) > g(s, \sigma)$ for all s, σ.

Step 2 (strict increase in depth). A symmetric construction yields strict increase in depth. If for some s, σ we had $g(s + 1, \sigma) \le g(s, \sigma)$, build h_m with $K_m = m$ targets at depth s and span σ so that, for $s \ge 1$, there is a root a_1 whose set of descendants is exactly the K_m targets, the σ shared leaves, and a support set R'_m with $|R'_m| \le \bar{c}$ (and all other nodes lie outside $\mathrm{Desc}(a_1)$) (Figure 3). If $s = 0$, omit a_1 and take $R'_m = \varnothing$.

Form $h'_m \in h_m \uplus h_m$ and add a cross-edge from an isolated node $w^\uparrow$ in the upper copy to $a_1^\downarrow$ in the lower copy (or to each target if $s = 0$). Since $w^\uparrow$ has no incoming edges, the only new reachability relations are paths originating at $w^\uparrow$, so depth increases for the targets, the σ shared leaves, and R'_m. Node $a_1^\downarrow$ also gains $w^\uparrow$ as a new ancestor. Define

$$\widetilde{R}'_m = (\text{both copies of } R'_m) \cup \{w^\uparrow\} \cup (\{a_1^\downarrow\} \text{ if } s \ge 1) \cup (\text{the } \sigma \text{ shared leaves in the lower copy}).$$

Then $|\widetilde{R}'_m| \le \sigma + 2\bar{c} + 2 = c'_0$, a finite constant independent of m, and every node with $\Delta g_m(v) > 0$ lies in $\widetilde{R}'_m$.

Applying the Step-1 order-statistic argument with residual size c'_0 yields

$$I(h'_m) - I(h_m) - M_m(c'_0) \le 0.$$

Just as in Step 1, taking $\liminf_{m\to\infty}$ directly contradicts IPG along the macroscopic sequence. Therefore $g(s + 1, \sigma) > g(s, \sigma)$ for all s, σ.

Step 3 (marginals bounded away from zero along fixed coordinates). Fix s. Suppose $\inf_\sigma \Delta_\sigma g(s, \sigma) = 0$, where $\Delta_\sigma g(s, \sigma) = g(s, \sigma + 1) - g(s, \sigma)$. Then there exists a sequence $\sigma_k \to \infty$ such that $\Delta_\sigma g(s, \sigma_k) \to 0$. Using the span gadget from Step 1 with targets at fixed depth s and span $\sigma_m = \sigma_{k(m)}$, choose $k(m)$ increasing such that

$$s + \sigma_{k(m)} + \bar{c} + 1 \le (q - 1)m,$$

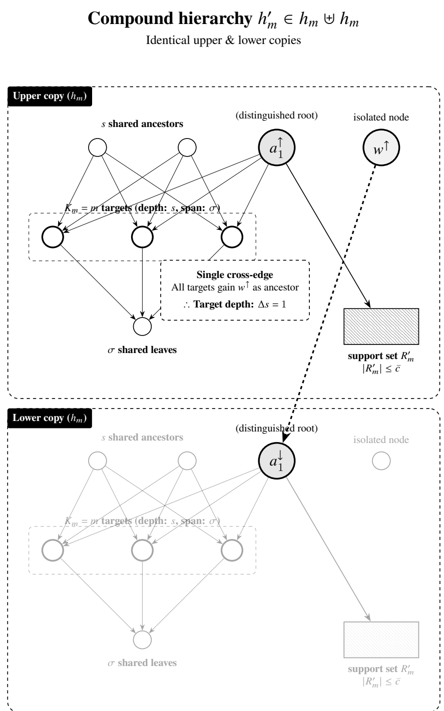

Figure 3: Compound hierarchy h'_m.

ensuring the construction fits within size qm while reserving an isolated node in h_m.

The same residual set bound gives a fixed constant c_0 (depending only on s and $\bar{c}$) such that all strictly positive gains lie among at most c_0 nodes. Hence

$$I(h'_m) - I(h_m) - M_m(c_0) \leq \frac{K_m}{|V(h'_m)|} \Delta_\sigma g(s, \sigma_m).$$

Since $K_m/|V(h'_m)| = 1/(2q)$, and $\Delta_\sigma g(s, \sigma_m) \to 0$, the right-hand side converges to 0, so the $\liminf$ is ≤ 0, contradicting IPG. Therefore $\inf_\sigma \Delta_\sigma g(s, \sigma) > 0$.

Fix σ and argue symmetrically using the depth gadget from Step 2: if $\inf_s \Delta_s g(s, \sigma) = 0$ with $\Delta_s g(s, \sigma) = g(s+1, \sigma) - g(s, \sigma)$, choose a sequence $s_k \to \infty$ with $\Delta_s g(s_k, \sigma) \to 0$, and then choose an increasing map $k(m)$ such that

$$s_{k(m)} + \sigma + \bar{c} + 1 \leq (q-1)m.$$

Placing the targets at depth $s_{k(m)}$ and span σ and applying the same reasoning (now with a fixed residual bound depending only on σ and $\bar{c}$) yields $\inf_s \Delta_s g(s, \sigma) > 0$ for each fixed σ.

This completes the proof. ∎

4.3.2 Axiomatic characterization

With the foundational structure and regularity conditions established, we can now state the main representation theorem for the family of average bilateral indices.

Theorem 3. *A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A, AS, and ETN, and its corresponding bilateral average representation satisfies IPG, if and only if there exists a function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for all $h \in \mathscr{H}$,*

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big),$$

and g has discrete marginals strictly bounded away from zero: there exists a constant $m > 0$ such that for all $s, \sigma \in \mathbb{Z}_+$,

$$m \le \Delta_s g(s, \sigma) \qquad \textit{and} \qquad m \le \Delta_\sigma g(s, \sigma).$$

Remark 7. A uniform lower bound $m > 0$ on the discrete marginals implies that g is strictly increasing in both arguments: any unit increase in s or σ raises g by at least m.

Proof of Theorem 3. **Necessity (⇒).** Suppose I satisfies A, AS, ETN, and its corresponding bilateral average representation satisfies IPG. By Lemma 1, the combination of A, AS, and ETN implies that I is a bilateral average index, admitting a representation of the form

$$I(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g(s_h(v), \sigma_h(v))$$

for some real-valued local evaluation function g.

Because I satisfies IPG, Proposition 1 dictates that g must be strictly increasing. We now show IPG implies the discrete marginals are uniformly bounded away from zero. Suppose, toward a contradiction, that $\inf_{s,\sigma} \Delta_s g(s, \sigma) = 0$. Then there exists a sequence of coordinate pairs (s_k, σ_k) such that $\Delta_s g(s_k, \sigma_k) \to 0$ as $k \to \infty$.

Following the gadget construction in the proof of Proposition 1, we can build a macroscopic sequence of compounds $h'_k \in h_k \uplus h_k$, where $\uplus$ is the partial-append operator (adding leaf-to-root cross-edges), applied here to two copies of h_k (one designated as the "upper" copy, one as the "lower" copy). Choose the gadget parameters such that the number of target nodes K_k satisfies $\lim_{k\to\infty} K_k/|V(h'_k)| = \bar{\rho}$ for some constant $\bar{\rho} > 0$. For each k, the hierarchy contains exactly K_k target nodes placed at depth s_k and span σ_k. Furthermore, the construction guarantees that all strictly positive coordinate shifts outside the targets are confined to a residual set $\widetilde{R}_k$ whose size is strictly bounded by a finite constant c_0 that is completely independent of k.

Identify each node $v \in V(h'_k)$ with its counterpart v_0 in the appropriate baseline copy of h_k. Since I is an AS-average, replication invariance yields $I(h_k) = I(h_k \cup h_k)$ (where $\cup$ denotes the disjoint union of the two baseline copies). Note that $|V(h'_k)| = |V(h_k \cup h_k)| = 2|V(h_k)|$. Hence, the difference in the index can be expressed precisely as the average of node-level evaluation increments:

$$I(h'_k) - I(h_k) = \frac{1}{|V(h'_k)|} \sum_{v \in V(h'_k)} \Delta g_k(v).$$

Let T_k be the set of targets. By construction, every node v outside $T_k \cup \widetilde{R}_k$ has $\Delta s_k(v) = \Delta\sigma_k(v) = 0$, and hence $\Delta g_k(v) = 0$. Thus,

$$I(h'_k) - I(h_k) = \frac{1}{|V(h'_k)|} \left(\sum_{v \in T_k} \Delta g_k(v) + \sum_{v \in \widetilde{R}_k} \Delta g_k(v) \right).$$

Moreover, $\Delta g_k(v) \le \Delta g_k^+(v) = \max\{\Delta g_k(v), 0\}$ for all v, which implies:

$$I(h'_k) - I(h_k) \le \frac{1}{|V(h'_k)|} \left(\sum_{v \in T_k} \Delta g_k(v) + \sum_{v \in \widetilde{R}_k} \Delta g_k^+(v) \right).$$

Because $|\widetilde{R}_k| \le c_0$, the sum of the positive gains in $\widetilde{R}_k$ is bounded by the sum of the c_0 largest positive gains across the entire hierarchy. Let

$$\Delta g^+_{k,(1)} \ge \Delta g^+_{k,(2)} \ge \cdots$$

denote the order statistics of the positive gains in h'_k. Then

$$\sum_{v \in \widetilde{R}_k} \Delta g_k^+(v) \le \sum_{i=1}^{c_0} \Delta g^+_{k,(i)}.$$

Dividing by $|V(h'_k)|$, this normalized sum is bounded directly by the definition of the outlier term $M_k(c_0)$ in IPG. Hence, we obtain the upper bound:

$$I(h'_k) - I(h_k) - M_k(c_0) \le \frac{1}{|V(h'_k)|} \sum_{v \in T_k} \Delta g_k(v).$$

Because every target node experiences a unit increase in depth and no change in span, its individual evaluation gain is exactly $\Delta_s g(s_k, \sigma_k) > 0$ (since g was established as strictly increasing). Therefore,

$$I(h'_k) - I(h_k) - M_k(c_0) \le \frac{K_k}{|V(h'_k)|} \Delta_s g(s_k, \sigma_k).$$

Because the sequence is macroscopic (with $\lim_{k\to\infty} K_k/|V(h'_k)| = \bar{\rho} > 0$) and we assumed the marginal gains shrink to zero ($\Delta_s g(s_k, \sigma_k) \to 0$), evaluating the intensive growth condition yields:

$$\liminf_{k\to\infty} \big[I(h'_k) - I(h_k) - M_k(c_0)\big] \le \lim_{k\to\infty} \frac{K_k}{|V(h'_k)|} \Delta_s g(s_k, \sigma_k) = 0.$$

This implies the macroscopic sequence fails to generate strictly positive intensive growth, directly contradicting IPG. Hence, $\inf_{s,\sigma} \Delta_s g(s, \sigma) > 0$. An identical argument applying the span gadget yields $\inf_{s,\sigma} \Delta_\sigma g(s, \sigma) > 0$. Letting

$$m = \min \left\{ \inf_{s,\sigma} \Delta_s g(s, \sigma), \inf_{s,\sigma} \Delta_\sigma g(s, \sigma) \right\} > 0$$

proves the existence of a uniform strictly positive lower bound m across the entire domain, establishing necessity.

Sufficiency (⇐). Suppose I admits the displayed representation where g has discrete marginals bounded below by a strictly positive constant $m > 0$. By the sufficiency direction of Lemma 1, any index possessing this functional dependence strictly on depth and span coordinates necessarily satisfies A, AS, and ETN.

To establish IPG, note first that because each compound $h'_k \in h_k \uplus h_k$ is formed by adding directed edges and deleting none, every node's depth and span are weakly nondecreasing (since adding edges preserves all existing directed paths, so ancestor and descendant sets can only expand): $\Delta s_k(v) \ge 0$ and $\Delta \sigma_k(v) \ge 0$ for all v. The total evaluation gain for any node can be decomposed into a telescoping sequence of unit steps (with the convention that an empty sum equals 0):

$$g(s+\Delta s,\sigma+\Delta\sigma)-g(s,\sigma)=\sum_{i=0}^{\Delta s-1}\Delta_s g(s+i,\sigma)+\sum_{j=0}^{\Delta\sigma-1}\Delta_\sigma g(s+\Delta s,\sigma+j).$$

Consequently, the uniform lower bound $m > 0$ implies that any strictly positive coordinate shift ($\Delta s + \Delta\sigma \geq 1$) yields a marginal evaluation gain of at least m. For any macroscopic sequence of compounds ($h'_k \in h_k \uplus h_k$), a non-vanishing fraction $\rho' > 0$ of nodes experiences such a shift. Thus, there are at least $\rho'|V(h'_k)|$ nodes that achieve a positive evaluation gain of at least m.

The outlier term $M_k(c)$ explicitly removes the c largest marginal gains from the normalized aggregate sum. Even if all c removed nodes are drawn directly from this macroscopic set, there remain at least $\rho'|V(h'_k)| - c$ nodes contributing a gain of at least m to the residual sum. Therefore, evaluating the full intensive growth condition yields:

$$\liminf_{k\to\infty}\left[I(h'_k)-I(h_k)-M_k(c)\right]\geq\lim_{k\to\infty}\frac{\rho'|V(h'_k)|-c}{|V(h'_k)|}m=\rho' m>0,$$

so the index successfully leaves a persistent per-capita structural trace and satisfies IPG. This establishes sufficiency. ∎

The characterization above is the core structural result for the average family. For further conceptual ramifications of this theorem—including the role of IPG in admissible specifications and the equivalence of unilateral and bilateral forms—along with non-redundancy examples and an axiom compliance discussion, see Carbonell-Nicolau (2026, Remarks 9–10, §§4.3.3–4.3.4).

4.4 A class of total bilateral indices

While Theorem 3 yields *average* bilateral indices, some applications call for *aggregate* structural volume. This requires replacing the population average in AS with the unnormalized sum ADD. As before, we impose ETN (now relative to the ADD witness f) to remove sensitivity to internal routing among supervisors/subordinates.

Lemma 2. *A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A, ADD, and ETN if and only if there* *exists a function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for every $h \in \mathscr{H}$,*

$$I(h) = \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

The proof repeats Lemma 1 with AS replaced by ADD: symmetrize f via A to obtain $\bar{f}$, use ETN to rewire to the canonical double-star $D_{s,\sigma}$, and set $g(s,\sigma) = \bar{f}(D_{s,\sigma})$. Thus, A+ADD+ETN force an unweighted depth–span sum.

Unlike the intensive case, a nonzero baseline for an isolated node would make total hierarchy scale with the size of an unstructured workforce; we therefore normalize isolated nodes to contribute zero.

Isolated Node Normalization (INN). A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies INN if $I(h) = 0$ whenever $h \in \mathscr{H}$ consists of a single isolated node.

4.4.1 Extensive Population Growth (EPG)

Definition 3. A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ is a *total bilateral index* if there exists $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for all $h \in \mathscr{H}$,

$$I(h) = \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

To require that macroscopic change yield a genuine aggregate increase, we impose a regularity condition for total indices. We use the notation of Section 4.3.1 for macroscopic sequences ($h'_k \in h_k \uplus h_k$), baseline matching $v \leftrightarrow v_0$, and gains $\Delta g_k(v)$ (hence $\Delta g_k^+(v) = \max\{\Delta g_k(v), 0\}$). Because totals are unnormalized sums, the outlier term is the *absolute* top-c accumulation: if $\Delta g^+_{k,(1)} \geq \Delta g^+_{k,(2)} \geq \cdots$ are the order statistics of $\{\Delta g_k^+(v) : v \in V(h'_k)\}$, define for $c \geq 1$

$$M_k^{\text{ext}}(c) = \sum_{i=1}^{c} \Delta g^+_{k,(i)}.$$

Condition 3 (EPG: Extensive Population Growth)**.** *A total bilateral index I satisfies EPG if for any macroscopic sequence ($h'_k \in h_k \uplus h_k$) with $|V(h_k)| \to \infty$ and any fixed $c \geq 1$,*

$$\liminf_{k\to\infty} \Big[\big(I(h'_k) - 2I(h_k)\big) - M_k^{\text{ext}}(c) \Big] > 0.$$

Remark 8. EPG is the extensive analogue of the trimming logic in IPG: after removing any fixed c outliers (via $M_k^{\mathrm{ext}}(c)$), macroscopic advancement must still raise the aggregate score. The baseline $2I(h_k)$ reflects additivity: two disjoint copies of h_k have total volume $2I(h_k)$.

Proposition 2. *If a total bilateral index satisfies EPG, its local evaluation function* $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ *must be strictly increasing in both arguments. Furthermore, for any fixed value of one coordinate, the discrete marginals must satisfy*

$$\liminf_{\sigma\to\infty} \sigma \Delta_\sigma g(s,\sigma) > 0 \quad \textit{and} \quad \liminf_{s\to\infty} s \Delta_s g(s,\sigma) > 0.$$

Proof. Suppose I is a total bilateral index characterized by $g(s,\sigma)$. We evaluate the aggregate structural shifts directly using the macroscopic compound sequences $h'_m \in h_m \uplus h_m$ constructed in the proof of Proposition 1.

Step 1 (strict increase in span). Assume, toward a contradiction, that g is not strictly increasing in span. Then there exist $s, \sigma \in \mathbb{Z}_+$ such that $g(s,\sigma+1) \le g(s,\sigma)$. Let $\delta = g(s,\sigma+1) - g(s,\sigma) \le 0$.

Apply the exact compound hierarchy sequence $h'_m \in h_m \uplus h_m$ constructed in Step 1 of the proof of Proposition 1, containing $K_m = m$ targets shifting from (s,σ) to $(s,\sigma+1)$. In the compound h'_m, the fraction of targets $K_m/|V(h'_m)|$ remains strictly bounded away from zero for all sufficiently large m, ensuring the sequence is macroscopic. By construction, all strictly positive gains are confined to the residual set $\widetilde{R}_m$. The size of this set is strictly bounded by a finite constant $c_0 = s + 2\bar{c} + 2$, which is independent of m.

Because the baseline volume of the two disjoint copies under the additive framework is exactly $2I(h_m)$, the aggregate change is

$$I(h'_m) - 2I(h_m) = \sum_{v\in V(h'_m)} \Delta g_m(v) \le K_m\,\delta + \sum_{v\in\widetilde{R}_m} \Delta g_m^+(v).$$

Evaluating EPG at $c = c_0$, the extensive trimming term $M_m^{\mathrm{ext}}(c_0)$ by definition sums the c_0 largest positive gains across the entire compound. Since $|\widetilde{R}_m| \le c_0$, we have $\sum_{v\in\widetilde{R}_m} \Delta g_m^+(v) \le M_m^{\mathrm{ext}}(c_0)$. Subtracting $M_m^{\mathrm{ext}}(c_0)$ thus removes at least the total positive contribution of $\widetilde{R}_m$ in the order-statistic sense. Since targets experience a gain of $\delta \le 0$ and all nodes outside both the targets and the residual set experience nonpositive gains, we obtain

$$\big(I(h'_m) - 2I(h_m)\big) - M_m^{\mathrm{ext}}(c_0) \le K_m\,\delta \le 0.$$

Taking $\liminf_{m\to\infty}$ strictly contradicts EPG. Hence $g(s, \sigma + 1) > g(s, \sigma)$ for all s, σ.

Step 2 (strict increase in depth). A symmetric argument applies. If $g(s+1, \sigma) \leq g(s, \sigma)$, we employ the sequence constructed in Step 2 of the proof of Proposition 1, wherein $K_m = m$ targets experience a unit increase in depth. All positive spillovers are confined to a residual set $\widetilde{R}'_m$ whose size is strictly bounded by a finite constant $c'_0 = \sigma + 2\bar{c} + 2$. Since $|\widetilde{R}'_m| \leq c'_0$, we have $\sum_{v\in\widetilde{R}'_m} \Delta g^+_m(v) \leq M^{\text{ext}}_m(c'_0)$. Subtracting the trimming term again removes at least the positive spillovers in the order-statistic sense, yielding a nonpositive aggregate gain and contradicting EPG. Thus $g(s + 1, \sigma) > g(s, \sigma)$.

Step 3 (decay rate of marginals). Fix s. Suppose the discrete marginals decay so rapidly in σ that $\liminf_{\sigma\to\infty} \sigma\Delta_\sigma g(s, \sigma) = 0$. Then there exists a strictly increasing sequence $\sigma_m \to \infty$ such that $\sigma_m \Delta_\sigma g(s, \sigma_m) \to 0$.

We reuse the span gadget from Step 1, placing $K_m = \sigma_m$ targets at fixed depth s and varying span σ_m. Because the targets share the σ_m leaves, we define the baseline hierarchy size as exactly $|V(h_m)| = 2\sigma_m + s + \bar{c} + 2$. In the compound h'_m, at least $K_m = \sigma_m$ nodes experience a strictly positive span shift. The total node count is $|V(h'_m)| = 2|V(h_m)|$. As $m \to \infty$, the fraction of shifting nodes $K_m/|V(h'_m)|$ converges to $1/4$, meaning it remains strictly bounded away from zero for all sufficiently large m, ensuring the sequence is macroscopic.

By Steps 1 and 2, g is strictly increasing, so the target nodes experience strictly positive gains. The aggregate target gain is exactly $K_m\Delta_\sigma g(s, \sigma_m) = \sigma_m\Delta_\sigma g(s, \sigma_m)$, which by assumption converges to 0. As in Step 1, all positive spillovers outside the targets are confined to a finite residual set $\widetilde{R}_m$ of size at most c_0, which is independent of the growing span σ_m. Because $|\widetilde{R}_m| \leq c_0$, their total contribution is mathematically upper-bounded by the trimming term: $\sum_{v\in\widetilde{R}_m} \Delta g_m(v) \leq \sum_{v\in\widetilde{R}_m} \Delta g^+_m(v) \leq M^{\text{ext}}_m(c_0)$. Thus, subtracting the trimming term dominates the residual spillovers, leaving the aggregate gain bounded above by the target contributions:

$$\big(I(h'_m) - 2I(h_m)\big) - M^{\text{ext}}_m(c_0) \leq \sigma_m\Delta_\sigma g(s, \sigma_m).$$

Taking the limit infimum as $m \to \infty$, the right-hand side converges to 0, which strictly contradicts the positivity required by EPG. Thus, we must have $\liminf_{\sigma\to\infty} \sigma\Delta_\sigma g(s, \sigma) > 0$.

A symmetric argument using the depth gadget from Step 2, wherein the fraction of shifting nodes again converges to a positive constant, yields $\liminf_{s\to\infty} s\Delta_s g(s,\sigma) > 0$ for each fixed σ. This completes the proof. ∎

4.4.2 Axiomatic characterization

Theorem 4. *A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A, ADD, ETN, INN, and its corresponding total bilateral representation satisfies EPG, if and only if there exists a graph-independent function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that $g(0,0) = 0$,*

$$I(h) = \sum_{v\in V(h)} g\big(s_h(v), \sigma_h(v)\big) \quad \textit{for all } h \in \mathscr{H},$$

and g satisfies the asymptotic growth condition: for every macroscopic sequence of compounds $(h'_k \in h_k \uplus h_k)$ with $|V(h_k)| \to \infty$ and every integer $c \geq 1$,

$$\liminf_{k\to\infty}\left(\sum_{v\in V(h'_k)} \Delta g_k(v) - \sum_{i=1}^{c} \Delta g^{+}_{k,(i)}\right) > 0.$$

Remark 9. One might wonder why the asymptotic growth condition is not replaced by the simpler axis-based bounds identified in Proposition 2 (i.e., strict monotonicity and $\liminf x\Delta_x g > 0$), or by a strict uniform lower bound mirroring the intensive framework (Theorem 3).

First, while the marginal bounds in Proposition 2 are *necessary* consequences of EPG, they are not generally *sufficient*. Because macroscopic shifts can distribute coordinate gains across highly irregular, asymmetric graph topologies, pointwise limits along isolated axes cannot perfectly bound the aggregate trimmed growth required across all possible sequences. The growth condition quantifies over these sequences to achieve exact sufficiency.

Second, imposing a uniform strict lower bound on an unweighted sum (e.g., $\Delta g \geq m > 0$) would mathematically outlaw diminishing marginal returns, forbidding natural sublinear evaluation functions such as logarithmic or square-root growth. In an aggregate measure, it is economically plausible that adding the hundredth layer of depth contributes marginally less to the total structural volume than adding the second layer. The growth condition deliberately accommodates this decay, enforcing only the asymptotic limit: marginal gains may shrink, but they must remain collectively large enough across a macroscopic shift to strictly offset the removal of any finite number of outliers.

The proof is a direct unpacking of Lemma 2 and the definition of EPG: under the total bilateral representation, EPG is equivalent to the stated asymptotic growth condition by algebraic substitution. See Carbonell-Nicolau (2026, §§4.4.2–4.4.4) for details and for additional remarks on admissible specifications and equivalence transformations (including unilateral reductions), as well as a detailed compliance summary.

4.5 A class of bilateral concentration indices

The preceding sections characterized average and total indices by applying linear (AS) and absolute (ADD) separability axioms. We now complete the structural trichotomy by characterizing the family of indices derived from Quadratic Separability (QS).

Rather than measuring average or aggregate hierarchy, indices in this class evaluate structural density—the concentration of authority relative to the population's theoretical capacity.

Combined with the quadratic density structure of QS and the anonymity of A, ETN restricts the index to evaluate hierarchy solely through individual depth and span—forcing local evaluations to ignore internal edge restructurings that leave supervisor and subordinate sets unchanged.

Lemma 3. *A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ satisfies A, QS, and ETN if and only if there exists a function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for every $h \in \mathscr{H}$,*

$$I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} g(s_h(v), \sigma_h(v)). \tag{4}$$

The proof is entirely analogous to that of Lemma 1 and therefore omitted.

4.5.1 Concentration Population Growth (CPG)

Definition 4. A hierarchical index $I : \mathscr{H} \to \mathbb{R}$ is a *bilateral concentration index* if there exists a local evaluation function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ such that for all $h \in \mathscr{H}$,

$$I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big).$$

To ensure the local evaluation function $g(s, \sigma)$ correctly rewards hierarchical depth and span, we require a regularity condition analogous to IPG.

However, we cannot apply IPG directly to concentration indices. A macroscopic sequence of compounds injects new structure proportional to the population size. Because a concentration index penalizes population quadratically, this denominator growth consistently overwhelms structural additions. Thus, as a general rule for density metrics where the numerator grows at most linearly in $|V(h)|$ along macroscopic compounding, the raw difference $I(h'_k) - I(h_k)$ typically becomes negative—quadratic denominator growth can easily dominate the linear structural additions—so applying IPG directly is inappropriate.

To evaluate structural monotonicity without density dilution, we multiply the index by the squared population to recover the absolute structural volume. Dividing the marginal change in this volume by the base population yields the *per-capita structural accumulation*, mirroring the mechanics of IPG.

Notation. We use the comparative framework from Section 4.3.1. In particular, for any compound $h'_k \in h_k \uplus h_k$, we use the same identification of each node with its baseline counterpart v_0 in h_k, and the same definitions of $\Delta s_k(v)$, $\Delta\sigma_k(v)$, $\Delta g_k(v)$, and $\Delta g_k^+(v)$, as well as the associated order statistics $\Delta g^+_{(1)} \geq \Delta g^+_{(2)} \geq \cdots$. For $c \geq 1$, define the (absolute) outlier term

$$M_k^{\mathrm{con}}(c) = \sum_{i=1}^{c} \Delta g^+_{(i)}.$$

Condition 4 (CPG: Concentration Population Growth)**.** *A bilateral concentration index I satisfies CPG if, for any macroscopic sequence $(h'_k \in h_k \uplus h_k)$ formed from a baseline hierarchy h_k such that $|V(h_k)| \to \infty$, the per-capita structural accumulation strictly outpaces the contribution of any finite set of outlier nodes. Specifically, for any fixed integer $c \geq 1$:*

$$\liminf_{k\to\infty} \frac{1}{|V(h_k)|} \left[|V(h'_k)|^2 I(h'_k) - 2|V(h_k)|^2 I(h_k) - M_k^{\mathrm{con}}(c) \right] > 0.$$

This CPG normalization prevents quadratic dilution from masking whether structural growth is being rewarded by g.

4.5.2 Axiomatic characterization

With the foundational structure and regularity conditions established, we can now state the main representation theorem for this family. This result parallels the characterization of average bilateral indices (Theorem 3), demonstrating that replacing the separability axiom completely redefines the macroscopic scaling of the index without altering the localized evaluation of hierarchy.

Lemma 4 (CPG–IPG Equivalence)**.** *Under the bilateral concentration representation* $I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} g(s_h(v), \sigma_h(v))$*, the index satisfies CPG if and only if the corresponding average bilateral index,*

$$I^{\text{avg}}(h) = \frac{1}{|V(h)|} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big),$$

satisfies IPG, where IPG is evaluated over the exact same macroscopic sequences of compounds $h'_k \in h_k \uplus h_k$ *used in CPG.*

By Lemma 4, CPG is equivalent to IPG for the corresponding average bilateral index defined by the same g; hence Proposition 1 implies that g is strictly increasing with discrete marginals bounded away from zero.

Theorem 5. *A hierarchical index* $I : \mathscr{H} \to \mathbb{R}$ *satisfies A, QS, ETN, and CPG if and only if there exists a function* $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{R}$ *such that for all* $h \in \mathscr{H}$*,*

$$I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} g\big(s_h(v), \sigma_h(v)\big),$$

and g *has discrete marginals strictly bounded away from zero: there exists a constant* $m > 0$ *such that for all* $s, \sigma \in \mathbb{Z}_+$*,*

$$m \leq \Delta_s g(s,\sigma) = g(s+1,\sigma) - g(s,\sigma) \quad \textit{and} \quad m \leq \Delta_\sigma g(s,\sigma) = g(s,\sigma+1) - g(s,\sigma).$$

Proof. By Lemma 3, I satisfies A, QS, and ETN if and only if it admits the stated representation $I(h) = \frac{1}{|V(h)|^2} \sum_{v \in V(h)} g(s_h(v), \sigma_h(v))$ for some local evaluation function g.

By the equivalence established in Lemma 4, this representation satisfies CPG if and only if the corresponding bilateral average index defined by the exact same evaluation function g satisfies IPG. By Theorem 3, this holds if and only if g has discrete marginals uniformly bounded below by a strictly positive constant $m > 0$. The result follows immediately. ∎

Additional remarks on admissible specifications and on unilateral/bilateral equivalences are provided in Carbonell-Nicolau (2026, §§4.5.3–4.5.4).

5 Hierarchical indices from the literature

We briefly benchmark prominent hierarchy indices against the axioms introduced above. Table 1 summarizes which axioms each index satisfies or violates. The main pattern is that several widely used measures fail separability or extension requirements once restricted to DAGs, while others are effectively calibrated to detect cycles and therefore become uninformative on hierarchies. A full discussion and examples are provided in Carbonell-Nicolau (2026, §5).

Summary of axiomatic compliance. Table 1 synthesizes the compliance of each literature index with our axiomatic framework.

Table 1: Axiomatic compliance of hierarchical indices from the literature

Axiom	Avg. Depth (a)	GRC (b)	Conn. (c)	Graph Hier. (c)	Graph Eff. (c)	LUB (c)
A	✓	✓	✓	✓[†]	✓	✓
AS	✓	×	×	×	×	×
ADD	×	×	×	×	×	×
QS	×	×	×	×	×	×
IRP	✓	×	×	✓[†]	✓	✓
ERP	×	×	×	×	×	×
CRP	×	✓	✓	×	×	×
SR	✓	×	×	×	×	×
UE/DE	×	×	×	×	×	×
UE*/DE*	✓	×	×	×	×	×
INN	✓	undef.	undef.	undef.	undef.	undef.

Notes: ✓ denotes satisfaction; × denotes violation; "undef." indicates the axiom cannot be evaluated. [†]Satisfied vacuously (index is constant on DAGs). The average depth index additionally satisfies conditions ETN and IPG.
Sources: (*a*) Carbonell-Nicolau (2025b); (*b*) Mones et al. (2012); (*c*) Krackhardt (1994).

6 Concluding remarks

This paper transitions from partial hierarchical pre-orders to a complete axiomatic characterization of three numerical index families. Grounding measurement in structural principles, we demonstrate that all three families share a common foundation of additive decomposition.

They diverge solely in how they normalize by organizational size, and this normalization choice is inseparably linked to the replication principle that governs each paradigm: absolute scaling defines an extensive view, linear normalization an intensive one, and quadratic normalization a concentration perspective. No single index can satisfy more than one replication principle, making the choice of measurement lens fundamentally normative rather than technical.

This trichotomy resolves the tensions revealed by our impossibility results. We prove that normalized separability axioms cannot coexist with strict vertical extension: population averaging dilutes the index when flat structures are appended, while quadratic normalization collapses density unconditionally. Even weakened partial extension axioms fail under quadratic scaling, as the mathematical void created by structural fragmentation cannot be offset by finite local linkages. Resolving these tensions requires accepting the trade-off embedded in the replication principles. Under intensive normalization, we characterize the average bilateral family, where hierarchy scales as a per-capita average invariant under replication. These indices satisfy relaxed extension and subordination removal but necessarily sacrifice strict monotonicity to avoid dilution. Under extensive normalization, the total bilateral family sums node-level evaluations without a population denominator. Immune to dilution, these indices satisfy strict extension and subordination removal while scaling linearly with replication. Under quadratic normalization, the concentration family evaluates authority density relative to potential dyadic structure. Because standard organizational expansion is inherently sparse, any genuine density metric must penalize vertical extensions; consequently, this family satisfies subordination removal and strict concentration under replication but violates all extension axioms.

Because each paradigm corresponds to a distinct replication principle, the characterization theorems provide uniquely identified tools tailored to the analyst's normative stance. Applying this lens to established indices reveals systematic axiom violations: most prominent measures either ignore directionality, collapse to constants on acyclic graphs, or rely on global normalization that violates structural separability. Among those examined, only the average depth index aligns with our core properties of hierarchy measure. Future research can leverage these characterizations empirically, testing how different organizational contexts align with intensive versus extensive scaling, while further exploring the empirical implications of alternative functional specifications for local depth and span evaluations.

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