Int J Game Theory (2015) 44:869–890
DOI 10.1007/s00182-014-0457-3

On equilibrium refinements in supermodular games

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Accepted: 20 November 2014 / Published online: 30 November 2014

Abstract We show that supermodular games satisfying sequential better-reply security possess a pure strategy perfect equilibrium and a strategically stable set of pure strategy equilibria. We illustrate that in continuous supermodular games, perfect equilibria may contain weakly dominated actions. Moreover, in discontinuous supermodular games satisfying sequential better-reply security, perfect equilibria may involve play of actions in the interior of the set of weakly dominated actions. We show that supermodular games satisfying sequential better-reply security possess pure strategy perfect equilibria outside the interior of the set of weakly dominated action profiles.

Keywords Supermodular game · Weakly dominated strategy · Perfect equilibrium · Strategically stable set

JEL Classification C72

1 Introduction

Supermodular games (Topkis 1979; Vives 1990; Milgrom and Roberts 1990; Milgrom and Shannon 1990) encompass a rich family of economic games, including models of oligopoly competition, macroeconomic coordination failures, arms races, bank runs, technology adoption and diffusion, R&D competition, pretrial bargaining, coordination in teams, and congestion games. While these games are known to have Nash equilibria, the existence of refined equilibrium points in supermodular games

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remains an open question. In this paper we provide existence results for two standard refinements of the Nash equilibrium concept: perfection (Selten 1975) and strategic stability (Kohlberg and Mertens 1986). In addition, we pinpoint a number of facts about the relationship between prefection and weakly dominated action profiles in supermodular games.

As pointed out in Carbonell-Nicolau (2011a), the existence of a perfect equilibrium in a normal form game depends crucially on the existence of Nash equilibria in the game's Selten perturbations (i.e., perturbations in which all players choose a completely mixed strategy with small but positive probability). In this paper, we show that the collection of supermodular games is closed under Selten perturbations (Lemma 5). Using this fact, and assuming *sequential better-reply security* (Carbonell-Nicolau and McLean 2013), a property of a game that guarantees that the Nash equilibrium correspondence of perturbed games has a closed graph, we prove (in Sect. 3.1) the existence of a pure strategy perfect equilibrium (Theorem 1). Using Zorn's Lemma, we then prove the existence of strategically stable sets of pure action profiles in supermodular games (Theorem 2).

Our existence results are followed by a discussion (in Sect. 3.2) of the relationship between perfection and weak domination. It is well-known that perfect equilibria in finite-action games are admissible. It is also well-known that standard refinements of the Nash equilibrium concept cease to satisfy certain "natural" properties in infinite-action games. For example, as shown by Simon and Stinchcombe (1995), infinite-action, continuous games may exhibit unique Nash and perfect equilibria in weakly dominated actions. For games in the special class of supermodular games, however, the set of Nash equilibria must contain at least one weakly undominated action profile. This has been shown by Kultti and Salonen (1997). We show that in continuous supermodular games (resp. supermodular, sequentially better-reply secure games), perfect equilibria may belong to the set of weakly dominated action profiles (resp. the interior of the set of weakly dominated action profiles). We then prove, in Sect. 3.2.1, that supermodular, sequentially better-reply secure games possess pure strategy perfect equilibria outside the interior of the set of weakly dominated action profiles.

2 Preliminaries

2.1 Supermodular games

This subsection presents terminology and results from the theory of lattices (see, for instance, Birkhoff (1967)) that will be needed in the formal definition of a supermodular game (Definition 5 below) and in the proofs of our main results.

Definition 1 A *lattice* is a pair $(A, \leq)$, where A is a nonempty set and $\leq$ is a partial order (i.e., a reflexive, antisymmetric, and transitive binary relation) in $A \times A$ such that for every $\{a, b\} \subseteq A$, the infimum of $\{a, b\}$, $\inf\{a, b\}$, and the supremum of $\{a, b\}$, $\sup\{a, b\}$, exist in A.

Definition 2 A *sublattice* of a lattice $(A, \leq)$ is a subset B of A together with the partial order induced by $\leq$ on B (i.e., the intersection of $\leq$ and $B \times B$) such that $\inf\{a, b\} \in B$ and $\sup\{a, b\} \in B$ whenever $a \in B$ and $b \in B$.

Definition 3 A *lattice* $(A, \leq)$ is *lattice complete* if $\inf B \in A$ and $\sup B \in A$ for every nonempty $B \subseteq A$.

Definition 4 A *topological lattice* is a lattice $(A, \leq)$ equipped with a topology for which the maps $(a, b) \mapsto \inf\{a, b\}$ and $(a, b) \mapsto \sup\{a, b\}$ from $A \times A$ (with the product topology) into A are continuous.

The *interval topology* on a lattice $(A, \leq)$ is defined by choosing the closed intervals $[a, b] := \{x \in A : a \leq x \leq b\}$ as a subbasis for the closed sets.

It is well-known that a lattice $(A, \leq)$, equipped with its interval topology, constitutes a topological lattice.

A *strategic form game* (or simply a *game*) is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $u_i : X \to \mathbb{R}$ represents player i's payoff function, where $X := \times_{i=1}^N X_i$.

If $G = (X_i, u_i)_{i=1}^N$ is a game and $Y_i \subseteq X_i$ for each i, we will write the game $(Y_i, u_i|_{Y_i})_{i=1}^N$ simply as $(Y_i, u_i)_{i=1}^N$. If each X_i is a nonempty metric (resp. compact) space, G is said to be a *metric* (resp. *compact*) *game*. If G is a metric game and if u_i is bounded and Borel measurable for each i, we say that G is a *metric*, *Borel game*.

Given a game $G = (X_i, u_i)_{i=1}^N$ such that each $(X_i, \leq_i)$ is a lattice, the pair $(X, \leq)$ is a lattice, where $\leq$ is the relation in $X \times X$ defined as follows:

$$x \leq y \Leftrightarrow \forall i \in \{1, ..., N\}\, (x_i \leq_i y_i)\,.$$

Given i, the product lattice $(X_{-i}, \leq_{-i})$ is defined similarly.

Definition 5 A game $(X_i, u_i)_{i=1}^N$ is *supermodular* if the following are satisfied:

- Each X_i is a compact metrizable topological lattice endowed with a topology τ_i at least as fine as the interval topology.
- Each u_i is bounded and Borel measurable.
- For each i and $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is upper semicontinuous on X_i.
- For each i and $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is *supermodular*, i.e., for each i, $x_{-i} \in X_{-i}$, and $\{x_i, y_i\} \subseteq X_i$,

$$u_i(\sup\{x_i, y_i\}, x_{-i}) + u_i(\inf\{x_i, y_i\}, x_{-i}) \geq u_i(x_i, x_{-i}) + u_i(y_i, x_{-i}).$$

- For each i, u_i exhibits *increasing differences in X_i and X_{-i}*, i.e., $u_i(x_i, y_{-i}) - u_i(x_i, x_{-i})$ is increasing in x_i for all $y_{-i} \geq_{-i} x_{-i}$.

Remark 1 Note that by definition a supermodular game is a compact, Borel, metric game.

Remark 2 The *order topology* on a lattice $(A, \leq)$ is defined by the concept of order convergence. A net (a^α) in A is said to *order-converge* to $a \in A$ if $\liminf a^\alpha =$

$\limsup a^\alpha = a$. The order topology is then defined by choosing a subset B of A to be closed if any order-convergent net in B converges to a point in B.

A lattice is compact in its interval topology if and only if it is lattice complete (Birkhoff 1967, Theorem 20, p. 250). Let $(A, \leq)$ be a topological lattice and suppose that $(A, \leq)$ has a metrizable compatible topology τ for which A is a compact space. If τ is stronger than the interval topology on $(A, \leq)$, then A must also be compact for the (weaker) interval topology. Consequently, by virtue of the above result, $(A, \leq)$ is lattice complete and it follows that τ must coincide with the order topology on $(A, \leq)$ (Strauss 1968, p. 221). If in addition the interval topology is Hausdorff, then the order topology must be equal to the interval topology on $(A, \leq)$ (Lawson 1973, Proposition 4(3)), and in this case τ must coincide with the order and interval topologies on $(A, \leq)$.

2.2 Sequential better-reply security

The following definition is taken from Carbonell-Nicolau and McLean (2013).

Definition 6 A metric game $G = (X_i, u_i)_{i=1}^N$ is *sequentially better-reply secure* if the following condition is satisfied: if $(x^n, u(x^n))$ is a sequence with $(x^n, u(x^n)) \in X \times \mathbb{R}^N$ for each n, if $(x^n, u(x^n))$ converges to $(x, \gamma) \in X \times \mathbb{R}^N$, and if x is not a Nash equilibrium of G, then there exist an i, an $\eta > \gamma_i$, a subsequence (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k \in X_i$ and $u_i(y_i^k, x_{-i}^k) \geq \eta$.

As shown in Carbonell-Nicolau and McLean (2013), sequential better-reply security implies that the Nash equilibrium correspondence, defined over the domain of payoff functions, has a closed graph. Since perfect equilibria and stable sets of equilibria are robust with respect to perturbations of the players' action spaces, and because these perturbations can be viewed as payoff perturbations, sequential better-reply security is a useful property when addressing questions of equilibrium refinement.

Remark 3 Within the class of compact, metric, lattice, Borel games, the class of sequentially better-reply secure, supermodular games subsumes the class of supermodular games as defined in Milgrom and Roberts (1990).

Remark 4 Sequential better-reply security is weaker than several conditions introduced in the literature on the existence of Nash equilibrium. See Carbonell-Nicolau and McLean (2013).

2.3 Perfect equilibria, strictly perfect equilibria and stable sets of equilibria

In this subsection we provide the formal definitions of perfection and stability.

Given a compact, metric game $G = (X_i, u_i)_{i=1}^N$, let $M(X_i)$ denote the set of regular Borel measures on X_i endowed with the weak* topology and let $\Delta(X_i)$ denote the set of regular Borel probability measures on X_i, endowed with the relative weak* topology. It is well known (e.g., Aliprantis and Border 2006) that $\Delta(X_i)$ is a compact space that is metrizable via the Prokhorov metric. Throughout the paper, the symbol $\otimes$ will be used to denote product measures on appropriately defined σ-fields.

The *mixed extension* of G is the strategic form game

$$\overline{G} := (\Delta(X_i), u_i)_{i=1}^N,$$

where $u_i : \times_{i=1}^N \Delta(X_i) \to \mathbb{R}$ is defined by

$$u_i(\mu) := \int_X u_i d\mu.$$

For each $x_i \in X_i$, let $\theta_{x_i}^i$ represent the Dirac measure on X_i with support $\{x_i\}$. The map $x_i \mapsto \theta_{x_i}^i$ (resp. $(x_1, .., x_N) \mapsto (\theta_{x_1}^1, .., \theta_{x_N}^N)$) is an embedding, so X_i (resp. X) can be topologically identified with a subspace of $\Delta(X_i)$ (resp. $\times_{i=1}^N \Delta(X_i)$). We sometimes abuse notation and refer to $\theta_{x_i}^i \in \Delta(X_i)$ (resp. $(\theta_{x_1}^1, .., \theta_{x_N}^N) \in \times_{i=1}^N \Delta(X_i)$) simply as x_i (resp. $(x_1, .., x_N) = x$).

Note that because the map $(x_1, .., x_N) \mapsto (\theta_{x_1}^1, .., \theta_{x_N}^N)$ is an embedding, a subset $S \subseteq X$ is compact (resp. closed in X) if and only if $\{(\theta_{x_1}^1, .., \theta_{x_N}^N) : x \in S\} \subseteq \times_{i=1}^N \Delta(X_i)$ is compact (resp. closed) in $\times_{i=1}^N \Delta(X_i)$.

Given $(\delta, \mu, \nu) \in [0, 1)^N \times [\times_{i=1}^N \Delta(X_i)] \times [\times_{i=1}^N \Delta(X_i)]$, define

$$(1-\delta)\nu + \delta\mu := ((1-\delta_1)\nu_1 + \delta_1\mu_1, ..., (1-\delta_N)\nu_N + \delta_N\mu_N),$$

where for each i, $(1-\delta_i)\nu_i + \delta_i\mu_i$ represents the measure σ_i in $\Delta(X_i)$ such that

$$\sigma_i(B) = (1-\delta_i)\nu_i(B) + \delta_i\mu_i(B), \quad \text{for all Borel subsets } B \text{ of } X_i.$$

A measure $\mu_i \in M(X_i)$ is said to be *strictly positive* if $\mu_i(U) > 0$ for every nonempty open set U in X_i.

For each i, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive members of $\Delta(X_i)$.

Given $(\delta, \mu) \in [0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$, let $G_{(\delta,\mu)}$ be a strategic form game defined as

$$G_{(\delta,\mu)} := (X_i, u_i^{(\delta,\mu)})_{i=1}^N,$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$\begin{aligned} u_i^{(\delta,\mu)}(x) &:= u_i((1-\delta_1)x_1 + \delta_1\mu_1, ..., (1-\delta_N)x_N + \delta_N\mu_N) \\ &= u_i\left((1-\delta_1)\theta_{x_1}^1 + \delta_1\mu_1, ..., (1-\delta_N)\theta_{x_N}^N + \delta_N\mu_N\right). \end{aligned}$$

The set of Nash equilibria of a compact, metric Borel game $G = (X_i, u_i)_{i=1}^N$ will be denoted $E(G)$ and the set of Nash equilibria of its mixed extension $\overline{G}$ will be denoted $E(\overline{G})$. Similarly, the set of Nash equilibria of the compact, metric Borel game $G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^N$ will be denoted $E(G_{(\delta,\mu)})$ and the set of Nash equilibria of its mixed extension $\overline{G}_{(\delta,\mu)}$ will be denoted $E(\overline{G}_{(\delta,\mu)})$. As usual, we may consider $E(G)$ (resp. $E(G_{(\delta,\mu)})$) to be a subset of $E(\overline{G})$ (resp. $E(\overline{G}_{(\delta,\mu)})$) when we identify pure strategy profiles with their corresponding profiles of Dirac measures in $\times_{i=1}^N \Delta(X_i)$.

Definition 7 A strategy profile $\sigma \in E(\overline{G})$ is a *trembling-hand perfect (thp) equilibrium* in $G = (X_i, u)_{i=1}^N$ if there exist sequences (δ^n), (μ^n), and (σ^n) such that $(0, 1)^N \ni \delta^n \to 0$, $\mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$, $\sigma^n \to \sigma$, and $\sigma^n \in E(\overline{G}_{(\delta^n,\mu^n)})$ for each n.

Definition 8 A strategy profile $\sigma \in E(\overline{G})$ is a *strictly perfect equilibrium* in $G = (X_i, u)_{i=1}^N$ if for all sequences (δ^n) and (μ^n) such that $(0, 1)^N \ni \delta^n \to 0$ and $\mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$, there exists a sequence (σ^n) satisfying $\sigma^n \in E(\overline{G}_{(\delta^n,\mu^n)})$ for each n and $\sigma^n \to \sigma$.

Every strictly perfect equilibrium is a trembling-hand perfect equilibrium. Furthermore, this definition of trembling-hand perfection is equivalent to an alternative definition in terms of perturbed sets of mixed strategies. If $\eta_i \in M(X_i)$ is strictly positive and $\eta_i(X_i) < 1$, we define the perturbed mixed-strategy set of player i as

$$\Delta(X_i, \eta_i) := \{\nu_i \in \Delta(X_i) : \nu_i \geq \eta_i\}.$$

Given a profile $\eta = (\eta_1, ..., \eta_N)$ of perturbations, we define the associated *Selten perturbation* of G to be the game

$$\overline{G}_\eta = (\Delta(X_i, \eta_i), u_i)_{i=1}^N .$$

Lemma 1 *Let $G = (X_i, u_i)_{i=1}^N$ be a game and let $\sigma \in E(\overline{G})$ be a strategy profile. The following are equivalent:*

(i) The profile σ is a trembling-hand perfect equilibrium in G.
(ii) There exist sequences (η^n) and (σ^n) such that $\eta^n \to 0$, $\sigma^n \to \sigma$, and σ^n is an equilibrium in the Selten perturbed game $\overline{G}_{\eta^n}$ for each n.

Next we record a useful characterization of strict perfection.

Lemma 2 *Let $G = (X_i, u_i)_{i=1}^N$ be a game and let $\sigma \in E(\overline{G})$ be a strategy profile. The following are equivalent:*

(i) The profile σ is a strictly perfect equilibrium in G.
(ii) For every $\varepsilon > 0$, there exists an $\alpha > 0$ such that the following holds: if $0 < \delta_i < \alpha$ for each i and if $\mu \in \times_{i=1}^N \widehat{\Delta}(X_i)$, then $E(\overline{G}_{(\delta,\mu)}) \cap B_\varepsilon(\sigma) \neq \varnothing$.

Definition 9 An equilibrium $(x_1, ..., x_N) \in E(G)$ is a *pure strategy perfect equilibrium* of $G = (X_i, u_i)_{i=1}^N$ if $(\theta_{x_1}^1, ..., \theta_{x_N}^N)$ is a perfect equiibrium in $E(\overline{G})$.

Definition 10 Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric Borel game. A subset $S \subseteq E(\overline{G})$ is *KM prestable* if S is closed and the following condition is satisfied: for every open set U containing S, there exists $\alpha \in (0, 1]$ such that for each $\delta \in (0, \alpha)^N$ and every $\mu \in \times_{i=1}^N \widehat{\Delta}(X_i)$,

$$E(\overline{G}_{(\delta,\mu)}) \cap U \neq \varnothing.$$

A subset $S \subseteq E(\overline{G})$ is a *KM stable set* if S is a minimal (with respect to set inclusion) KM prestable set.

Remark 5 As a consequence of Lemma 2, an equilibrium $\sigma \in E(\overline{G})$ is strictly perfect if and only if the set $E = \{\sigma\}$ is a KM stable set.

The notion of stability was introduced for finite-action games by Kohlberg and Mertens (1986).[1]

3 Equilibrium refinements in supermodular games

This section presents our main results. In Sect. 3.1 we establish the existence of pure strategy perfect equilibria and stable sets of pure strategy equilibria in supermodular, sequentially better-reply secure games. In Sect. 3.2 we show that in continuous supermodular games (resp. supermodular, sequentially better-reply secure games), perfect equilibria may belong to the set of weakly dominated action profiles (resp. the interior of the set of weakly dominated action profiles). We then provide, in Sect. 3.2.1, conditions under which supermodular games possess pure strategy perfect equilibria outside the interior of the set of weakly dominated action profiles.

3.1 Perfection and stability

The following is our first main result.

Theorem 1 *Any supermodular game G satisfying sequential better-reply security has a pure strategy perfect equilibrium, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact in X.*

The proof of Theorem 1 relies on the following two lemmas.

Lemma 3 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there are sequences (δ^n) and (μ^n) satisfying the following:*

- $\delta^n \in (0, 1)^N$ *and* $\mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$ *for each n;*
- $\delta^n \to 0$*; and*
- $E(G_{(\delta^n, \mu^n)}) \neq \varnothing$ *for each n.*

Then G has a pure strategy perfect equilibrium, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact in X.

Lemma 4 *If G is supermodular, then $E(G_{(\delta,\mu)}) \neq \varnothing$ for every $(\delta, \mu) \in [0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$.*

Lemma 3 is a statement about general games. This lemma follows from the analysis in Carbonell-Nicolau and McLean (2013). We relegate its proof to Appendix.

[1] Stability in infinite-action games is studied in Al-Najjar (1995) (for the case of continuous games) and in Carbonell-Nicolau (2011d) (for the case of discontinuous games).

Lemma 4 states that Selten perturbations of supermodular games have Nash equilibria. Essential for the proof of this lemma is the fact the collection of supermodular games is closed under Selten perturbations.[2]

Lemma 5 *If $G = (X_i, u_i)_{i=1}^N$ is supermodular, then $G_{(\delta,\mu)}$ is supermodular for every $(\delta, \mu) \in [0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$.*

The proofs of Lemmas 4 and 5 are presented in Appendix.

Proof of Theorem 1 Fix a supermodular game $G = (X_i, u_i)_{i=1}^N$ satisfying sequential better-reply security. Take sequences (δ^n) and (μ^n) such that

$$\delta^n \in (0, 1)^N \quad \text{and} \quad \mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i), \quad \text{for each } n,$$

and $\delta^n \to 0$. Each $G_{(\delta^n,\mu^n)}$ has a Nash equilibrium (Lemma 4). Therefore, the conclusion follows from Lemma 3. □

We now present our second main result, a strengthening of Theorem 1 in terms of stable sets of equilibria.

A subset $S \subseteq E(G)$ is a *KM stable set of pure strategy profiles* if $\left\{(\theta^1_{x_1}, .., \theta^N_{x_N}) \in \times_{i=1}^N \Delta(X_i) : (x_1, .., x_N) \in S\right\}$ is a KM stable set in $E(\overline{G})$.

Theorem 2 *Any supermodular game G satisfying sequential better-reply security has a KM stable set of pure strategy profiles. Furthermore, any KM stable set of pure strategy profiles of G is a compact subset of the set of pure strategy perfect equilibria of G, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact.*

The proof of Theorem 2 is based on Lemma 4 and on the following analogue of Lemma 3.

Lemma 6 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there exists $\alpha \in (0, 1)$ such that for every $(\delta, \mu) \in (0, \alpha]^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$, $E(G_{(\delta,\mu)}) \neq \varnothing$. Then G has a stable set of pure strategy profiles. Furthermore, any stable set of pure strategy profiles of G is a compact subset of the set of pure strategy perfect equilibria of G, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact.*

The proof of Lemma 6 is relegated to Appendix.

Proof of Theorem 2 Fix a supermodular game $G = (X_i, u_i)_{i=1}^N$ satisfying sequential better-reply security. Take any $\delta \in (0, 1)^N$ and any $\mu \in \widehat{\Delta}(X)$. Then $G_{(\delta,\mu)}$ has a Nash equilibrium (Lemma 4). Now apply Lemma 6. □

[2] In non-supermodular games, the conditions needed to ensure that Selten perturbations possess Nash equilibria are stronger than those needed to guarantee existence of Nash equilibria in the original game. See Carbonell-Nicolau (2011a, b).

Remark 6 The reader may wonder whether Theorem 1 and Theorem 2 follow from extant results. Existence results regarding the existence of (pure and mixed) perfect equilibria and stable sets in strategic-form games (e.g., Carbonell-Nicolau (2011a, b, c, d, 2014) and Carbonell-Nicolau and McLean (2013)) require conditions stronger than the notion of payoff security introduced in Reny (1999), and it is easy to see that there are supermodular, sequentially better-reply secure games that fail payoff security.[3] For instance, the game given in Example 2 (Sect. 3.2) is supermodular and sequentially better-reply secure but violates payoff security at the strategy profile $(\frac{1}{2}, 0)$.

3.2 Perfection and (limit) admissibility

In this subsection we study the relationship between perfection and weak domination in supermodular games. We begin by defining the notions of admissibility and limit admissibility.

Definition 11 A strategy $x_i \in X_i$ is *weakly dominated for* i if there exists a strategy $\mu_i \in \Delta(X_i)$ such that $u_i(\mu_i, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, with strict inequality for some x_{-i}.

Definition 12 A strategy profile $\mu \in \times_{i=1}^{N} \Delta(X_i)$ is *admissible* if $\mu_i(D_i) = 0$ for all i, where D_i denotes the set of strategies weakly dominated for i.

Definition 13 A strategy profile $\mu \in \times_{i=1}^{N} \Delta(X_i)$ is *limit admissible* if $\mu_i(int(D_i)) = 0$ for all i, where $int(D_i)$ denotes the interior of the set of strategies weakly dominated for i.

It is well-known that perfect equilibria in finite-action games are admissible. It is also well-known that standard refinements of the Nash equilibrium concept cease to satisfy certain "natural" properties in infinite-action games. For example, there are continuous games whose unique Nash and perfect equilibrium is not admissible (Simon and Stinchcombe (1995), Example 2.1). Furthermore, perfect equilibria in discontinuous games need not be limit admissible (Carbonell-Nicolau (2011c), Example 1).[4]

In supermodular, sequentially better-reply secure games, perfect equilibria may also be nonadmissible. In fact, consider the well-known Bertrand duopoly game of price competition with constant average (and marginal) cost and continuous monopoly profit function. This game is supermodular and sequentially better-reply secure.[5] Furthermore, this game has a unique Nash equilibrium in weakly dominated strategies: the strategy profile in which both firms post a zero price. Since the game has a perfect

[3] A metric game $(X_i, u_i)_{i=1}^{N}$ is *payoff secure* if for each i, $\varepsilon > 0$, and $x \in X$, there exist $y_i \in X_i$ and a neighborhood V of x_{-i} such that $u_i(y_i, y_{-i}) > u_i(x) - \varepsilon$ for all $y_{-i} \in V$.

[4] See also Carbonell-Nicolau (2011e) for additional "anomalies" of perfect equilibria and stable sets in discontinuous games.

[5] Reny (1999, p. 1033) notes that this game satisfies better-reply security, a condition stronger than sequential better-reply security.

equilibrium by Theorem 1, it follows that it has a unique perfect equilibrium in weakly dominated strategies.

Example 2.1 in Simon and Stinchcombe (1995) is a version of a duopoly game of price competition with differentiated products. This is a continuous game which also has the property that the unique Nash and perfect equilibrium is nonadmissible. While this game is continuous, it is not supermodular, so the reader may wonder whether there are continuous, supermodular games whose Nash and perfect equilibria are nonadmissible. A partial answer is given by Kultti and Salonen (1997), who prove that supermodular games must have at least one admissible Nash equilibrium (Kultti and Salonen (1997), Theorem 1 and Proposition 1). The following example demonstrates that continuous, supermodular games may have perfect, nonadmissible equilibria.

Example 1 Let (α^n) be a sequence with $\alpha^n \in (0, \frac{1}{2})$ for each n and $\alpha^n \nearrow \frac{1}{2}$. Let (γ^n) be a sequence with $\gamma^n \in (\frac{1}{4}, 1)$ and $1 - 2\gamma^{2n} > 0$ for each n and $\gamma^n \searrow \frac{1}{4}$.

Consider the game

$$G := (X_i, u_i)_{i=1}^2 = \left((\cup_n\{\alpha^n\}) \cup \left\{\frac{1}{2}, 1\right\}, (\cup_n\{\gamma^n\}) \cup \left\{\frac{1}{8}, \frac{1}{4}\right\}, u_1, u_2\right),$$

where u_2 is identically zero and

$$u_1(x_1, x_2) := \begin{cases} -1 + \frac{x_2 - \gamma^{2n}}{(1-2\gamma^{2n})n} & \text{if } x_1 = \alpha^n, n = 1, ..., \\ -1 & \text{if } x_1 = \frac{1}{2}, \\ -1 & \text{if } x_1 = 1 \text{ and } x_2 \in (\frac{1}{4}, 1], \\ -4x_2 & \text{if } x_1 = 1 \text{ and } x_2 \in [0, \frac{1}{4}]. \end{cases}$$

Let $\leq_1$ be the usual order on $\mathbb{R}$, and define $\leq_2$ as follows:

$$a \leq_2 b \Leftrightarrow a \geq b.$$

If, for each i, player i's action space is endowed with the order $\leq_i$ and the resulting lattice $(X_i, \leq_i)$ is endowed with the interval topology (which coincides with the order topology (recall the discussion in Remark 2) and the relativization of the Euclidean topology on $\mathbb{R}$) then the first bullet point in Definition 5 is clearly satisfied. On the other hand, it is easy to see that each u_i is continuous so the second and third bullet points in Definition 5 are satisfied (recall that each X_i is compact). Finally, it is routine to verify that for each i and $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is supermodular and that u_1 has increasing differences in X_1 and X_2 with respect to $\leq_1$ and $\leq_2$ (u_2 clearly has increasing differences in X_2 and X_1 with respect to $\leq_1$ and $\leq_2$).

The strategy profile $(\frac{1}{2}, \frac{1}{4})$ is a Nash equilibrium of G . This follows from the fact that u_2 is identically zero and

$$u_1\left(\frac{1}{2}, \frac{1}{4}\right) = -1 = u_1\left(1, \frac{1}{4}\right) \geq u_1\left(\alpha^n, \frac{1}{4}\right) = -1 + \frac{\frac{1}{4} - \gamma^{2n}}{(1 - 2\gamma^{2n})n}, \quad \text{for all } n.$$

On the other hand,

$$u_1\left(\frac{1}{2}, x_2\right) \leq u_1(1, x_2), \quad \text{for all } x_2,$$

with strict inequality if $x_2 = \frac{1}{8}$, so the action $\frac{1}{2}$ is weakly dominated for player 1.

We next show that $(\frac{1}{2}, \frac{1}{4})$ is perfect. Given $l \in \mathbb{N}$, choose $\beta_l > 1$ such that

$$\left(1 - \frac{1}{\beta_l}\right) A_l > \frac{1}{\beta_l} 2B, \tag{1}$$

where

$$A_l := \frac{\gamma^{2l+1} - \gamma^{2l+2}}{(1 - 2\gamma^{2(l+1)})(l+1)}$$

and $|u_1(x_1, x_2)| \leq B$ for all $(x_1, x_2) \in X_1 \times X_2$. Take a sequence of trembles (μ_2^l) with $\mu_2^l \in \widehat{\Delta}(X_2)$ for each l satisfying the following for each l:

$$\mu_2^l\Big(\Big\{\frac{1}{4}\Big\}\Big) + \mu_2^l\Big(\Big\{\frac{1}{8}\Big\}\Big) + \sum_{k \neq 2l+1} \mu_2^l(\{\gamma^k\}) = \frac{1}{\beta_l},$$
$$\mu_2^l(\{\gamma^{2l+1}\}) = 1 - \frac{1}{\beta_l}.$$

For each l, we have

$$\begin{aligned}
u_1(\alpha^{l+1}, \mu_2^l) - u_1(1, \mu_2^l) &= \mu_2^l\Big(\Big\{\frac{1}{8}\Big\}\Big)\left(-\frac{1}{2} + \frac{\frac{1}{8} - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)}\right) \\
&\quad + \sum_{x_2 \in \{\frac{1}{4}\} \cup (\bigcup_{k \neq 2l+1} \{\gamma^k\})} \mu_2^l(\{x_2\}) \left(\frac{x_2 - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)}\right) \\
&\quad + \left(1 - \frac{1}{\beta_l}\right)\left(\frac{\gamma^{2l+1} - \gamma^{2(l+1)}}{(1 - 2\gamma^{2(l+1)})(l+1)}\right) \\
&\geq \left(1 - \frac{1}{\beta_l}\right) A_l - \frac{1}{\beta_l} 2B \\
&> 0,
\end{aligned}$$

where the last inequality follows from (1). Therefore, given a sequence (δ^l) with $\delta^l \in (0, 1)$ for each l and $\delta^l \to 0$, and given $\mu_1 \in \widehat{\Delta}(X_1)$, we have, for each l,

$$\begin{aligned}
&u_1((1 - \delta^l)\alpha^{l+1} + \delta^l \mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l) \\
&\quad - u_1((1 - \delta^l)1 + \delta^l \mu_1, (1 - \delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l) \\
&= (1 - \delta^l)\left(u_1(\alpha^{l+1}, \mu_2^l) - u_1(1, \mu_2^l)\right) > 0.
\end{aligned}$$

Hence, if for each l and every $n \in \{1, ..., l\}$ we have

$$\begin{aligned} &u_1((1-\delta^l)\alpha^{l+1} + \delta^l \mu_1, (1-\delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l) \\ &\quad - u_1((1-\delta^l)\alpha^n + \delta^l \mu_1, (1-\delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l) \\ &\quad = (1-\delta^l)\left(u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l)\right) > 0, \end{aligned} \tag{2}$$

then, for each l,

$$\arg\max_{x_1} u_1^{(\delta^l,(\mu_1,\mu_2^l))}(x_1, \gamma^{2(l+1)}) \in \{\alpha^{l+1}, \alpha^{l+2}, ...\} \cup \left\{\frac{1}{2}\right\}.$$

This means that, given l, if player 2 plays $\gamma^{2(l+1)}$ in the game $G_{(\delta^l,(\mu_1,\mu_2^l))}$, player 1 best responds by choosing a member of the set

$$\{\alpha^{l+1}, \alpha^{l+2}, ...\} \cup \left\{\frac{1}{2}\right\}.$$

Consequently, since $\gamma^{2(l+1)}$ is always optimal for player 2 in $G_{(\delta^l,(\mu_1,\mu_2^l))}$ (recall that u_2 is identically zero), it follows that $G_{(\delta^l,(\mu_1,\mu_2^l))}$ has an equilibrium $(x_1^l, \gamma^{2(l+1)})$ for some

$$x_1^l \in \{\alpha^{l+1}, \alpha^{l+2}, ...\} \cup \left\{\frac{1}{2}\right\}.$$

But then the sequence

$$\left((1-\delta^l)x_1^l + \delta^l \mu_1, (1-\delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l\right)$$

converges to $(\frac{1}{2}, \frac{1}{4})$, and for each l,

$$\left((1-\delta^l)x_1^l + \delta^l \mu_1, (1-\delta^l)\gamma^{2(l+1)} + \delta^l \mu_2^l\right)$$

is a Nash equilibrium of $\overline{G}_{\delta^l * (\mu_1,\mu_2^l)}$, so $(\frac{1}{2}, \frac{1}{4})$ is perfect.

We conclude that if (2) holds for each l and every $n \in \{1, ..., l\}$, then $(\frac{1}{2}, \frac{1}{4})$ is a perfect equilibrium of G. Hence, it suffices to show that $u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l) > 0$ for each l and every $n \in \{1, ..., l\}$. Fix l and $n \in \{1, ..., l\}$. Then

$$\begin{aligned} &u_1(\alpha^{l+1}, \mu_2^l) - u_1(\alpha^n, \mu_2^l) \\ &\quad = \sum_{\substack{x_2 \in \{\frac{1}{4}, \frac{1}{8}\} \\ \cup(\bigcup_{k \neq 2l+1}\{\gamma^k\})}} \mu_2^l(\{x_2\}) \left(\frac{x_2 - \gamma^{2(l+1)}}{(1-2\gamma^{2(l+1)})(l+1)} - \frac{x_2 - \gamma^{2n}}{(1-2\gamma^{2n})n}\right) \end{aligned}$$

$$+\left(1-\frac{1}{\beta_l}\right)\left(\frac{\gamma^{2l+1}-\gamma^{2(l+1)}}{(1-2\gamma^{2(l+1)})(l+1)}-\frac{\gamma^{2l+1}-\gamma^{2n}}{(1-2\gamma^{2n})n}\right)$$
$$\geq\left(1-\frac{1}{\beta_l}\right)A_l-\frac{1}{\beta_l}2B>0,$$

where the last inequality uses (1).

While continuous supermodular games may have perfect equilibria that fail admissibility, continuity ensures that perfect equilibria are always limit admissible. In fact, as shown in Carbonell-Nicolau (2011c), the following property suffices for a compact, metric normal form game $G=(X_i,u_i)_{i=1}^N$ to have only limit admissible perfect equilibria: for each i, if $x_i\in X_i$ is weakly dominated in G for player i, then for some $\mu_i\in\Delta(X_i)$ that weakly dominates x_i, there exists $y_{-i}\in X_{-i}$ with $u_i(\mu_i,z_{-i})>u_i(x_i,z_{-i})$ for all z_{-i} in some neighborhood of y_{-i}. If this condition is not fulfilled, perfect equilibria in supermodular games may fail limit admissibility. This is illustrated in the following example, which presents a supermodular, sequentially better-reply secure game with a perfect equilibrium that is not limit admissible.[6]

Example 2 Consider the two-player game $G:=(X_1,X_2,u_1,u_2)$, where $X_1=X_2=[0,1]$,

$$u_1(x_1,x_2):=\begin{cases}-1 & \text{if}\quad x_1\in\left[0,\frac{1}{2}\right)\text{ and }x_2=0,\\ x_2 & \text{elsewhere,}\end{cases}$$

and

$$u_2(x_1,x_2):=\begin{cases}1 & \text{if}\quad x_2=1,\\ 0 & \text{elsewhere.}\end{cases}$$

Let $\leq_2$ be the usual order on $\mathbb{R}$, and define $\leq_1$ as follows:

$$a\leq_1 b\Leftrightarrow a\geq b.$$

It is routine to verify that the game G endowed with $\leq_1$ and $\leq_2$ is supermodular.

We next show that G satisfies sequential better-reply security. Suppose that $(x^n,u(x^n))$ is a convergent sequence of elements in $X\times\mathbb{R}^2$ with limit $(x,\gamma)\in X\times\mathbb{R}^2$. Suppose that x is not a Nash equilibrium of G. We must show that there exist an i, an $\eta>\gamma_i$, a subsequence (x^k) of (x^n), and a sequence (y_i^k) such that for each k, $y_i^k\in X_i$ and $u_i(y_i^k,x_{-i}^k)\geq\eta$. This is clearly satisfied if x is a point of continuity of u. Suppose that x is a point of discontinuity of u. Then x must be a member of the set

$$Z:=\left\{(z_1,z_2)\in X:z_2=0\text{ and }z_1\in\left[0,\frac{1}{2}\right)\right\}\cup\{(z_1,z_2)\in X:z_2=1\}.$$

The set of pure strategy Nash equilibria of G in Z is $\{(z_1,z_2)\in X:z_2=1\}$. Consequently, since x is not a Nash equilibrium of G, and because $x\in Z$, we must

[6] Example 1 in Carbonell-Nicolau (2011c) illustrates that perfect equilibria in discontinuous games need not be limit admissible, but the example given in Carbonell-Nicolau (2011c) is not supermodular.

have $x_2 = 0$. Since $x_2 = 0$, and because $(x^n, u(x^n)) \to (x, \gamma)$, the definition of u_2 entails $\gamma_2 = 0$. But then, for each n we have

$$u_2(x_1^n, 1) = 1 > 0 = \gamma_2.$$

Next, observe that the strategy profile (0, 1) is not limit admissible. In fact, any action in $\left[0, \frac{1}{2}\right)$ is weakly dominated by any action in $\left[\frac{1}{2}, 1\right]$ for player 1. In addition, the point (0, 1) is a perfect equilibrium. To see this, take $\mu_2 \in \widehat{\Delta}(X_2)$ with $\mu_2(\{0\}) = 0$. From the definition of u_2 it is clear that for any $(\delta, \mu_1) \in [0, 1) \times \widehat{\Delta}(X_1)$, 1 is a best response for player 2 to any $x_1 \in X_1$ in the game $G_{(\delta,(\mu_1,\mu_2))}$. Moreover, since $u_1(0, x_2) = u_1(x_1, x_2)$ for all $(x_1, x_2) \in [0, 1] \times (0, 1]$ and $\mu_2(\{0\}) = 0$, we have

$$u_1((1-\delta)0 + \delta\mu_1, (1-\delta)1 + \delta\mu_2) \geq u_1((1-\delta)x_1 + \delta\mu_1, (1-\delta)1 + \delta\mu_2)$$

for all $x_1 \in [0, 1]$. Hence (0, 1) is a Nash equilibrium of $G_{(\delta,\mu)}$, and so (0, 1) is a perfect equilibrium of G.

3.2.1 Limit admissible perfect equilibria

In Example 2, it was shown that perfect equilibria in supermodular games may fail limit admissibility. In this subsection we show that supermodular games satisfying sequential better-reply security have limit admissible, perfect equilibria.

Theorem 3 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a supermodular game satisfying sequential better-reply security. Then G possesses a limit admissible, pure strategy perfect equilibrium. Moreover, the set of limit admissible, pure strategy perfect equilibria of G is compact.*

Proof Choose $\mu \in \times_{i=1}^N \widehat{\Delta}(X_i)$ and a sequence (δ^n) with $\delta^n \in (0, 1)^N$ for each n and $\delta^n \to 0$.

Recall from Remark 2 that X_i is a complete lattice. Since τ_i is a compact metrizable topology, it follows from the Corollary on p. 221 of Strauss (1968) that τ_i coincides with the order topology on X_i. Therefore, each $u_i^{(\delta^n,\mu)}(\cdot, x_{-i})$ is upper semicontinuous with respect to the order topology on X_i. These observations, together with Lemma 5, imply that $G = (X_i, u_i^{(\delta^n,\mu)})_{i=1}^N$ is a quasi-supermodular game as defined in Kultti and Salonen (1997).[7] Applying Proposition 1 in Kultti and Salonen (1997), we conclude that, for each n, the game $G_{(\delta^n,\mu)}$ has a pure strategy Nash equilibrium x^n that is undominated against mixed strategies. In particular, $x^n \in \times_{j=1}^N (X_j \setminus int(D_j))$.

Since (x^n) lies in X and X is sequentially compact, we may write (passing to a subsequence if necessary) $x^n \to x$. Since $u_i^{(\delta^n,\mu)}$ converges uniformly to u_i for each i, and because x^n is a Nash equilibrium of $G_{(\delta^n,\mu)}$ for each n and $x^n \to x$, it follows from Lemma 7 in the Appendix that x is a Nash equilibrium of G. Furthermore, because

[7] That is, X_i is a complete lattice, supermodularity imples quasi-supermodularity, increasing differences implies the single crossing property and upper semi-continuity with respect to the order topology implies order upper-semicontinuity.

$x^n \in \times_{j=1}^N (X_j \setminus int(D_j))$ for each n, and since $x^n \to x$ and $\times_{j=1}^N (X_j \setminus int(D_j))$ is closed, we have $x \in \times_{j=1}^N (X_j \setminus int(D_j))$ so x is limit admissible.

Finally, the set of limit admissible, pure strategy perfect equilibria of G can be written as the intersection of the set of pure strategy perfect equilibria of G and the compact set $\times_{j=1}^N (X_j \setminus int(D_j))$. Hence, since the set of pure strategy perfect equilibria of G is closed (hence compact) in the compact space X, it follows that the set of limit admissible, pure strategy perfect equilibria of G is a compact. □

Acknowledgments We thank an Associate Editor and two anonymous referees for their comments.

Appendix

Proof of Lemma 3

We first state some preparatory lemmas.

The following lemma is a weakening of Theorem 2 in Carbonell-Nicolau and McLean (2013).[8]

Lemma 7 *Suppose that* $G = (X_i, u_i)_{i=1}^N$ *is a metric game satisfying sequential better-reply security and suppose suppose that* $(X_i, u_i^n)_{i=1}^N$ *is a sequence of games such that* $(u_1^n, ..., u_N^n)$ *converges uniformly to* $(u_1, ..., u_N)$*. If* (ε^n) *is a sequence in* $(0, \infty)$ *with* $\varepsilon^n \to 0$*, if* (x^n) *is a sequence such that* x^n *is an* ε^n*-equilibrium of* $(X_i, u_i^n)_{i=1}^N$ *for each n, and if* $x^n \to x$ *for some* $x \in X$*, then* x *is a Nash equilibrium of* G*.*

Remark 7 Observe that Lemma 7 implies in particular that the set of pure strategy Nash equilibria in a sequentially better-reply secure game is closed in X.

Lemma 8 *Suppose that* $G = (X_i, u_i)_{i=1}^N$ *is a compact, metric Borel game satisfying sequential better-reply security. Then the set of pure strategy perfect equilibria of G is closed in* $\times_{i=1}^N \Delta(X_i)$*.*

Proof Let (x^n) be a sequence of pure strategy perfect equilibria of G such that $x^n \to x$ for some $x \in X$. We show that x is a perfect equilibrium of G. Note that, for each n (since x^n is a perfect equilibrium of G), there exist $\mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$, $\delta^n \in (0, 1)^N$ such that $\delta_i^n < \frac{1}{n}$ for each i and $\sigma^n \in E(\overline{G}_{(\delta^n, \mu^n)})$ for each i such that $\lambda(\sigma_i^n, x_i^n) < \frac{1}{n}$ where λ denotes the Prokhorov metric on $\Delta(X_i)$. Therefore, $\sigma_i^n \to x_i$ implying that

$$(1 - \delta_i^n)\sigma_i^n + \delta_i^n \mu_i^n \to x_i \text{ for each } i,$$

and it remains to show that x is a Nash equilibrium of G. Observe that each x^n is a Nash equilibrium of G and $x^n \to x$. Consequently, because the set of Nash equilibria in G is closed in X (Remark 7), it follows that x is a Nash equilibrium of G. □

We are now ready to prove Lemma 3.

[8] The reader is referred to Carbonell-Nicolau and McLean (2013) for the fully fledged version.

Lemma 3 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there are sequences (δ^n) and (μ^n) satisfying the following:*

- $\delta^n \in (0, 1)^N$ *and* $\mu^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$ *for each* n*;*
- $\delta^n \to 0$*; and*
- $E(G_{(\delta^n,\mu^n)}) \neq \varnothing$ *for each* n.

Then G has a pure strategy perfect equilibrium, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact in X.

Proof For each n, let x^n be a Nash equilibrium of $G_{(\delta^n,\mu^n)}$. Since $x^n \in X$ for each n, and because X is sequentially compact, we may write (passing to a subsequence if necessary) $x^n \to x$ for some $x \in X$. Because x^n is a Nash equilibrium of $G_{(\delta^n,\mu^n)}$ for each n, the strategy profile

$$(1 - \delta^n)x^n + \delta^n \mu^n := \left((1 - \delta_1^n)x_1^n + \delta_1^n \mu_1^n, ..., (1 - \delta_N^n)x_N^n + \delta_N^n \mu_N^n\right)$$

is a Nash equilibrium of $\overline{G}_{(\delta^n,\mu^n)}$ for each n. Hence, since $x^n \to x$ and $\delta^n \to 0$, we have

$$(1 - \delta^n)x^n + \delta^n \mu^n \to x,$$

and it follows that x is a perfect profile. In addition, since $u_i^{(\delta^n,\mu^n)}$ converges uniformly to u_i for each i and $x^n \to x$, and because x^n is a Nash equilibrium of $G_{(\delta^n,\mu^n)}$ for each n, it follows from Lemma 7 that x is a Nash equilibrium of G.

It remains to show that the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is a compact. By Lemma 8 (resp. Remark 7), the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is closed in X, and hence compact. □

Proof of Lemma 5

Prior to proving Lemma 5 we need a preparatory lemma. This lemma states that the extension of a function $f : X \to \mathbb{R}$ to the domain $\times_{i=1}^N \Delta(X_i)$ inherits upper semicontinuity in own strategies from f. Since we have not found precisely this result in the literature, a proof is provided.

Lemma 9 *Let $f : X \to \mathbb{R}$ be a bounded Borel measurable function on $X = \times_{i=1}^N X_i$, where each X_i is a compact metric space. Given i, if $f(\cdot, x_{-i})$ is upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$, then for each $\mu_{-i} \in \times_{j \neq i} \Delta(X_j)$, the map*

$$\mu_i \mapsto \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i})\mu_i(dx_i)\mu_{-i}(dx_{-i})$$

defined on $\Delta(X_i)$ is upper semicontinuous.

Proof Fix i. Because $f(\cdot, x_{-i})$ is upper semicontinuous on X_i for every $x_{-i} \in X_{-i}$, $f(\cdot, x_{-i})$ is upper semicontinuous on $\Delta(X_i)$ for every $x_{-i} \in X_{-i}$ (e.g., Aliprantis and

Border (2006), Theorem 15.5). Therefore, given $x_{-i} \in X_{-i}$, if (ν_i^n) is a sequence with $\nu_i^n \in \Delta(X_i)$ for each n and $\nu_i^n \to \nu_i$, then

$$\limsup_{n\to\infty} \int_{X_i} f(x_i, x_{-i})\nu_i^n(dx_i) \leq \int_{X_i} f(x_i, x_{-i})\nu_i(dx_i).$$

Consequently, for every $\nu_{-i} \in \times_{j\neq i}\Delta(X_j)$,

$$\int_{X_{-i}} \left[\limsup_{n\to\infty} \int_{X_i} f(x_i, x_{-i})\nu_i^n(dx_i)\right] \nu_{-i}(dx_{-i}) \\ \leq \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i})\nu_i(dx_i)\nu_{-i}(dx_{-i}). \tag{3}$$

Fix a sequence (ν_i^n) with $\nu_i^n \in \Delta(X_i)$ for each n and $\nu_i^n \to \nu_i$. For each n, define $\phi_i^n : X_{-i} \to \mathbb{R}$ by

$$\phi_i^n(x_{-i}) := \int_{X_i} f(x_i, x_{-i})\nu_i^n(dx_i).$$

Given $\nu_{-i} \in \times_{j\neq i}\Delta(X_j)$, we have, by Fatou's lemma,

$$\limsup_{n\to\infty} \int_{X_i} \left[\int_{X_{-i}} f(x_i, x_{-i})\nu_{-i}(dx_{-i})\right] \nu_i^n(dx_i) = \limsup_{n\to\infty} \int_{X_{-i}} \phi_i^n(x_{-i})\nu_{-i}(dx_{-i}) \\ \leq \int_{X_{-i}} \left[\limsup_{n\to\infty} \phi_i^n(x_{-i})\right] \nu_{-i}(x_{-i}).$$

This, combined with (3), gives

$$\limsup_{n\to\infty} \int_{X_i} \int_{X_{-i}} f(x_i, x_{-i})\nu_{-i}(dx_{-i})\nu_i^n(dx_i) \leq \int_{X_{-i}} \int_{X_i} f(x_i, x_{-i})\nu_i(dx_i)\nu_{-i}(dx_{-i}).$$

This establishes the result. □

We are now ready to prove Lemma 5.

Lemma 5 *If $G = (X_i, u_i)_{i=1}^N$ is supermodular, then $G_{(\delta,\mu)}$ is supermodular for every $(\delta, \mu) \in [0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$.*

Proof Suppose that $G = (X_i, u_i)_{i=1}^N$ is supermodular. Fix

$$((\delta_1, ..., \delta_N), \mu) \in [0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)].$$

Since G is a compact lattice game, it is clear that $G_{(\delta,\mu)}$ is a compact lattice game. Because $u_i(\cdot, x_{-i})$ is upper semicontinuous on X_i for each i and $x_{-i} \in X_{-i}$, Lemma 9 implies that $u_i(\cdot, \nu_{-i})$ is upper semicontinuous on $\Delta(X_i)$ for each i and every $\nu_{-i} \in \times_{j\neq i}\Delta(X_j)$. Consequently, for each i and every $x_{-i} \in X_{-i}$, $u_i^{(\delta,\mu)}(\cdot, x_{-i})$ is upper semicontinuous on X_i.

Next, we show that $u_i^{(\delta,\mu)}(\cdot, x_{-i})$ is supermodular for each i and every $x_{-i} \in X_{-i}$. Fix i and $\{x_i, y_i\} \subseteq X_i$. Define the product measure

$$\nu_{-i} := \underset{j \neq i}{\otimes} \left[(1-\delta_j)x_j + \delta_j \mu_j\right].$$

Since

$$u_i(\sup\{x_i, y_i\}, z_{-i}) + u_i(\inf\{x_i, y_i\}, z_{-i}) \geq u_i(x_i, z_{-i}) + u_i(y_i, z_{-i})$$

for each $z_{-i} \in X_{-i}$, it follows that

$$u_i(\sup\{x_i, y_i\}, \nu_{-i}) + u_i(\inf\{x_i, y_i\}, \nu_{-i}) \geq u_i(x_i, \nu_{-i}) + u_i(y_i, \nu_{-i}).$$

Therefore,

$$\begin{aligned}
u_i^{(\delta,\mu)}&(\sup\{x_i, y_i\}, x_{-i}) + u_i^{(\delta,\mu)}(\inf\{x_i, y_i\}, x_{-i}) \\
&= (1-\delta_i)\left[u_i(\sup\{x_i, y_i\}, \nu_{-i}) + u_i(\inf\{x_i, y_i\}, \nu_{-i})\right] \\
&\quad + \delta_i\left[u_i(\mu_i, \nu_{-i}) + u_i(\mu_i, \nu_{-i})\right] \\
&\geq (1-\delta_i)\left[u_i(x_i, \nu_{-i}) + u_i(y_i, \nu_{-i})\right] + \delta_i\left[u_i(\mu_i, \nu_{-i}) + u_i(\mu_i, \nu_{-i})\right] \\
&= u_i^{(\delta,\mu)}(x_i, x_{-i}) + u_i^{(\delta,\mu)}(y_i, x_{-i}).
\end{aligned}$$

It remains to show that $u_i^{(\delta,\mu)}$ exhibits increasing differences in X_i and X_{-i} for each i. Fix i and $\{y_{-i}, x_{-i}\} \subseteq X_{-i}$ with $y_{-i} \geq_{-i} x_{-i}$. Choose $\{x_i, y_i\} \subseteq X_i$ with $y_i \geq_i x_i$. We must show that

$$u_i^{(\delta,\mu)}(y_i, y_{-i}) - u_i^{(\delta,\mu)}(y_i, x_{-i}) \geq u_i^{(\delta,\mu)}(x_i, y_{-i}) - u_i^{(\delta,\mu)}(x_i, x_{-i}).$$

Let $\overline{N} = \{1, .., N\}$. Fix i, $I \subseteq \overline{N}\backslash i$, and $J \subseteq \overline{N}\backslash(I \cup i)$ and let $x_I = (x_j)_{j \in I} \in \times_{j \in I} X_i$ and $y_I = (y_j)_{j \in I} \in \times_{j \in I} X_i$ denote the projections of x_{-i} and y_{-i}. If $z_J \in \times_{j \in J} X_j$, then

$$u_i(y_i, (y_I, z_J)) - u_i(x_i, (y_I, z_J)) \geq u_i(y_i, (x_I, z_J)) - u_i(x_i, (x_I, z_J))$$

implying that

$$u_i(y_i, (y_I, \mu_J)) - u_i(x_i, (y_I, \mu_J)) \geq u_i(y_i, (x_I, \mu_J)) - u_i(x_i, (x_I, \mu_J)).$$

This implies that

$$\begin{aligned}
&\left[u_i^{(\delta,\mu)}(y_i, y_{-i}) - u_i^{(\delta,\mu)}(y_i, x_{-i})\right] - \left[u_i^{(\delta,\mu)}(x_i, y_{-i}) - u_i^{(\delta,\mu)}(x_i, x_{-i})\right] \\
&\quad = \sum_{I \subseteq \overline{N}\backslash i} \left[\prod_{j \in I}(1-\delta_j) \prod_{j \in \overline{N}\backslash(I \cup i)} \delta_j\right] \left[u_i\left((1-\delta_i)y_i + \delta_i \mu_i, (y_I, \mu_{\overline{N}\backslash(I \cup i)})\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-u_i\Big((1-\delta_i)x_i+\delta_i\mu_i,(y_I,\mu_{\overline{N}\setminus(I\cup i)})\Big)\Big]\\
&-\left[\sum_{I\subseteq\overline{N}\setminus i}\left[\prod_{j\in I}(1-\delta_j)\prod_{j\in\overline{N}\setminus(I\cup i)}\delta_j\right]\Big[u_i\big((1-\delta_i)y_i+\delta_i\mu_i,(x_I,\mu_{\overline{N}\setminus(I\cup i)})\big)\right.\\
&\left.-u_i\big((1-\delta_i)x_i+\delta_i\mu_i,(x_I,\mu_{\overline{N}\setminus(I\cup i)})\big)\Big]\right]\\
&=(1-\delta_i)\sum_{I\subseteq\overline{N}\setminus i}\left[\prod_{j\in I}(1-\delta_j)\prod_{j\in\overline{N}\setminus(I\cup i)}\delta_j\right]\\
&\quad\times\Big[u_i\big(y_i,(y_I,\mu_{\overline{N}\setminus(I\cup i)})\big)-u_i\big(x_i,(y_I,\mu_{\overline{N}\setminus(I\cup i)})\big)\Big]\\
&\quad-(1-\delta_i)\left[\sum_{I\subseteq\overline{N}\setminus i}\left[\prod_{j\in I}(1-\delta_j)\prod_{j\in\overline{N}\setminus(I\cup i)}\delta_j\right]\right.\\
&\quad\left.\times\Big[u_i\big(y_i,(x_I,\mu_{\overline{N}\setminus(I\cup i)})\big)-u_i\big(x_i,(x_I,\mu_{\overline{N}\setminus(I\cup i)})\big)\Big]\right]\geq 0,
\end{aligned}
$$

and we conclude that $G_{(\delta,\mu)}$ is supermodular. □

Proof of Lemma 4

Lemma 4 *If G is supermodular, then $G_{(\delta,\mu)}$ has a Nash equilibrium for every* $(\delta,\mu)\in[0,1)^N\times[\times_{i=1}^N\widehat{\Delta}(X_i)]$.

Proof If $G=(X_i,u_i)_{i=1}^N$ is supermodular, then $G_{(\delta,\mu)}$ is supermodular for every $(\delta,\mu)\in[0,1)^N\times(\times_{i=1}^N\widehat{\Delta}(X_i))$ (Lemma 5). Moreover, since X_i is endowed with a topology at least as fine as the interval topology, it follows from Theorem 4.2 in Vives (1990) that $G_{(\delta,\mu)}$ has a Nash equilibrium. □

Proof of Lemma 6

Lemma 6 *Suppose that $G=(X_i,u_i)_{i=1}^N$ is a compact, metric, Borel game satisfying sequential better-reply security. Suppose that there exists $\alpha\in(0,1)$ such that for every $(\delta,\mu)\in(0,\alpha]^N\times[\times_{i=1}^N\widehat{\Delta}(X_i)]$, $G_{(\delta,\mu)}$ has a Nash equilibrium. Then G has a stable set of pure strategy profiles. Furthermore, any stable set of pure strategy profiles of G is a compact subset of the set of pure strategy perfect equilibria of G, and the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact.*

Proof Suppose that $G = (X_i, u_i)_{i=1}^N$ is a supermodular game G satisfying sequential better-reply security. Then $E(G)$ is nonempty (Theorem 1). We will prove that

$$\Theta = \left\{(\theta_{x_1}^1, .., \theta_{x_N}^N) \in \Delta(X) : (x_1, .., x_N) \in E(G)\right\}$$

contains a KM stable set for G. First we show that Θ is KM prestable. For each $(\delta, \mu) \in (0, 1)^N \times [\times_{i=1}^N \widehat{\Delta}(X_i)]$, let $G_{(\delta,\mu)}$ be the game defined in Section 2.3 as

$$G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^N,$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is given by

$$u_i^{(\delta,\mu)}(x) := u_i\left((1-\delta_1)x_1 + \delta_1\mu_1, ..., (1-\delta_N)x_N + \delta_N\mu_N\right).$$

Let

$$\Theta^{(\delta,\mu)} := \left\{(\theta_{x_1}^1, .., \theta_{x_N}^N) \in \times_{i=1}^N \Delta(X_i) : (x_1, .., x_N) \in E(G_{(\delta,\mu)})\right\}.$$

Then $E(G_{(\delta,\mu)}) \neq \varnothing$ (Lemma 4). Since $E(G_{(\delta,\mu)}) \subseteq E(\overline{G}_{(\delta,\mu)})$, it suffices to prove that, for every open set U containing Θ, there exists an $\alpha > 0$ such that the following condition holds: for every $(\delta_1, .., \delta_N)$ with $0 < \delta_i < \alpha$ for each i and for every $(\mu_1, .., \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i,

$$\Theta^{(\delta,\mu)} \cap U \neq \varnothing.$$

To see this, suppose not. Then there exists an open set U containing Θ and, for each n, there exist numbers $0 < \delta_i^n < \frac{1}{n}$ and probability measures $\mu_i^n \in \times_{i=1}^N \widehat{\Delta}(X_i)$ such that $\Theta^{(\delta^n,\mu^n)} \cap U = \varnothing$. Since u is the uniform limit of the sequence $(u^{(\delta^n,\mu^n)})$ and X is compact, we can apply the same argument as that used in the proof of Theorem 1 and conclude that there exists a subsequence $(u^{(\delta^{n_k},\mu^{n_k})})$ and a sequence $x^k \in E(G_{(\delta^{n_k},\mu^{n_k})})$ such that $x^k \to x$ and $x \in E(G)$. This contradiction establishes the claim. Sequential better-reply security implies that $E(G)$ is closed in X and it follows that $E(G)$ is compact in X. Applying Theorem 14.8 in Aliprantis and Border (2006), we conclude that Θ is compact hence closed in $\times_{i=1}^N \Delta(X_i)$, implying that Θ is KM prestable.

To complete the proof, we show that Θ contains a minimal KM prestable set by applying Zorn's Lemma in a standard way. Let $\mathcal{F}$ be defined as the collection of sets W of Nash equilibria of G (identified with their corresponding profiles of Dirac measues) satisfying (i) $W \subseteq \Theta$ and (ii) W is KM prestable in G. Next, suppose that $\mathcal{F}$ is ordered by set inclusion and suppose that $\mathcal{C}$ is a totally ordered subcollection of $\mathcal{F}$. The collection $\mathcal{C}$ has the finite intersection property. Therefore, $S := \cap\{W : W \in \mathcal{C}\}$ is compact and nonempty since each member of $\mathcal{C}$ is closed and Θ is compact. To show that S is KM prestable, suppose that U is open and $S \subseteq U$. Then there exist $W' \in \mathcal{C}$ such that $W' \subseteq U$. Otherwise, $\{W \backslash U : W \in \mathcal{C}\}$ is a collection of closed subsets of Θ satisfying the finite intersection property. This implies that $S \backslash U = \cap\{W \backslash U : W \in \mathcal{C}\} \neq \varnothing$, an impossibility. Since W' is KM prestable, it follows that S is KM prestable. The existence of a minimal KM prestable set in G contained in Θ now follows from Zorn's Lemma.

Next, we show that each element of a KM stable set S of pure strategy profiles is a pure strategy perfect equilibrium. Let

$$\widehat{S} = \left\{(\theta^1_{x_1}, .., \theta^N_{x_N}) \in \Delta(X) : (x_1, .., x_N) \in S\right\}.$$

If $|\widehat{S}| = 1$, then the one member of $\widehat{S}$ is a strictly perfect equilibrium, hence a trembling-hand perfect equilibrium. So suppose that $|\widehat{S}| > 1$. Choose $x \in \widehat{S}$ and choose $\varepsilon > 0$ so that $\widehat{S} \backslash B_\varepsilon(x) \neq \varnothing$ where $B_\varepsilon(x)$ is the open ball of radius ε centered at x associated with the Prokhorov metric on $\times_{i=1}^N \Delta(X_i)$. Since S is KM stable and $S \backslash B_\varepsilon(x)$ is closed and nonempty, it follows from minimality that $S \backslash B_\varepsilon(x)$ is not KM stable. Therefore, there exists an open set U containing $S \backslash B_\varepsilon(x)$ such that, for every k, there exist $0 < \delta_i^k < \frac{1}{k}$ and $\mu_k \in \times_{i=1}^N \widehat{\Delta}(X_i)$ such that $E(\overline{G}_{(\delta^k, \mu^k)}) \cap U = \varnothing$. Next, note that $S \subseteq U \cup B_\varepsilon(x)$ and $U \cup B_\varepsilon(x)$ is open. Since S is prestable, it follows that $E(\overline{G}_{(\delta^k, \mu^k)}) \cap [U \cup B_\varepsilon(x)] \neq \varnothing$ for sufficiently large k. In particular, $E(\overline{G}_{(\delta^k, \mu^k)}) \cap B_\varepsilon(x) \neq \varnothing$ for sufficiently large k and we conclude that x is trembling-hand perfect.

Finally, stable sets are by definition closed in the set of Nash equilibria of the mixed extension $\overline{G}$, so stable sets of pure strategy profiles in G are closed (hence compact) in the compact space X. The fact that the set of pure strategy perfect equilibria (resp. the set of pure strategy Nash equilibria) of G is compact follows from Theorem 1. $\square$

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