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RESEARCH ARTICLE

Nash and Bayes–Nash equilibria in strategic-form games with intransitivities

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Abstract
We study games with intransitive preferences that admit skew-symmetric representations. We introduce the notion of surrogate better-reply security for discontinuous skew-symmetric games and elucidate the relationship between surrogate better-reply security and other security concepts in the literature. We then prove existence of behavioral strategy equilibrium for discontinuous skew-symmetric games of incomplete information (and, in particular, existence of mixed-strategy equilibrium for discontinuous skew-symmetric games of complete information), generalizing extant results.

Keywords Skew-symmetric game · Bayesian game · Existence of Nash equilibrium · Discontinuous game · Behavioral strategy

JEL classification C72

1 Introduction

A number of important applications of Game Theory involve discontinuous payoff functions. Building on previous work of Dasgupta and Maskin (1986), Simon (1987), and others, Reny (1999) derived a number of existence results for games with discontinuous payoffs using various weakenings of upper semicontinuity of payoffs (such as Simon's (1987) reciprocal upper semicontinuity or Dasgupta and Maskin's (1986) upper semicontinuity of the sum of payoffs) and lower semicontinuity of payoffs (such as the notion of payoff security). If strategy sets are convex and payoffs are

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quasiconcave in own actions, then these weakenings of upper and lower semicontinuity can be applied to derive pure-strategy existence results.[1] The mixed extension of a game will satisfy the convexity and quasiconcavity assumptions so these pure-strategy existence results can be applied to the mixed extension if the mixed extension itself satisfies the Reny weakenings of upper and lower semicontinuity. It is however useful to identify conditions on the primitives of a complete information game implying that the mixed extension will satisfy the Reny conditions (therefore implying the existence of a mixed-strategy equilibrium). Such conditions are typically easier to verify and one such condition, called uniform payoff security in Monteiro and Page (2007), guarantees that the mixed extension of a strategic-form game is payoff secure. In Carbonell-Nicolau and McLean (2018), this mixed-strategy result is generalized in order to obtain the existence of equilibrium in behavioral and distributional strategies in games of incomplete information with discontinuous payoffs. In related work, He and Yannelis (2016a) extend the mixed-strategy result of Allison and Lepore (2014) for games satisfying disjoint payoff matching to the incomplete information setup.

All of the aforementioned results are formulated in the framework of strategic-form games with a utility representation, i.e., games in which each player's preference order defined on the set of all strategy profiles is represented by a real-valued payoff function. Several recent papers have investigated the extent to which these results for games with discontinuous payoff functions can be extended to the case in which a player's preference order need not be representable by a utility function. Reny (2016a) introduces the notions of point security and correspondence security for games in which players' preference relations are complete, reflexive, and transitive, and generalizes existence results for strategic-form games with payoff functions found in Reny (1999), Barelli and Meneghel (2013), and McLennan et al. (2011). Carmona and Podczeck (2016) introduce the notions of point target security and correspondence target security and provide several further generalizations of these results. For related results in games and models of abstract economies in which agents' preferences need not be representable by utility functions, see Reny (2016c) and He and Yannelis (2016b).

The existence results in these papers, while generalizing many results for strategic-form games, are pure-strategy existence results that weaken the assumption that preferences are representable by payoff functions but retain the assumption of convexity of the players' strategy sets and, in several results, the convexity assumption of preferences. In the absence of convexity of strategy sets and/or preferences, one naturally looks for an equilibrium in mixed strategies. It is the goal of this paper to generalize certain of the aforementioned results in a framework that weakens the assumption that discontinuous preferences are representable by utility functions while still allowing for a tractable theory of mixed-strategy equilibrium in the absence of convexity. Consequently, our approach is based on complete, reflexive preference relations that need not satisfy transitivity but do admit a skew-symmetric representation.

To introduce this idea, let S be a nonempty set. A function $\varphi : S \times S \to \mathbb{R}$ is *skew-symmetric* if $\varphi(x, y) = -\varphi(y, x)$ for all $(x, y) \in S \times S$. Obviously, skew symmetry implies that $\varphi(x, x) = 0$ for all $x \in S$. A relation $\succsim$ in $S \times S$ has a skew-symmetric

[1] For an excellent survey of this literature including extensive references, see Carmona (2013). For more recent results, see Reny (2016b).

representation if there exists a skew-symmetric function $\varphi : S \times S \to \mathbb{R}$ satisfying

$$y \succsim x \Leftrightarrow \varphi(y, x) \geq 0.$$

From the definition, it follows that every relation $\succsim$ admitting a skew-symmetric representation is reflexive and complete, and if $\succsim$ admits a utility representation $u : S \to \mathbb{R}$, then $\succsim$ admits the skew-symmetric representation $\varphi(x, y) = u(x) - u(y)$.

In the case of a consumer with a preference relation defined on $S = \mathbb{R}^L_+$ for some $L > 0$, Shafer (1974) characterized preferences admitting a skew-symmetric representation in terms of comparability (reflexivity and completeness), convexity, and continuity axioms. In the case of decision making in the presence of risk, Fishburn extended the linear theory of von Neumann and Morgenstern to the case of intransitive preferences and in several papers, Fishburn studies the axiomatic structure of skew-symmetric bilinear utility theory and its properties. This work is presented in the comprehensive monograph of Fishburn (1988b) that also presents his skew-symmetric bilinear extension of Savage's (1972) subjective expected utility theory.

Intransitive preferences in games arise naturally in strategic interactions among groups of agents taking collective actions. These situations can often be modeled as games admitting a skew-symmetric representation, and we illustrate this idea in the context of Bayesian games.

Summarizing, we wish to weaken the order assumption on preferences but still retain sufficient structure so as to allow for mixed-strategy equilibria in the absence of convex preferences or convex strategy sets. In addition, we want to recover as special cases the extant results concerning mixed-strategy equilibria of discontinuous complete information games and distributional/behavioral strategy equilibria of discontinuous incomplete information games. To that end, we introduce the notion of surrogate function that will form the basis for the various security definitions that we will present in the context of games with preferences represented by a skew-symmetric function.

Following a presentation of basic definitions in Sect. 2, we define, in Sect. 3, the basic concept of *surrogate better-reply security* and its generalizations, *surrogate point security* and *surrogate correspondence security*. We note that surrogate better-reply security (resp. surrogate point and correspondence security) generalizes the notion of better-reply security (resp. point and correspondence security) defined in Reny (1999) (resp. Reny (2016a)) in the case of strategic-form games, and we record several existence theorems for skew-symmetric discontinuous games satisfying these surrogate security definitions. Informally, our surrogate security concepts allow us to replace a game defined by skew-symmetric evaluation functions with a new "surrogate" game satisfying reflexivity, completeness and transitivity whose equilibria yield equilibria of the original game. We can then apply an existence result in Reny (2016a) or Nessah and Tian (2016) to the surrogate game, thus establishing the existence of equilibrium in the original problem. In Sect. 4, we introduce *uniform surrogate payoff security* for skew-symmetric games of incomplete information as a generalization of the notion of *uniform payoff security* defined in Carbonell-Nicolau and McLean (2018) for strategic-form games, and we provide a behavioral strategy equilibrium existence result for discontinuous incomplete information skew-symmetric games. A simple application illustrates the existence results in Sect. 4.4. In the case of strategic-form games, our results for games of incomplete information can be specialized to games of complete information, resulting in existence theorems that strictly generalize the mixed-strategy result in Monteiro and Page (2007). In the case of skew-symmetric games, the analysis extends that of Fishburn and Rosenthal (1986), who proved the existence of a mixed-strategy Nash equilibrium in strategic-form games with finitely many actions.

2 Preliminaries

Given $i \in \{1, \ldots, N\}$ and sets $X_1, \ldots, X_N$, define $X_{-i} := \times_{j \neq i} X_j$; given i, the set $\times_{j=1}^N X_j$ is sometimes represented as $X_i \times X_{-i}$, and we sometimes write $z = (z_i, z_{-i}) \in X_i \times X_{-i}$ for a member z of $\times_{j=1}^N X_j$.

Definition 1 A ***game*** is a collection $G = (X_i, \succsim_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i, and $\succsim_i$ is a preference relation for player i defined on the set $X := \times_{i=1}^N X_i$ of action profiles, i.e., $\succsim_i$ is a binary relation in $X \times X$.

We say that a game $G = (X_i, \succsim_i)_{i=1}^N$ has a ***skew-symmetric (SSYM) representation*** if for each i there exists a skew-symmetric map $\varphi_i : X \times X \to \mathbb{R}$ satisfying[2]

$$y \succsim_i x \Leftrightarrow \varphi_i(y, x) \geq 0, \quad \text{for all } (x, y) \in X \times X.$$

A ***skew-symmetric (SSYM) game*** is a collection $G = (X_i, \varphi_i)_{i=1}^N$, where each $\varphi_i : X \times X \to \mathbb{R}$ is skew-symmetric and each X_i is a topological space.

A ***strategic-form (SF) game*** is a collection $G = (X_i, u_i)_{i=1}^N$, where each $u_i : X \to \mathbb{R}$ is a payoff function and each X_i is a topological space.

Obviously, every SF game $G = (X_i, u_i)_{i=1}^N$ has an equivalent representation as an SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ where $\varphi_i(x, y) = u_i(x) - u_i(y)$ for each $(x, y) \in X \times X$ and i.

Definition 2 We say that $G = (X_i, \varphi_i)_{i=1}^N$ is ***compact*** (resp. ***metric***) if each X_i is a compact (resp. metric) space. The game $G = (X_i, \varphi_i)_{i=1}^N$ is ***quasiconcave*** if for each i, X_i is a convex subset of a linear space and the map $x_i \mapsto \varphi_i((x_i, z_{-i}), z)$ is quasiconcave on X_i for each $z \in X$.[3] The game $G = (X_i, \varphi_i)_{i=1}^N$ is ***bounded*** if for each i, $\varphi_i : X \times X \to \mathbb{R}$ is bounded.

Definition 3 A ***Nash equilibrium*** of an SSYM game $(X_i, \varphi_i)_{i=1}^N$ is a strategy profile $(z_1, \ldots, z_N) \in \times_{i=1}^N X_i$ such that for each i,

$$\varphi_i((x_i, z_{-i}), z) \leq 0, \quad \text{for all } x_i \in X_i.$$

[2] The map φ_i is *skew-symmetric* if $\varphi_i(x, y) = -\varphi_i(y, x)$ for all $(x, y) \in X \times X$.

[3] In the special case of SF games, this notion of quasiconcavity reduces to the standard notion of quasiconcavity in own strategies.

When $(X_i, \varphi_i)_{i=1}^N$ has a utility representation, Definition 3 reduces to the standard notion of equilibrium for strategic-form games.

When each X_i is a finite set, Fishburn and Rosenthal (1986) proved the existence of a mixed-strategy equilibrium by mimicking Nash's "adjustment function" argument that only utilizes Brouwer's Fixed Point Theorem. However, the classical argument for existence of a pure-strategy equilibrium based on the Kakutani Fixed Point Theorem also trivially applies if enough continuity is assumed. For example, suppose that each X_i is a compact, nonempty, convex subset of $\mathbb{R}^{m_i}$ for some $m_i \geq 1$. In addition, suppose that each φ_i is continuous on $X \times X$ and $x_i \mapsto \varphi_i((x_i, z_{-i}), z)$ is quasiconcave for each $z \in X$. Now define

$$\mu_i(z) := \arg\max_{x_i \in X_i} \varphi_i((x_i, z_{-i}), z), \quad \text{for each } z \in X.$$

Then, combining Berge's Maximum Theorem and the Kakutani Fixed Point Theorem, it follows that there exists $x^* \in X$ such that for each $x \in X$, we have

$$x^* \in \mu_1(x^*) \times \cdots \times \mu_n(x^*),$$

i.e.,

$$\varphi_i((x_i, x^*_{-i}), x^*) \leq \varphi_i(x^*, x^*) = 0.$$

A mixed-strategy equilibrium result is similarly straightforward.

Note the assumption here that φ_i is continuous on $X \times X$, which allows us to apply Berge's Maximum Theorem in the usual way. This raises some delicate issues later when trying to formulate a discontinuous generalization of the better-reply security condition introduced in Reny (1999) that includes Reny's original definition in the special case where $\varphi_i(x, y) = u_i(x) - u_i(y)$ for each $(x, y) \in X \times X$.

Of course, there is also an obvious approach to existence using some version of the Ky-Fan inequality. Let

$$F(x, z) = \sum_{i=1}^{N} \varphi_i((x_i, z_{-i}), z).$$

Then $x^* \in X$ is an equilibrium if and only if

$$F(x, x^*) \leq 0, \quad \text{for all } x \in X.$$

Consequently, any assumptions that guarantee the existence of a solution to the Ky-Fan inequality will yield an equilibrium even in the SSYM generalization. For example, we can deduce the existence of an equilibrium from the Ky-Fan inequality if for each i, $x_i \mapsto \varphi_i((x_i, z_{-i}), z)$ is concave for each $z \in X$ and $z \mapsto \varphi_i((x_i, z_{-i}), z)$ is lower semicontinuous for each $x_i \in X_i$. Of course, one can ask whether an equilibrium exists when $x_i \mapsto \varphi_i((x_i, z_{-i}), z)$ is quasiconcave for each $z \in X$ and $z \mapsto \varphi_i((x_i, z_{-i}), z)$ is lower semicontinuous for each $x_i \in X_i$, and we can answer this in the affirmative as consequence of our main result below.

3 Complete information games: pure-strategy equilibrium

Throughout this section, we will assume that for each i, X_i is a nonempty subset of a locally convex Hausdorff topological vector space.

3.1 Surrogate better-reply security

We begin by recalling the basic notion of better-reply security for SF games. Given an SF game $G = (X_i, u_i)_{i=1}^N$, the *graph* of G is the set

$$\Gamma_G := \{(x, \alpha) \in X \times \mathbb{R}^N : (u_1(x), \ldots, u_N(x)) = \alpha\}.$$

The closure of Γ_G in $X \times \mathbb{R}^N$ is denoted by $\overline{\Gamma}_G$.

Definition 4 (Reny 1999) An SF game $G = (X_i, u_i)_{i=1}^N$ is ***better-reply secure*** if whenever $(x^*, \alpha^*) \in \overline{\Gamma}_G$ and x^* is not a Nash equilibrium of G, there exist $i, \overline{x}_i \in X_i$, and an open set V_{x^*} containing x^* such that

$$\inf_{z \in V_{x^*}} u_i(\overline{x}_i, z_{-i}) > \alpha_i^*.$$

We wish to extend this definition to SSYM games so that, when specialized to SF games with bounded payoffs , we recover the definition of Reny (1999) .

Given a map $H : X \to \mathbb{R}^N$, define the *graph* of H by

$$\Gamma_H := \{(x, \alpha) \in X \times \mathbb{R}^N : H(x) = \alpha\},$$

and let $\overline{\Gamma}_H$ represent the closure of Γ_H in $X \times \mathbb{R}^N$.

Definition 5 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate better-reply secure*** if there exists a bounded function $H : X \to \mathbb{R}^N$ such that, whenever $(x^*, \alpha^*) \in \overline{\Gamma}_H$ and x^* is not a Nash equilibrium of G, then there exist $i, \overline{x}_i \in X_i$, and an open set V_{x^*} containing x^* such that

$$\inf_{z \in V_{x^*}} [\varphi_i((\overline{x}_i, z_{-i}), z) + H_i(z)] > \alpha_i^*.$$

A function $H : X \to \mathbb{R}^N$ satisfying the condition in Definition 5 is called a *surrogate function* for the game $G = (X_i, \varphi_i)_{i=1}^N$ in which case we will say that the game $G = (X_i, \varphi_i)_{i=1}^N$ is surrogate better-reply secure with respect to H. A similar convention will be followed for the successively more general notions of surrogate security that we will define in this paper.

Remark 1 An SF game $G = (X_i, u_i)_{i=1}^N$ with bounded payoff functions is better-reply secure if and only if the associated SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ with $\varphi_i(x, y) = u_i(x) - u_i(y)$ is surrogate better-reply secure with respect to H where $H_i = u_i$ for each i. In the following example, we show that a strategic-form game can satisfy surrogate better-reply security but not better-reply security. That is, surrogate better-reply security strictly generalizes better-reply security even for strategic-form games.

Example 1 Consider the two-player SF game $G = ([0, 1], [0, 1], u_1, u_2)$, where $u_2 \equiv 0$ and

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } x_1 \in [0, 1) \text{ and } x_2 = 0, \\ 0 & \text{if } x_1 \in [0, 1) \text{ and } x_2 \neq 0, \\ 2 & \text{if } x_1 = 1 \text{ and } x_2 = 0, \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 \neq 0. \end{cases}$$

Note that G is compact and quasiconcave. Compactness is clear. Quasiconcavity follows from the fact that $u_1(\cdot, x_2)$ is nondecreasing for each $x_2 \in [0, 1]$.

The game G fails better-reply security. To see this, note that $(x^*, u(x^*)) = ((0, 0), (1, 0)) \in \overline{\Gamma}_G$. Suppose that V_{x^*} is open in X and $(0, 0) \in V_{x^*}$. If there exists an $\overline{x}_1$ such that

$$\inf_{z \in V_{x^*}} u_1(\overline{x}_1, z_2) > 1$$

then $\overline{x}_1 = 1$. However, every open set containing $(0, 0)$ contains a point z such that $z_2 > 0$. Since $u_1(1, z_2) = 1$ if $z_2 > 0$, it follows that

$$\inf_{z \in V_{x^*}} u_1(\overline{x}_1, z_2) \leq 1.$$

A strategy pair (x_1, x_2) is not an equilibrium if and only if $(x_1, x_2) \in [0, 1) \times [0, 1]$. To see that G satisfies surrogate better-reply security, let $H_i \equiv 0$ for each i. We claim that G is surrogate better-reply secure with respect to H. In particular, we must show that, for every $(x_1^*, x_2^*) \in [0, 1) \times [0, 1]$, there exists $\overline{x}_1 \in [0, 1]$ and an open set V_{x^*} with $x^* \in V_{x^*}$ such that

$$\inf_{z \in V_{x^*}} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2)] > 0.$$

Choose $(x_1^*, x_2^*) \in [0, 1) \times [0, 1]$. Let $\overline{x}_1 = 1$ and choose an open set V_{x^*} with $x^* \in V_{x^*}$ so that $z_1 < 1$ for every $z \in V_{x^*}$. If $z \in V_{x^*}$ then

$$u_1(1, z_2) - u_1(z_1, z_2) = 1,$$

implying that

$$\inf_{z \in V_{x^*}} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2)] \geq 1.$$

Remark 2 In the case of an SF game $G = (X_i, u_i)_{i=1}^N$ with bounded payoff functions, surrogate better-reply security with respect to H where $H_i \equiv 0$ for each i is equivalent to the notion of weak transfer continuity of Nessah and Tian (see Definition 3.1 in Nessah (2011)).

As a consequence of a more general result (Theorem 1) that we will prove later, we have the following pure-strategy existence theorem for SSYM games that generalizes Theorem 3.1 in Reny (1999).

Proposition 1 *Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is a bounded, compact, quasiconcave SSYM game satisfying surrogate better-reply security. Then, G possesses a Nash equilibrium.*

The following corollary is immediate since an SSYM game is surrogate better-reply secure (with respect to the surrogate function $H \equiv 0$) if $x \mapsto \varphi_i((y_i, x_{-i}), x)$ is lower semicontinuous for each i and each $y_i \in X_i$.

Corollary 1 (to Proposition 1) *Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is a bounded, compact, quasiconcave SSYM game with the property that $x \mapsto \varphi_i((y_i, x_{-i}), x)$ is lower semicontinuous on X for each i and each $y_i \in X_i$. Then G possesses a Nash equilibrium.*

3.2 Surrogate point security

Definition 6 (Reny 2016a) aN SF game $G = (X_i, u_i)_{i=1}^N$ is ***point secure*** if whenever x^* is not a Nash equilibrium of G, there exist $\overline{x} \in X$ and an open set U containing x^* such that for each $y \in U$ there is a player i such that

$$u_i(\overline{x}_i, x'_{-i}) > u_i(y), \quad \text{for all } x' \in U.$$

Definition 7 (Reny 2016a) An SF game $G = (X_i, u_i)_{i=1}^N$ is ***point secure**** if whenever x^* is not a Nash equilibrium of G, there exist $\overline{x} \in X$ and an open set U containing x^* such that for each $y \in U$ there is a player i such that

$$y_i \notin \text{co}\{w_i \in X_i : u_i(\overline{x}_i, x'_{-i}) \leq u_i(w_i, y_{-i})\}, \quad \text{for all } x' \in U.$$

Remark 3 If the SF game $G = (X_i, u_i)_{i=1}^N$ is point secure and for each i the function $x_i \mapsto u_i(x_i, x_{-i})$ is quasiconcave for each x_{-i}, then G is point secure*.

Remark 4 Example 1 violates point security. To see this, note that $(0, 0)$ is not a Nash equilibrium and since $u_1(0, 0) = 1$, in this game the above definition requires that $u_1(1, x''_2) > 1$ for all x'' in some neighborhood of $(0, 0)$. This is impossible since for x'' arbitrarily close to $(0, 0)$ with $x''_2 \neq 0$ we have $u_1(1, x''_2) = 1$.

Definition 8 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate point secure*** if there exists a function $H : X \to \mathbb{R}^N$ such that, whenever x^* is not a Nash equilibrium of G, there exist $\overline{x} \in X$ and an open set U containing x^* such that for each $y \in U$ there is an i such that

$$\varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x') > H_i(y), \quad \text{for all } x' \in U.$$

Definition 9 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate point secure**** if there exists a function $H : X \to \mathbb{R}^N$ such that, whenever x^* is not a Nash equilibrium of G, there exist $\overline{x} \in X$ and an open set U containing x^* such that for each $y \in U$ there is an i such that

$$y_i \notin \text{co}\{w_i \in X_i : \varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x') \leq \varphi_i((w_i, y_{-i}), y) + H_i(y)\}$$

for all $x' \in U$.

Remark 5 If the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is surrogate point secure and for each i the function $x_i \mapsto \varphi_i((x_i, z_{-i}), z)$ is quasiconcave for each z, then G is surrogate point secure*.

Remark 6 It is easy to see that for SF games, point security implies surrogate point security of the associated SSYM game by choosing the surrogate function H with $H_i = u_i$ for each i.

Before proving that surrogate better-reply security implies surrogate point security, we provide an intermediate surrogate security definition that generalizes the notion of B-security in McLennan et al. (2011).

Definition 10 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate*** B***-secure*** if there exists a bounded function $H : X \to \mathbb{R}^N$ such that, whenever $x^* \in X$ is not a Nash equilibrium of G, there exist an open set V_{x^*} containing x^*, $\alpha \in \mathbb{R}^N$, and $\varepsilon > 0$ such that the following is satisfied: (a) there exists $\overline{x} \in X$ such that for each i, $\varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x') \geq \alpha_i + \varepsilon$ for each $x' \in V_{x^*}$; and (b) for each $x' \in V_{x^*}$, there is a player i with $H_i(x') < \alpha_i - \varepsilon$.

Lemma 1 (i) *If the SSYM game* $G = (X_i, \varphi_i)_{i=1}^N$ *is bounded and surrogate better-reply secure, then G is surrogate* B*-secure.* (ii) *If the SSYM game* $G = (X_i, \varphi_i)_{i=1}^N$ *is surrogate* B*-secure, then G is surrogate point secure.*

Proof (i) Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is bounded and surrogate better-reply secure with (bounded) surrogate function H. Suppose that $x^* \in X$ is not a Nash equilibrium of G. Define, for each i,

$$\beta_i := \sup_{\overline{x}_i \in X_i} \sup_{U \ni x^*} \inf_{x' \in U} [\varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x')].$$

Applying the argument in the proof of Lemma 2.5 in McLennan et al. (2011), we see that there exists $\varepsilon > 0$ such that if $x^n \to x^*$ and $H(x^n) \to \alpha'$ then there is an i with $\beta_i > \alpha'_i + 2\varepsilon$. Defining $\alpha \in \mathbb{R}^N$ by $\alpha_i := \beta_i - \varepsilon$, there exist V_{x^*} and $\overline{x} \in X$ such that for each i , $\varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x') \geq \alpha_i + \varepsilon$ for each $x' \in V_{x^*}$. This establishes item (a) of Definition 10. The proof that (b) of Definition 10 holds is a verbatim transcription of the argument in the last paragraph of the proof of Lemma 2.5 in McLennan et al. (2011).

(ii) Suppose that SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is surrogate B-secure with surrogate function H. Suppose that $x^* \in X$ is not an equilibrium of G. Then, there exist an open set V_{x^*} containing x^*, $\alpha \in \mathbb{R}^N$, and $\varepsilon > 0$ such that the following is satisfied: (a) there exists $\overline{x} \in X$ such that for each i, $\varphi_i((\overline{x}_i, x'_{-i}), x') + H_i(x') \geq \alpha_i + \varepsilon$ for each $x' \in V_{x^*}$; and (b) for each $x' \in V_{x^*}$, there is a player i with $H_i(x') < \alpha_i - \varepsilon$. Fix $x' \in V_{x^*}$. Then, there is a player i for whom

$$\varphi_i((\overline{x}_i, x''_{-i}), x'') + H_i(x'') \geq \alpha_i + \varepsilon > \alpha_i - \varepsilon > H_i(x'), \quad \text{for all } x'' \in V_{x^*}.$$

Thus, G is surrogate point secure with respect to H. □

We now extend Theorems 3.4 and Theorem 5.5 in Reny (2016a) to relations admitting an SSYM representation satisfying surrogate point security.

Proposition 2 *Suppose that the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is compact and surrogate point secure* with X_i convex for each i. Then, G possesses a Nash equilibrium.*

Proposition 2 follows from the main existence result, Theorem 1, which is presented in Sect. 3.3.

Corollary 2 (to Proposition 2) *Suppose that the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is compact, quasiconcave, and surrogate point secure. Then, G possesses a Nash equilibrium.*

Adapting the definition of Tian (1992) to the SSYM framework, we have:

Definition 11 A skew-symmetric function $\varphi_i : X \times X \to \mathbb{R}$ is ***0-transfer lower semicontinuous in** X **with respect to** X_i* if for each $(x_i, z) \in X_i \times X$ satisfying $\varphi_i((x_i, z_{-i}), z) > 0$ there exist $\overline{x}_i \in X_i$ and an open set U containing x such that

$$\varphi_i((\overline{x}_i, x'_{-i}), x') > 0, \quad \text{for all } x' \in U.$$

Generalizing the definition of Prokopovych (2013, Example 1) to the skew-symmetric framework, we have:

Definition 12 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ satisfies the ***single player deviation property*** if whenever x^* is not a Nash equilibrium of G, there exist $\overline{x} \in X$ and an open set U containing x^* and an i such that

$$\varphi_i((\overline{x}_i, x'_{-i}), x') > 0, \quad \text{for all } x' \in U.$$

If $z \mapsto \varphi_i((x_i, z_{-i}), z)$ is lower semicontinuous for each $x_i \in X_i$, then φ_i is 0-transfer lower semicontinuous in X with respect to X_i. If $G = (X_i, \varphi_i)_{i=1}^N$ is an SSYM game and if each φ_i is 0-transfer lower semicontinuous in X with respect to X_i, then G satisfies the single player deviation property. Finally, note that G satisfies the single player deviation property if and only if G is surrogate point secure with respect to surrogate function H where $H_i \equiv 0$ for each i. Consequently, we have the following generalization of our Corollary 1, the existence result presented in Example 1 of Prokopovych (2013), and (consequently) Corollary 3.1 in Nessah (2011).

Corollary 3 (to Proposition 2) *Suppose that the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is compact and quasiconcave. If $G = (X_i, \varphi_i)_{i=1}^N$ satisfies the single player deviation property, then G possesses a Nash equilibrium.*

We conclude this subsection with an example illustrating the existence of quasiconcave, surrogate point secure SF games that fails to satisfy point security and surrogate point security with $H \equiv 0$ and also fails to satisfy point security and surrogate point security with $H \equiv u$.

Example 2 Let $G = (X_i, u_i)_{i=1}^2$. Here, for each i, $X_i := [0, 2]$, and for $x = (x_1, x_2) \in [0, 1]^2$, the payoff u_i is defined as follows:

$$u_i(x_1, x_2) := \begin{cases} 1 - x_i & \text{if } x_i > x_{-i}, \\ 0 & \text{if } x_i \leq x_{-i}. \end{cases}$$

For $x \in [1, 2]^2 \setminus \{(1, 1)\}$, define $u_2 \equiv 0$ and

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } 1 \leq x_1 < 2 \text{ and } x_2 = 2, \\ 0 & \text{if } 1 \leq x_1 < 2 \text{ and } 1 \leq x_2 < 2, \\ 2 & \text{if } x_1 = 2 \text{ and } x_2 = 2, \\ x_2 - 1 & \text{if } x_1 = 2 \text{ and } 1 \leq x_2 < 2. \end{cases}$$

Everywhere else in $[0, 2] \times [0, 2]$, the payoffs are identically zero.

This example fails to satisfy surrogate point security with $H = u$, i.e., the example fails to satisfy point security. Therefore, this example fails to satisfy surrogate better-reply security with $H = u$, i.e., better-reply security. To see this, note that $x^* = (1, 2)$ is not a Nash equilibrium and that $(u_1(1, 2), u_2(1, 2)) = (1, 0)$. Suppose that V_{x^*} is open in X and $(1, 2) \in V_{x^*}$ and let $y = (1, 2)$. For each $\overline{x}_1 \in [0, 2]$, there exists a $z \in V_{x^*}$ with $z_2 < 2$ such that $u_1(\overline{x}_1, 2) \in \{0, 1\}$. For each $\overline{x}_2 \in [0, 2]$, let $z = (1, 2)$ and note that $(1, 2) \in V_{x^*}$ and $u_2(1, \overline{x}_2) = 0$.

This example fails to satisfy surrogate point security with $H \equiv 0$. Therefore, this example fails to satisfy surrogate better-reply security with $H \equiv 0$. To see this, choose $x = (0, 0)$ and an open set V containing x. Note that x is not an equilibrium and choose $\overline{x}_1 \in [0, 2]$. We must show that there exists a $z \in V$ such that $u_1(x_1, z_2) - u_1(z_1, z_2) \leq 0$.

If $\overline{x}_1 = 0$, choose $z_2 = 0$ and z_1 so that $(z_1, 0) \in V$ and $0 < z_1 < 1$. Then

$$u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) = u_1(0, 0) - u_1(z_1, 0) = 0 - (1 - z_1) < 0.$$

If $0 < \overline{x}_1 < 1$, choose $z_2 = 0$ and z_1 so that $(z_1, 0) \in V$ and $0 < z_1 < \overline{x}_1$. Then

$$u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) = u_1(\overline{x}_1, 0) - u_1(z_1, 0) = (1 - \overline{x}_1) - (1 - z_1) < 0.$$

If $1 \leq \overline{x}_1 \leq 2$, choose $z_2 = 0$ and z_1 so that $(z_1, 0) \in V$ and $0 < z_1 < 1$. Then

$$u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) = u_1(\overline{x}_1, 0) - u_1(z_1, 0) = 0 - (1 - z_1) < 0.$$

A completely symmetric argument for player 2 establishes the claim.

Finally, in Sect. A.3 of the Appendix, we show that this example satisfies surrogate better-reply security for H defined as $H(x) := u(x)$ for all $x \in [0, 1]^2$ and $H(x) := 0$ elsewhere.

3.3 Surrogate correspondence security

For two subsets A and B of topological vector spaces, we call a correspondence $F : A \rightrightarrows B$ *co-closed* if the correspondence $x \mapsto \text{co}(F(x))$ has closed graph in the relative topology on $A \times B$.

Definition 13 (Reny 2016a) An SF game $G = (X_i, u_i)_{i=1}^N$ is ***correspondence secure*** if whenever x^* is not a Nash equilibrium of G, there exist an open set U containing x^* and a closed correspondence $d : U \rightrightarrows X$ with nonempty convex values such that for each $y \in U$ there is a player i such that

$$u_i(z_i, x'_{-i}) > u_i(y), \quad \text{for all } x' \in U \text{ and } z_i \in d_i(x').$$

Definition 14 (Reny 2016a) An SF game $G = (X_i, u_i)_{i=1}^N$ is ***correspondence secure****** if whenever x^* is not a Nash equilibrium of G, there exist an open set U containing x^* and a co-closed correspondence $d : U \rightrightarrows X$ with nonempty values such that for each $y \in U$ there is a player i such that

$$y_i \notin \text{co}\{w_i \in X_i : u_i(z_i, x'_{-i}) \leq u_i(w_i, y_{-i})\}, \text{ for all } x' \in U \text{ and } z_i \in d_i(x').$$

The following are the surrogate generalizations of Definitions 13 and 14.

Definition 15 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate correspondence secure*** if there exists a function $H : X \to \mathbb{R}^N$ such that, whenever x^* is not a Nash equilibrium of G, there exist an open set U containing x^* and a closed correspondence $d : U \rightrightarrows X$ with nonempty convex values such that for each $y \in U$ there is a player i such that

$$\varphi_i((z_i, x'_{-i}), x') + H_i(x') > H_i(y), \quad \text{for all } x' \in U \text{ and } z_i \in d_i(x').$$

Definition 16 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is ***surrogate correspondence secure****** if there exists a function $H : X \to \mathbb{R}^N$ such that, whenever x^* is not a Nash equilibrium of G, there exist an open set U containing x^* and a co-closed correspondence $d : U \rightrightarrows X$ with nonempty values such that for each $y \in U$ there is a player i such that

$$y_i \notin \text{co}\{w_i \in X_i : \varphi_i((z_i, x'_{-i}), x') + H_i(x') \leq \varphi_i((w_i, y_{-i}), y) + H_i(y)\}$$

for all $x' \in U$ and for all $z_i \in d_i(x')$.

Generalizing Definition 3.2 in Nessah (2011), we also have:

Definition 17 An SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ is said to be ***generalized weakly transfer continuous*** if whenever $x^* \in X$ is not a Nash equilibrium of G, there exist an open set U containing x^*, a closed correspondence $d : U \rightrightarrows X$ with nonempty convex values and a player i such that

$$\inf_{(x', z_i) \in \Gamma_{d_i}} \varphi_i((z_i, x'_{-i}), x') > 0$$

where Γ_{d_i} denotes the graph of the correspondence $d_i : U \rightrightarrows X_i$.

Note that if $G = (X_i, \varphi_i)_{i=1}^N$ is generalized weakly transfer continuous, then G is surrogate correspondence secure with respect to surrogate function $H : X \to \mathbb{R}^N$, where $H_i \equiv 0$ for each i.

We now present our main existence result for pure-strategy equilibria and then discuss the proof technique and the relationship between Theorem 1, Theorem 5.6 in Reny (2016a), and Theorem 4 in Nessah and Tian (2016). Note that Theorem 1 extends Theorem 5.6 in Reny (2016a) to reflexive, complete relations that admit a skew-symmetric representation satisfying surrogate correspondence security*. In addition, Theorem 1 provides an alternative route to Theorem 3.1 of Nessah (2011) using our surrogate security concept rather than the qualitative games approach of Theorem 5 in Prokopovych (2013) (see Example 1 in Prokopovych (2013)).

Theorem 1 *Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is a compact SSYM game with X_i convex for each i. If G satisfies surrogate correspondence security*, then G possesses a Nash equilibrium.*

There are two ways to prove Theorem 1. In the first, we could use the surrogate function to construct a surrogate game based on that found in the proof of Theorem 5.6 in Reny (2016a), and then use the argument in that proof to establish existence. Alternatively, we can prove Theorem 1 as an application of Theorem 4 of Nessah and Tian (2016) by using the surrogate function to construct an "inherited correspondence" as defined in that paper. For the sake of completeness, however, we will provide a self-contained proof that adapts the argument of Theorem 4 of Nessah and Tian (2016) directly to our framework. This proof highlights the role of the surrogate function in defining a surrogate game whose evaluations satisfy reflexivity, completeness and transitivity and to which Reny's (2016a) Theorem 5.6 can be applied. The reader can find the proof of Theorem 1 in Sect. A.1 of the Appendix.

Remark 7 The hypothesis of Theorem 1 is sufficient for existence but not necessary. Indeed, the quasiconcave SF game $G = (X_i, u_i)_{i=1}^2$, where $X_1 = X_2 := [0, 1]$, $u_2 \equiv 0$, and

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } (x_1, x_2) = (1, 0), \\ 0 & \text{elsewhere,} \end{cases}$$

can be shown to violate surrogate correspondence security*, and yet the point $(1, 0)$ is a Nash equilibrium.

3.4 Surrogate payoff security and reciprocal upper semicontinuity

Better-reply security for strategic-form games (and its extension to SSYM games) is quite general so it is useful to identify conditions that are stronger than better-reply security but easier to check. In the case of strategic-form games, Reny (1999) identifies two conditions whose conjunction implies better-reply security. We next provide surrogate generalizations of these conditions that can be applied in the SSYM framework.

Definition 18 A function $H : X \to \mathbb{R}^N$ satisfies ***reciprocal upper semicontinuity*** if $(x, \eta) \in \overline{\Gamma}_H$ and $H(x) \le \eta$ imply that $H(x) = \eta$.

Remark 8 If $\sum_i H_i : X \to \mathbb{R}^N$ is upper semicontinuous, then H satisfies reciprocal upper semicontinuity.

Definition 19 An SSYM game $(X_i, \varphi_i)_{i=1}^N$ is ***surrogate payoff secure*** if there exists a function $H : X \to \mathbb{R}^N$ such that for each $x \in X$, $y \in X$, $\varepsilon > 0$ and i, there exists an $\overline{x}_i \in X_i$ and an open set V_x with $x \in V_x$ such that

$$\varphi_i((\overline{x}_i, z_{-i}), z) + H_i(z) \geq \varphi_i((y_i, x_{-i}), x) + H_i(x) - \varepsilon, \quad \text{for all } z \in V_x.$$

Proposition 3 *Suppose that $H : X \to \mathbb{R}^N$ is bounded and satisfies reciprocal upper semicontinuity. If $G = (X_i, \varphi_i)_{i=1}^N$ is surrogate payoff secure with surrogate function H, then G is surrogate better-reply secure with surrogate function H.*

Proof Suppose that $(x^*, \alpha^*) \in \overline{\Gamma}_H$ and x^* is not an equilibrium. Then, reciprocal upper semicontinuity implies that $H(x^*) = \alpha^*$ or $H_i(x^*) > \alpha_i^*$ for some i. Suppose that $H(x^*) = \alpha^*$. Since x^* is not an equilibrium, there exists an i and $y_i \in X_i$ such that

$$\varphi_i((y_i, x_{-i}^*), x^*) > 0.$$

Choose $\varepsilon > 0$ so that $\varphi_i((y_i, x_{-i}^*), x^*) - \varepsilon = \gamma > 0$. Applying surrogate payoff security, there exists an $\overline{x}_i \in X_i$ and an open set V_{x^*} with $x^* \in V_{x^*}$ such that

$$\varphi_i((\overline{x}_i, z_{-i}), z) + H_i(z) \geq \varphi_i((y_i, x_{-i}^*), x^*) + H_i(x^*) - \varepsilon = \alpha_i^* + \gamma$$

for all $z \in V_{x^*}$. Suppose that $H_i(x^*) > \alpha_i^*$ for some i. Choose $\varepsilon > 0$ and $\gamma > 0$ so that $H_i(x^*) - \varepsilon = \alpha_i^* + \gamma$. Then applying surrogate payoff security again, there exists an $\overline{x}_i \in X_i$ and an open set V_{x^*} with $x^* \in V_{x^*}$ such that

$$\varphi_i((\overline{x}_i, z_{-i}), z) + H_i(z) \geq \varphi_i((y_i, x_{-i}^*), x^*) + H_i(x^*) - \varepsilon > H_i(x^*) - \varepsilon = \alpha_i^* + \gamma$$

for all $z \in V_{x^*}$. □

4 Games of incomplete information: behavioral strategy equilibrium

We borrow notation and terminology from Carbonell-Nicolau and McLean (2018) wherever possible. If S is a compact metric space, then $\mathcal{B}(S)$ denotes the σ-algebra of Borel subsets of S, and $\Delta(S)$ represents the set of Borel probability measures on S. In addition, $C(S)$ denotes the set of all real-valued continuous maps on S.

4.1 Bayesian games

We begin with a general formulation of a Bayesian game. We then introduce the notion of a Bayesian game with a skew-symmetric (resp. utility) representation.

Definition 20 A ***Bayesian game*** is a collection

$$\Gamma = \left((T_i, \mathcal{T}_i), X_i, \succsim_i, p\right)_{i=1}^N,$$

where

- $\{1, \dots, N\}$ is a finite set of players;
- $(T_i, \mathcal{T}_i)$ is a measurable space, where T_i is player i 's nonempty type space;[4]
- X_i is player i's action space, a nonempty compact metric space;
- $\succsim_i$ is a binary relation on the set of Borel probability measures on $T \times X$, $\Delta(T \times X)$, where $T := \times_{i=1}^N T_i$ and $X := \times_{i=1}^N X_i$; and
- p is a probability measure on $(T, \otimes_{i=1}^N \mathcal{T}_i)$, denoting the common prior over type profiles.

We denote by p_i the marginal probability measure induced by p on T_i, i.e., p_i is a probability measure on $(T_i, \mathcal{T}_i)$ defined by

$$p_i(S) := p(S \times T_{-i}).$$

For each $(T_i, \mathcal{T}_i)$ and X_i, $\mathcal{C}(T_i, X_i)$ will denote the space of integrably bounded Carathéodory integrands on $T_i \times X_i$, i.e., the functions $f : T_i \times X_i \to \mathbb{R}$ that are integrably bounded and $(\mathcal{T}_i \otimes \mathcal{B}(X_i), \mathcal{B}(\mathbb{R}))$-measurable with $f(t_i, \cdot) \in C(X_i)$ for each $t_i \in T_i$.[5]

The product σ-algebra $\otimes_{i=1}^N \mathcal{T}_i$ will be denoted by $\mathcal{T}$, and $\Delta(T, \mathcal{T})$ will represent the set of probability measures on the measurable space $(T, \mathcal{T})$.

4.2 Strategies

Definition 21 Let $\Gamma = \big((T_i, \mathcal{T}_i), X_i, \succsim_i, p\big)_{i=1}^N$ be a Bayesian game. A ***pure strategy*** for a player i in Γ is a $(\mathcal{T}_i, \mathcal{B}(X_i))$-measurable map $s_i : T_i \to X_i$ with the interpretation that, upon learning her type $t_i \in T_i$, a player i selects the action $s_i(t_i)$ from the set X_i.

Let $\mathcal{P}_i$ denote the set of pure strategies for player i, and set $\mathcal{P} := \times_{i=1}^N \mathcal{P}_i$.

Definition 22 Let $\Gamma = \big((T_i, \mathcal{T}_i), X_i, \succsim_i, p\big)_{i=1}^N$ be a Bayesian game. A ***behavioral strategy*** for player i in Γ is a *transition probability* with respect to $(T_i, \mathcal{T}_i)$ and $(X_i, \mathcal{B}(X_i))$, i.e., a mapping

$$\sigma_i : \mathcal{B}(X_i) \times T_i \to [0, 1],$$

where $\sigma_i(\cdot|t_i) \in \Delta(X_i)$ for each $t_i \in T_i$ and $\sigma_i(A|\cdot) : T_i \to \mathbb{R}$ is a $(\mathcal{T}_i, \mathcal{B}(\mathbb{R}))$-measurable function for each $A \in \mathcal{B}(X_i)$.

The set of behavioral strategies for player i will be denoted by $\mathcal{Y}_i$, and $\mathcal{Y}$ will represent the Cartesian product $\times_{i=1}^N \mathcal{Y}_i$.

A Bayesian game $\Gamma = \big((T_i, \mathcal{T}_i), X_i, \succsim_i, p\big)_{i=1}^N$ admits ***a skew-symmetric (SSYM) representation*** if, for each i, there exists a bounded and

[4] Observe that no topological structure is imposed on T_i.

[5] An $(\mathcal{T}_i \otimes \mathcal{B}(X_i), \mathcal{B}(\mathbb{R}))$-measurable function $f : T_i \times X_i \to \mathbb{R}$ is integrably bounded if there exists a p_i-integrable function φ satisfying $|f(t_i, x_i)| \le \varphi(t_i)$ for all $(t_i, x_i) \in T_i \times X_i$.

$\left([\otimes_{i=1}^N \mathcal{T}_i] \otimes [\otimes_{i=1}^N \mathcal{B}(X_i)] \otimes [\otimes_{i=1}^N \mathcal{B}(X_i)], \mathcal{B}(\mathbb{R})\right)$ -measurable map $\psi_i : T \times X \times X \to \mathbb{R}$ satisfying

$$\psi_i(t, x, y) = -\psi_i(t, y, x), \quad \text{for all } (t, x, y) \in T \times X \times X,$$

such that the map $\boldsymbol{\psi}_i : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ defined by

$$\boldsymbol{\psi}_i(\sigma, \mu) := \int_T \int_{X_N} \cdots \int_{X_1} \int_{X_N} \cdots \int_{X_1} \psi_i(t, x, y) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \mu_j(dy_j|t_j)\right] p(\mathrm{d}t) \tag{1}$$

satisfies the following:

$$[\otimes_j \sigma_j] * p \succsim_i [\otimes_j \mu_j] * p \Leftrightarrow \boldsymbol{\psi}_i(\sigma, \mu) \geq 0, \quad \text{for all } (\sigma, \mu) \in \mathcal{Y} \times \mathcal{Y},$$

where, given $\nu \in \mathcal{Y}$, $[\otimes_j \nu_j] * p$ denotes the probability measure in $\Delta(T \times X)$ defined by

$$([\otimes_j \nu_j] * p)(A \times B_1 \times \cdots \times B_N) := \int_A \left[\prod_{j=1}^N \nu_j(B_j|t_j)\right] p(\mathrm{d}t).$$

In this case, we write $\Gamma = ((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ and we call Γ an ***SSYM Bayesian game***.

The game Γ has a ***utility representation*** if there exists, for each i, a payoff function $u_i : T \times X \to \mathbb{R}$ such that $\psi_i(t, x, y) = u_i(t, x) - u_i(t, y)$ for each $(x, y) \in X \times X$. In this case we write $\Gamma = ((T_i, \mathcal{T}_i), X_i, u_i, p)_{i=1}^N$ and we call Γ a ***Bayesian game with a utility representation***.

For every pure-strategy $s_i \in \mathcal{P}_i$, there is a corresponding "pure" behavioral strategy $\sigma_i^{s_i} \in \mathcal{Y}_i$ defined by

$$\sigma_i^{s_i}(A|t_i) := \delta_{s_i(t_i)}(A),$$

where $\delta_{s_i(t_i)} \in \Delta(X_i)$ denotes the Dirac measure with mass point $s_i(t_i)$.

For $s_i \in \mathcal{P}_i$ and $\sigma_{-i} \in \mathcal{Y}_{-i}$, define

$$\boldsymbol{\psi}_i(s_i, \sigma_{-i}) := \boldsymbol{\psi}_i(\sigma_i^{s_i}, \sigma_{-i}).$$

To define the topology for the sets $\mathcal{Y}_i$, let $\widehat{\mathcal{L}}_i$ be the space of uniformly finite transition measures with respect to $(T_i, \mathcal{T}_i)$ and $(X_i, \mathcal{B}(X_i))$. Recall that $\mathcal{C}(T_i, X_i)$ denotes the space of integrably bounded Carathé odory integrands on $T_i \times X_i$.

Definition 23 The ***narrow topology*** on $\widehat{\mathcal{L}}_i$ is the weakest topology with respect to which all functionals in the set

$$\left\{\zeta_f : f \in \mathcal{C}(T_i, X_i)\right\}$$

are continuous, where $\zeta_f : \widehat{\mathcal{L}_i} \to \mathbb{R}$ is defined for each $f \in \mathcal{C}(T_i, X_i)$ as

$$\zeta_f(\mu) := \int_{T_i} \int_{X_i} f(t_i, x_i)\mu(dx_i|t_i)p_i(\mathrm{d}t_i).$$

We view $\mathcal{Y}_i$ as a subspace of $\widehat{\mathcal{L}_i}$ endowed with its relative topology, and the Cartesian product $\mathcal{Y}$ is endowed with the corresponding product topology. The following lemma is a consequence of Theorem 2.3 in Balder (1988).

Lemma 2 *$\mathcal{Y}_i$ is a compact, convex subspace of the topological vector space $\widehat{\mathcal{L}_i}$.*

4.3 Equilibrium existence

Definition 24 A ***Bayes–Nash*** equilibrium of a Bayesian game $\Gamma = ((T_i, \mathcal{T}_i), X_i, \succsim_i, p)_{i=1}^N$ is a profile $(\sigma_1, \ldots, \sigma_N) \in \mathcal{Y}$ such that for each i and $\mu_i \in \mathcal{Y}_i$,

$$[\otimes_j \sigma_j] * p \succsim_i [\mu_i \otimes (\otimes_{j \neq i}\sigma_j)] * p.$$

It is easy to see that a Bayes–Nash equilibrium of an SSYM Bayesian game $\Gamma = ((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ is a Nash equilibrium of the SSYM game G_Γ defined by

$$G_\Gamma := \left(\mathcal{Y}_i, \boldsymbol{\psi}_i\right)_{i=1}^N, \tag{2}$$

where $\boldsymbol{\psi}_i : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$ is given by (1), i.e., a profile $(\sigma_1, \ldots, \sigma_N) \in \mathcal{Y}$ such that for each i,

$$\boldsymbol{\psi}_i((\mu_i, \sigma_{-i}), \sigma) \leq 0, \quad \text{for all } \mu_i \in \mathcal{Y}_i.$$

Carbonell-Nicolau and McLean (2018) introduced a notion of uniform payoff security for games of incomplete information, and the following extensions are proposed here.

Definition 25 An SSYM Bayesian game $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ is ***uniformly surrogate payoff secure*** if there exists a bounded function $H : T \times X \to \mathbb{R}^N$ such that for each i, $\varepsilon > 0$, and $s_i \in \mathcal{P}_i$, there exists $s_i^* \in \mathcal{P}_i$ such that for all $(t, x, y) \in T \times X \times X$, there exist neighborhoods V_x and V_y of x and y, respectively, such that

$$\psi_i(t, (s_i^*(t_i), x'_{-i}), y') + H_i(t, y') > \psi_i(t, (s_i(t_i), x_{-i}), y) + H_i(t, y) - \varepsilon,$$
$$\text{for all } (x', y') \in V_x \times V_y.$$

Remark 9 Suppose that $\Gamma = ((T_i, \mathcal{T}_i), X_i, u_i, p)_{i=1}^N$ is a Bayesian game with a utility representation. If Γ is uniformly payoff secure in the sense of Carbonell-Nicolau and McLean (2018, Definition 9), then Γ satisfies both uniform surrogate payoff security and weak uniform surrogate payoff security for the surrogate function $H = u$.

For SSYM Bayesian games, our existence results are presented in terms of Definition 25. In the special case of Bayesian games, we can prove stronger results in terms of a weaker notion (Definition 26 below).

Fix an SSYM Bayesian game $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ and suppose that for each i, $H_i : T \times X \to \mathbb{R}$ is bounded and $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$ -measurable. Then one may define $\boldsymbol{H}_i : \mathcal{Y} \to \mathbb{R}$ as follows:

$$\boldsymbol{H}_i(\sigma) := \int_T \int_{X_N} \cdots \int_{X_1} H_i(t, x) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] p(\mathrm{d}t).$$

The following two results are instrumental in proving our first main existence result. Surrogate payoff security for surrogate function $\boldsymbol{H}$ (in Lemma 3) and upper semicontinuity of the map $\sum_{i=1}^N \boldsymbol{H}_i(\cdot)$ (in Lemma 4) are defined with respect to the narrow topology (see Definition 23). The proof of Lemma 3 is relegated to Sect. A.2. Lemma 4 is an immediate consequence of Lemma 3 in Carbonell-Nicolau and McLean (2018).

Lemma 3 *Suppose that the SSYM Bayesian game $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ satisfies uniform surrogate payoff security with surrogate function H. If $H_i : T \times X \to \mathbb{R}$ is bounded and $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$-measurable for each i, and if p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$, then the game G_Γ defined in (2) is surrogate payoff secure with surrogate function $\boldsymbol{H}$.*

Lemma 4 *Given an SSYM Bayesian game $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$, suppose that for each $t \in T$, the map $\sum_{i=1}^N H_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous. Suppose further that p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$. Then the map $\sum_{i=1}^N \boldsymbol{H}_i(\cdot) : \mathcal{Y} \to \mathbb{R}$ is upper semicontinuous.*

Theorem 2 *Suppose that the SSYM Bayesian game $\Gamma = ((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ satisfies uniform surrogate payoff security with surrogate function H. Suppose that $H_i : T \times X \to \mathbb{R}$ is bounded and $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$-measurable for each i. Suppose further that for each $t \in T$, the map $\sum_{i=1}^N H_i(t, \cdot) : X \to \mathbb{R}$ is upper semicontinuous. If p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$, then Γ possesses a Bayes–Nash equilibrium.*

Proof For each i, $\mathcal{Y}_i$ is a compact, convex subspace of a topological vector space (Lemma 2), and for each $\sigma \in \mathcal{Y}$, the map $\boldsymbol{\psi}_i((\cdot, \sigma_{-i}), \sigma) : \mathcal{Y}_i \to \mathbb{R}$ is quasiconcave. Hence, because the map $\sum_{i=1}^N \boldsymbol{H}_i(\cdot) : \mathcal{Y} \to \mathbb{R}$ is upper semicontinuous (Lemma 4), and the game G_Γ defined in (2) is surrogate payoff secure for $\boldsymbol{H}$ (Lemma 3), it follows from Proposition 3 and Remark 8 that Γ possesses a Bayes–Nash equilibrium. □

The analysis for SSYM Bayesian games with a utility representation is analogous to that for general SSYM games. In this case, we can prove a stronger existence result in terms of the following definition.

Definition 26 A Bayesian game with a utility representation $((T_i, \mathcal{T}_i), X_i, u_i, p)_{i=1}^N$ satisfies ***weak uniform surrogate payoff security*** if there exists a bounded function

$H : T \times X \to \mathbb{R}^N$ such that for each i, $\varepsilon > 0$, and $s_i \in \mathcal{P}_i$, there exists $s_i^* \in \mathcal{P}_i$ such that for all $(t, x) \in T \times X$, there exists a neighborhood V_x of x such that

$$u_i(t, (s_i^*(t_i), z_{-i})) - u_i(t, z) + H_i(t, z) \\ > u_i(t, (s_i(t_i), x_{-i})) - u_i(t, x) + H_i(t, x) - \varepsilon, \quad \text{for all } z \in V_x.$$

The following lemma is the analog of Lemma 3 for Bayesian games.

Lemma 5 *Suppose that the Bayesian game with a utility representation* $((T_i, \mathcal{T}_i), X_i, u_i, p)_{i=1}^N$ *satisfies weak uniform surrogate payoff security with surrogate function H. If* $H_i : T \times X \to \mathbb{R}$ *is bounded and* $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$*-measurable for each i and if p is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$ *, then the game* G_Γ *defined in (2) is surrogate payoff secure with surrogate function* $\boldsymbol{H}$.

Theorem 3 *Suppose that the Bayesian game with a utility representation* $\Gamma = ((T_i, \mathcal{T}_i), X_i, u_i, p)_{i=1}^N$ *satisfies weak uniform surrogate payoff security with surrogate function H . Suppose that* $H_i : T \times X \to \mathbb{R}$ *is bounded and* $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)],\right.$ $\left.\mathcal{B}(\mathbb{R})\right)$*-measurable for each i . Suppose further that for each* $t \in T$*, the map* $\sum_{i=1}^N H_i(t, \cdot) : X \to \mathbb{R}$ *is upper semicontinuous. If p is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$*, then* Γ *possesses a Bayes–Nash equilibrium.*

Proof For each i, $\mathcal{Y}_i$ is a compact, convex subspace of a topological vector space (Lemma 2), and for each $\sigma \in \mathcal{Y}$, the map $\boldsymbol{\psi}_i((\cdot, \sigma_{-i}), \sigma) : \mathcal{Y}_i \to \mathbb{R}$ is quasiconcave. Hence, because the map $\sum_{i=1}^N \boldsymbol{H}_i(\cdot) : \mathcal{Y} \to \mathbb{R}$ is upper semicontinuous (Lemma 4), and the game G_Γ defined in (2) is surrogate payoff secure for $\boldsymbol{H}$ (Lemma 5), it follows from Proposition 3 and Remark 8 that Γ possesses a Bayes–Nash equilibrium. □

4.4 Example

Intransitivities arise naturally in games when the players' actions reflect the preferences of a group of individuals. We have in mind strategic interactions among groups that take collective actions. In this section, we provide a simple example illustrating this idea as an immediate application of Theorem 2.

There are two groups of individuals (or organizations), A and B, which are viewed as "the players." Group $i \in \{A, B\}$ has n_i members. Each group $i \in \{A, B\}$ observes a private signal t_i from an arbitrary measurable type space $(T_i, \mathcal{T}_i)$ and makes a collective choice from an action space X_i. The groups' actions are taken simultaneously. To keep matters simple, assume that each X_i is finite. The preferences of an individual $k \in \{1, \ldots, n_i\}$ of group $i \in \{A, B\}$ over profiles of types and actions in $T \times X$, where $T := T_A \times T_B$ and $X := X_A \times X_B$, are represented by a $\left([\otimes_{i \in \{A,B\}} \mathcal{T}_i] \otimes [\otimes_{i \in \{A,B\}} \mathcal{B}(X_i)], \mathcal{B}(\mathbb{R})\right)$ -measurable utility function $u_{(i,k)} : T \times X \to \mathbb{R}$.

The game played by the two players A and B is

$$\Gamma = \left((T_i, \mathcal{T}_i), X_i, \succsim_i, p\right)_{i \in \{A,B\}},$$

where, for each $i \in \{A, B\}$ and $(\sigma, \mu) \in \mathcal{Y} \times \mathcal{Y}$,

$$[\otimes_j \sigma_j] * p \precsim_i [\otimes_j \mu_j] * p \Leftrightarrow \boldsymbol{\psi}_i(\sigma, \mu) \geq 0,$$

where $\boldsymbol{\psi}_i$ is defined as in (1) and where $\psi_i : T \times X \times X \to \mathbb{R}$ is defined by

$$\psi_i(t, x, y) := \#\left\{k : u_{(i,k)}(t, x) > u_{(i,k)}(t, y)\right\} - \#\left\{k : u_{(i,k)}(t, x) < u_{(i,k)}(t, y)\right\}.$$

Note that, according to these preferences, the net plurality within group i (defined as the difference between the number of group members preferring one action profile over another) determines the group's aggregate preferences over action profiles. It is well known that majority voting yields—except in very special cases—an intransitive aggregate preference relation. Consequently, the game defined above features intransitivities that cannot be handled by the extant literature on the existence of equilibrium in Bayesian games.

The game G admits a skew-symmetric representation. Indeed, for each $i \in \{A, B\}$ and $(t, x, y) \in T \times X \times X$, $\psi_i(t, x, y) = -\psi_i(t, y, x)$. Thus, $\Gamma = ((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i \in \{A,B\}}$ is the associated SSYM Bayesian game. An immediate implication of Theorem 2 is that Γ possesses a Bayes–Nash equilibrium.

4.5 Mixed-strategy equilibria in complete information games

Recall that if S is a compact metric space, then $\mathcal{B}(S)$ denotes the σ-algebra of Borel subsets of S, and $\Delta(S)$ represents the set of Borel probability measures on S.

Definition 27 Given an SSYM, compact, metric game $G = (X_i, \varphi_i)_{i=1}^N$, the ***mixed extension*** of G is the SSYM game $(\Delta(X_i), \boldsymbol{\varphi}_i)_{i=1}^N$, where

$$\boldsymbol{\varphi}_i : \left[\times_{j=1}^N \Delta(X_j)\right] \times \left[\times_{j=1}^N \Delta(X_j)\right] \to \mathbb{R}$$

is defined by

$$\boldsymbol{\varphi}_i(\mu, \nu) = \int_{X_N} \cdots \int_{X_1} \int_{X_N} \cdots \int_{X_1} \varphi_i(x, y) \mu_1(dx_1) \cdots \mu_N(dx_N) \nu_1(dy_1) \cdots \nu_N(dy_N).$$

Definition 28 below specializes Definition 25 to the case of complete information SSYM games and Theorem 4 below follows immediately from Theorem 2.

Definition 28 An SSYM game $(X_i, \varphi_i)_{i=1}^N$ satisfies ***uniform surrogate payoff security*** if there exists a bounded function $H : X \to \mathbb{R}^N$ such that for every i, $\eta_i \in X_i$, and $\varepsilon > 0$, there exists $\overline{\eta}_i \in X_i$ such that for all $(x, y) \in X \times X$ there exist neighborhoods V_x and V_y of x and y, respectively, such that

$$\varphi_i((\overline{\eta}_i, x'_{-i}), y') + H_i(y') > \varphi_i((\eta_i, x_{-i}), y) + H_i(y) - \varepsilon, \quad \text{for all } (x', y') \in V_x \times V_y.$$

Theorem 4 *Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is a compact, metric SSYM game with φ_i bounded and $(\mathcal{B}(X), \mathcal{B}(\mathbb{R}))$-measurable for each i and suppose that G satisfies uniform surrogate payoff security with surrogate function H. If $H : X \to \mathbb{R}^N$ is bounded and $(\mathcal{B}(X), \mathcal{B}(\mathbb{R}^N))$-measurable, and if $\sum_{i=1}^N H_i : X \to \mathbb{R}$ is upper semicontinuous, then the mixed extension $(\Delta(X_i), \boldsymbol{\varphi}_i)_{i=1}^N$ has a Nash equilibrium.*

Definition 29 below specializes Definition 26 to case of complete information strategic-form games and Theorem 5 is an immediate consequence of Theorem 3.

Definition 29 An SF game $(X_i, u_i)_{i=1}^N$ satisfies ***weak uniform surrogate payoff security*** if there exists a bounded function $H : X \to \mathbb{R}^N$ such that for every i, $\eta_i \in X_i$, and $\varepsilon > 0$, there exists $\overline{\eta}_i \in X_i$ such that for all $x \in X$ there exists a neighborhood V_x of x such that

$$u_i(\overline{\eta}_i, z_{-i}) - u_i(z) + H_i(z) > u_i(\eta_i, x_{-i}) - u_i(x) + H_i(x) - \varepsilon, \quad \text{for all } z \in V_x.$$

Theorem 5 *Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact, metric SF game with u_i bounded and $(\mathcal{B}(X), \mathcal{B}(\mathbb{R}))$-measurable for each i and suppose that G satisfies weak uniform surrogate payoff security with surrogate function $H : X \to \mathbb{R}^N$. If H is $(\mathcal{B}(X), \mathcal{B}(\mathbb{R}^N))$-measurable, and if $\sum_{i=1}^N H_i : X \to \mathbb{R}$ is upper semicontinuous, then the mixed extension $(\Delta(X_i), \boldsymbol{u}_i)_{i=1}^N$ has a Nash equilibrium.*

Remark 10 If $G = (X_i, u_i)_{i=1}^N$ is an SF game satisfying uniform surrogate payoff security (Definition 25) with surrogate function H, then G satisfies weak uniform surrogate payoff security (Definition 29) with surrogate function H. To see this, fix i, $\eta_i \in X_i$, and $\varepsilon > 0$. Then, there exists $\overline{\eta}_i \in X_i$ such that for all $x \in X$ there exist neighborhoods V'_x and V''_x of x such that

$$u_i(\overline{\eta}_i, x'_{-i}) - u_i(y'_i) + H_i(y') > u_i(\eta_i, x_{-i}) - u_i(x) + H_i(x) - \varepsilon,$$
$$\text{for all } (x', y') \in V'_x \times V''_x.$$

Defining $V_x = V'_x \cap V''_x$, it follows that, for each $z \in V_x$,

$$u_i(\overline{\eta}_i, z_{-i}) - u_i(z) + H_i(z) > u_i(\eta_i, x_{-i}) - u_i(x) + H_i(x) - \varepsilon.$$

Example 3 There are SF games that violate Monteiro and Page's (2007) uniform payoff security but satisfy weak uniform surrogate payoff security. Indeed, consider the two-player game from Example 1, $G = ([0, 1], [0, 1], u_1, u_2)$, where $u_2 \equiv 0$ and

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } x_1 \in [0, 1) \text{ and } x_2 = 0, \\ 0 & \text{if } x_1 \in [0, 1) \text{ and } x_2 \neq 0, \\ 2 & \text{if } x_1 = 1 \text{ and } x_2 = 0, \\ 1 & \text{if } x_1 = 1 \text{ and } x_2 \neq 0. \end{cases}$$

This game fails Reny's (2016a) point security (Remark 4). Since uniform payoff security implies point security, it follows that G also fails uniform payoff security.[6]

To see that G satisfies weak uniform surrogate payoff security with surrogate function $H \equiv 0$, choose $y_1 \in [0, 1]$, $\varepsilon > 0$, and $x \in [0, 1]^2$. If $x_2 > 0$ and $x_1 < 1$ then for $z \in [0, 1]^2$ with $z_2 > 0$ and $z_1 < 1$ we have

$$u_1(1, z_2) - u_1(z) = 1 \geq 1 - \varepsilon \geq u_1(y_1, x_2) - u_1(x) - \varepsilon.$$

If $x_2 = 0$ and $x_1 < 1$ then for $z \in [0, 1]^2$ with $z_1 < 1$ we have

$$u_1(1, z_2) - u_1(z) = 1 \geq 1 - \varepsilon \geq u_1(y_1, x_2) - u_1(x) - \varepsilon.$$

If $x_2 = 0$ and $x_1 = 1$ then for $z \in [0, 1]^2$ we have

$$u_1(1, z_2) - u_1(z) \geq 0 \geq 0 - \varepsilon \geq u_1(y_1, x_2) - u_1(x) - \varepsilon.$$

If $x_2 > 0$ and $x_1 = 1$ then for $z \in [0, 1]^2$ with $z_2 > 0$ we have

$$u_1(1, z_2) - u_1(z) = 0 \geq 0 - \varepsilon \geq u_1(y_1, x_2) - u_1(x) - \varepsilon.$$

A Appendix

A.1 Proof of Theorem 1

Theorem 1 is restated here for the convenience of the reader.

Theorem 1. *Suppose that $G = (X_i, \varphi_i)_{i=1}^N$ is a compact SSYM game with X_i convex for each i. If G satisfies surrogate correspondence security*, then G possesses a Nash equilibrium.*

Proof Suppose that G satisfies surrogate correspondence security* with surrogate function H. Suppose that G has no Nash equilibrium. We will adapt the proof of Theorem 4 in Nessah and Tian (2016) and construct a new game G^* with two players α and β, each with the same strategy set X. We will then show that the game G^* satisfies the hypotheses of Reny's (2016a) Theorem 5.6, implying that G^* has a Nash equilibrium, and that this implies that the game G has a Nash equilibrium. This contradiction establishes the result.

Suppose that G has no equilibrium.

Step 1 Define a new game $G^* = (X, X, \succsim_\alpha, \succsim_\beta)$ with two players, each of whom has strategy set X. The preferences of player α are defined as follows:

$$(\xi, \eta) \succsim_\alpha (x, y) \text{ if and only if } u_\alpha(\eta, \xi) \geq u_\alpha(y, x),$$

[6] It is easily seen that G also fails Allison and Lepore's (2014) disjoint payoff matching. However, as pointed out by a referee, this game does satisfy uniform diagonal security of Prokopovych and Yannelis (2014).

where $u_\alpha : X \times X \to \mathbb{R}$ is defined as

$$u_\alpha(\xi, \eta) := \begin{cases} 1 & \text{if } \xi = \eta, \\ 0 & \text{if } \xi \neq \eta. \end{cases}$$

The preferences of player β are defined as follows:

$$(\xi, \eta) \succsim_\beta (x, y) \text{ if and only if } f_i(\eta_i, \xi) \geq f_i(y_i, x) \text{ for all } i,$$

where $f_i : X_i \times X \to R$ is defined as follows:

$$f_i(\xi_i, \eta) = \varphi_i((\xi_i, \eta_{-i}), \eta) + H_i(\eta) \text{ for each } i \text{ and each } (\xi_i, \eta) \in X_i \times X.$$

Applying Definition 5.4 in Reny (2016a), we claim that the game $G^* = (X, X, \succsim_\alpha, \succsim_\beta)$ is correspondence secure with respect to $I = \{\beta\}$. To see this, suppose that G^* is not correspondence secure with respect to $I = \{\beta\}$. As per the definition of B_I on p. 556 of Reny (2016a), note that

$$B_{\{\beta\}} = \{(x, y) \in X \times X : x = y\}.$$

Then, there exists an $(x_\alpha, x_\beta) \in B_{\{\beta\}}$ such that (x_α, x_β) is not a Nash equilibrium in G^* and the following holds: for every neighborhood U of (x_α, x_β) and every co-closed correspondence $(d_\alpha, d_\beta) : U \to X \times X$ with nonempty values, there exists a $(y_\alpha, y_\beta) \in U \cap B_{\{\beta\}}$ such that, for some $(x'_\alpha, x'_\beta) \in U \cap B_{\{\beta\}}$ and some $z \in d_\beta(x'_\alpha, x'_\beta)$,

$$y_\beta \in \text{co}\{w \in X : (y_\alpha, w) \succsim_\beta (x'_\alpha, z)\}.$$

Given the definition of $B_{\{\beta\}}$, we conclude that there exists $x^* \in X$ such that (x^*, x^*) is not a Nash equilibrium of G^* and the following holds: for every neighborhood U of (x^*, x^*) in $X \times X$ and every co-closed correspondence $(d_\alpha, d_\beta) : U \to X \times X$ with nonempty values, there exists a $y^* \in X$ with $(y^*, y^*) \in U$ such that for some $x' \in X$ satisfying $(x', x') \in U$ and some $z \in d_\beta(x', x')$,

$$y^* \in \text{co}\{w \in X : (y^*, w) \succsim_\beta (x', z)\}.$$

Step 2 Note that x^* is not a Nash equilibrium of G since (x^*, x^*) is not a Nash equilibrium of G^*. Applying surrogate correspondence security* (Definition 16), there exists an open set V containing x^* and a co-closed correspondence $\delta : V \to X$ such that the following holds: for every $y \in V$ there exists a player j such that

$$y_j \notin \text{co}\{w_j \in X_j : f_j(\zeta_j, \xi) \leq f_j(w_j, y)\}$$

whenever $\xi \in V$ and $\zeta_j \in \delta_j(\xi)$. Let $U := V \times V$ and define $(d_\alpha(x, y), d_\beta(x, y)) := (\delta(x), \delta(y))$ for all $(x, y) \in V \times V$. Then, U is an open set in $X \times X$ containing (x^*, x^*) and it is easily verified that $(d_\alpha, d_\beta) : U \to X \times X$ is co-closed with nonempty values. Applying Step 1, there exists $y^* \in V$ such that for some $x' \in V$ and some $z \in d_\beta(x', x')$,

$$y^* \in \text{co}\{w \in X : (y^*, w) \succsim_\beta (x', z)\}. \tag{3}$$

Since $y^* \in V$, $x' \in V$, and $z_i \in \delta_i(x')$ for each i, it follows that there exists a player j such that

$$y^* \notin \text{co}\{w_j \in X_j : f_j(z_j, x') \leq f_j(w_j, y^*)\}. \tag{4}$$

Since

$$\begin{aligned}\text{co}\{w \in X : (y^*, w) \succsim_\beta (x', z)\} &= \text{co}\{w \in X : f_i(w_i, y^*) \geq f_i(z_i, x') \text{ for each } i\} \\ &= \text{co}\left(\bigcap_{i=1}^{N}\{w \in X : f_i(w_i, y^*) \geq f_i(z_i, x')\}\right) \\ &\subseteq \bigcap_{i=1}^{N} \text{co}\{w \in X : f_i(w_i, y^*) \geq f_i(z_i, x')\},\end{aligned}$$

(3) implies that

$$y^* \in \text{co}\{w \in X : f_j(w_j, y^*) \geq f_j(z_j, x')\},$$

contradicting (4). This establishes that the game $G^* = (X, X, \succsim_\alpha, \succsim_\beta)$ is correspondence secure with respect to $I = \{\beta\}$ (according to Definition 5.4 in Reny (2016a)).
Step 3 The game $G^* = (X, X, \succsim_\alpha, \succsim_\beta)$ satisfies the assumptions of Theorem 5.6 in Reny (2016a). Therefore, G^* admits a Nash equilibrium $(\overline{x}, \overline{x}) \in X \times X$, i.e., for each i and for all $y_i \in X_i$,

$$f_i(\overline{x}_i, \overline{x}) \geq f_i(y_i, \overline{x}) \text{ for all } i.$$

Therefore, $\varphi_i((y_i, \overline{x}_{-i}), \overline{x}) \leq 0$ for each i and for all $y_i \in X_i$ implying that $\overline{x}$ is a Nash equilibrium in G. This last contradiction proves the theorem. □

A.2 Proof of Lemma 3

A.2.1 Preliminary lemma

Lemma 6 *Suppose that the SSYM Bayesian game $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ satisfies uniform surrogate payoff security with surrogate function H. Suppose that $H_i : T \times X \to \mathbb{R}$ is bounded and $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$-measurable for each i. If p is absolutely continuous with respect to $p_1 \otimes \cdots \otimes p_N$, then for each i, $\varepsilon > 0$, and $s_i \in \mathcal{P}_i$, then there exists $s_i^* \in \mathcal{P}_i$ such that for every $\sigma \in \mathcal{Y}$, there exists a neighborhood V_σ of σ such that*

$$\boldsymbol{\psi}_i((s_i^*, \nu_{-i}), \nu) + \boldsymbol{H}_i(\nu) > \boldsymbol{\psi}_i((s_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) - \varepsilon, \quad \textit{for all } \nu \in V_\sigma. \tag{5}$$

Proof Fix $i, \varepsilon > 0$, and $s_i \in \mathcal{P}_i$. Let f be a density of p with respect to $p_1 \otimes \cdots \otimes p_N$. To lighten the notation, let $P := \otimes_{j=1}^N p_j$. Let $\mathcal{T}^*(P)$ denote the P-completion of $\mathcal{T}$ and let P^* denote the unique extension of P to $\mathcal{T}^*(P)$. Let

$$\mathcal{T}^* := \bigcap_{P \in \Delta(T,\mathcal{T})} \mathcal{T}^*(P)$$

denote the universal completion of $\mathcal{T}$. Note that $\mathcal{T} \subseteq \mathcal{T}^* \subseteq \mathcal{T}^*(P)$ and, abusing notation slightly, we will use P^* for the restriction of P^* to $\mathcal{T}^*$. Note that if $h : T \to \mathbb{R}$ is a bounded, $(\mathcal{T}, \mathcal{B}(\mathbb{R}))$-measurable map, then h is a bounded $(\mathcal{T}^*, \mathcal{B}(\mathbb{R}))$-measurable map and

$$\int_T h(t)P(\mathrm{d}t) = \int_T h(t)P^*(\mathrm{d}t).$$

Uniform surrogate payoff security gives $s_i^* \in \mathcal{P}_i$ such that for every $(t, x, y) \in T \times X \times X$, there are neighborhoods V_x and V_y of x and y, respectively, such that

$$\begin{aligned}\psi_i(t, (s_i^*(t_i), x'_{-i}), y') &+ H_i(t, y') \\ &> \psi_i(t, (s_i(t_i), x_{-i}), y) + H_i(t, y) - \frac{\varepsilon}{2}, \\ &\text{for all } (x', y') \in V_x \times V_y.\end{aligned}$$

Therefore, for every $(t, x, y) \in T \times X \times X$, there are neighborhoods V_x and V_y of x and y, respectively, such that

$$\begin{aligned}[\psi_i(t, (s_i^*(t_i), x'_{-i}), y') &+ H_i(t, y')]f(t) \\ &\geq \left[\psi_i(t, (s_i(t_i), x_{-i}), y) + H_i(t, y) - \frac{\varepsilon}{2}\right] f(t), \\ &\text{for all } (x', y') \in V_x \times V_y.\end{aligned} \tag{6}$$

Define $\xi : T \times X \times X \to \mathbb{R}$ by

$$\xi(t, x, y) := \sup_{n \in \mathbb{N}} \inf_{(x',y') \in N_{\frac{1}{n}}(x) \times N_{\frac{1}{n}}(y)} [u_i(t, (s_i^*(t_i), x'_{-i})) - u_i(t, y') + H_i(t, y')]f(t).$$

Using an argument analogous to that in Step 3 of the proof of Lemma 5 in Carbonell-Nicolau and McLean (2018), one can show that there exists an open set V_σ in $\mathcal{Y}$ (open with respect to the product topology generated by the p_i-narrow topology on each factor $\mathcal{Y}_i$) containing σ such that

$$\begin{aligned}&\int_T \int_X \int_X \xi(t, x, y) \left[\prod_{j=1}^N \nu_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \nu_j(dy_j|t_j)\right] P^*(\mathrm{d}t) \\ &\quad > \int_T \int_X \int_X \xi(t, x, y) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \sigma_j(dy_j|t_j)\right] P^*(\mathrm{d}t) - \frac{\varepsilon}{2}\end{aligned}$$

for all $(\nu_1, \ldots, \nu_N) \in V_\sigma$.

Because for each $(t, x) \in T \times X$ there are neighborhoods V_x and V_y of x and y, respectively, such that (6) holds, $(t, x, y) \in T \times X \times X$ implies that

$$[\psi_i(t, (s_i^*(t_i), x_{-i}), y) + H_i(t, y)]f(t) \\ \geq \xi(t, x, y) \geq \left[\psi_i(t, (s_i(t_i), x_{-i}), y) + H_i(t, y) - \frac{\varepsilon}{2}\right] f(t).$$

This, together with the conclusion in the preceding paragraph, implies that for every $(\nu_1, .., \nu_N) \in V_\sigma$,

$$\begin{aligned}
&\boldsymbol{\psi}_i((s_i^*, \nu_{-i}), \nu) + \boldsymbol{H}_i(\nu) \\
&\quad = \int_T \int_X \int_X [\psi_i(t, (s_i^*(t_i), x_{-i}), y) \\
&\qquad + H_i(t, y)]f(t) \left[\prod_{j=1}^N \nu_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \nu_j(dy_j|t_j)\right] P(\mathrm{d}t) \\
&\quad = \int_T \int_X \int_X [\psi_i(t, (s_i^*(t_i), x_{-i}), y) \\
&\qquad + H_i(t, y)]f(t) \left[\prod_{j=1}^N \nu_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \nu_j(dy_j|t_j)\right] P^*(\mathrm{d}t) \\
&\quad \geq \int_T \int_X \int_X \xi(t, x, y) \left[\prod_{j=1}^N \nu_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \nu_j(dy_j|t_j)\right] P^*(\mathrm{d}t) \\
&\quad > \int_T \int_X \int_X \xi(t, x, y) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \sigma_j(dy_j|t_j)\right] P^*(\mathrm{d}t) - \frac{\varepsilon}{2} \\
&\quad \geq \int_T \int_X \int_X \left[\psi_i(t, (s_i(t_i), x_{-i}), y)\right. \\
&\qquad \left. + H_i(t, y)\right] f(t) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \sigma_j(dy_j|t_j)\right] P^*(\mathrm{d}t) - \varepsilon \\
&\quad = \int_T \int_X \int_X [\psi_i(t, (s_i(t_i), x_{-i}), y) \\
&\qquad + H_i(t, y)]f(t) \left[\prod_{j=1}^N \sigma_j(dx_j|t_j)\right] \left[\prod_{j=1}^N \sigma_j(dy_j|t_j)\right] P(\mathrm{d}t) - \varepsilon \\
&\quad = \boldsymbol{\psi}_i((s_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) - \varepsilon.
\end{aligned}$$

This establishes (5). □

A.2.2 Proof of Lemma 3

We restate Lemma 3 here for the convenience of the reader.

Lemma 3. *Suppose that the SSYM Bayesian game* $((T_i, \mathcal{T}_i), X_i, \psi_i, p)_{i=1}^N$ *satisfies uniform surrogate payoff security with surrogate function* H. *If* $H_i : T \times X \to \mathbb{R}$ *is bounded and* $\left([\otimes_{j=1}^N \mathcal{T}_j] \otimes [\otimes_{j=1}^N \mathcal{B}(X_j)], \mathcal{B}(\mathbb{R})\right)$*-measurable for each* i, *and if* p *is absolutely continuous with respect to* $p_1 \otimes \cdots \otimes p_N$, *then the game* G_Γ *defined in* (2) *is surrogate payoff secure with surrogate function* $\boldsymbol{H}$.

Proof Fix $(\sigma, \mu) \in \mathcal{Y} \times \mathcal{Y}$, i, and $\varepsilon > 0$. Let f be a density of p with respect to $P := p_1 \otimes \cdots \otimes p_N$. We must show that there exist $\sigma_i^* \in \mathcal{Y}_i$ and a neighborhood V_σ of σ such that

$$\begin{aligned} &\boldsymbol{\psi}_i((\sigma_i^*, \nu_{-i}), \nu) + \boldsymbol{H}_i(\nu) \\ &\quad > \boldsymbol{\psi}_i((\mu_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) - \varepsilon, \quad \text{for every } \nu \in V_\sigma. \end{aligned} \tag{7}$$

By an argument analogous to that in the proof of Lemma 2 of Carbonell-Nicolau and McLean (2018), there exists $s_i \in \mathcal{P}_i$ such that $\boldsymbol{\psi}_i((s_i, \sigma_{-i}), \sigma) \geq \boldsymbol{\psi}_i((\mu_i, \sigma_{-i}), \sigma) - \frac{\varepsilon}{2}$. Consequently, there exists $s_i \in \mathcal{P}_i$ such that

$$\boldsymbol{\psi}_i((s_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) \geq \boldsymbol{\psi}_i((\mu_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) - \frac{\varepsilon}{2}. \tag{8}$$

By Lemma 6, there exist $s_i^* \in \mathcal{P}_i$ and a neighborhood V_σ of σ such that

$$\boldsymbol{\psi}_i((s_i^*, \nu_{-i}), \nu) + \boldsymbol{H}_i(\nu) > \boldsymbol{\psi}_i((s_i, \sigma_{-i}), \sigma) + \boldsymbol{H}_i(\sigma) - \frac{\varepsilon}{2}, \quad \text{for all } \nu \in V_\sigma.$$

This, together with (8), gives (7) for some $\sigma_i^* \in \mathcal{Y}_i$. □

A.3 Proof that the game in Example 2 satisfies surrogate better-reply security

The game was defined as $G = (X_i, u_i)_{i=1}^2$, where for each i, $X_i := [0, 2]$, and for $x = (x_1, x_2) \in [0, 1]^2$, the payoff u_i is defined as follows:

$$u_i(x_1, x_2) := \begin{cases} 1 - x_i & \text{if } x_i > x_{-i}, \\ 0 & \text{if } x_i \leq x_{-i}. \end{cases}$$

For $x \in [1, 2]^2 \setminus \{(1, 1)\}$, define $u_2 \equiv 0$ and

$$u_1(x_1, x_2) := \begin{cases} 1 & \text{if } 1 \leq x_1 < 2 \text{ and } x_2 = 2, \\ 0 & \text{if } 1 \leq x_1 < 2 \text{ and } 1 \leq x_2 < 2, \\ 2 & \text{if } x_1 = 2 \text{ and } x_2 = 2, \\ x_2 - 1 & \text{if } x_1 = 2 \text{ and } 1 \leq x_2 < 2. \end{cases}$$

Everywhere else in [0, 2], the payoffs are identically zero.

Suppose that $H(x) := u(x)$ for all $x \in [0, 1]^2$ and $H(x) := 0$ elsewhere. Suppose that $x = (x_1, x_2)$ is not a Nash equilibrium. We consider six cases.

Case 1 $x_2 = 2$, $0 \le x_1 < 2$. Let $\overline{x}_1 := 2$. Choose ε so that $z_1 < 2$ and $z_2 > \frac{3}{2}$ whenever $z \in B_\varepsilon(x)$ (here $B_\varepsilon(x)$ represents the open neighborhood of x with radius ε). Note that $H(z) = 0$ for all $z \in B_\varepsilon(x)$ and that $(x, \alpha) \in \overline{\Gamma}_H$ implies that $\alpha = 0$. Suppose that $z \in B_\varepsilon(x)$. Then

$$z_2 < 2 \Rightarrow u_1(2, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) = (z_2 - 1) - 0 + 0 > \frac{1}{2}$$

and

$$z_2 = 2 \Rightarrow u_1(2, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) = 2 - \max\{0, 1\} + 0 \ge 1,$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2)] > 0.$$

Case 2 $1 < x_2 < 2$, $0 \le x_1 < 2$. Let $\overline{x}_1 := 2$. Choose ε so that $z_1 < 2$ and $z_2 > \frac{x_2+1}{2}$ whenever $z \in B_\varepsilon(x)$. Note that $H(z) = 0$ for all $z \in B_\varepsilon(x)$ and that $(x, \alpha) \in \overline{\Gamma}_H$ implies that $\alpha = 0$. Suppose that $z \in B_\varepsilon(x)$. Then

$$u_1(2, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) = (z_2 - 1) - 0 + 0 > \frac{x_2 + 1}{2} - 1 = \frac{x_2 - 1}{2},$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2)] > 0.$$

Case 3 $x_2 = 1$, $0 \le x_1 < 1$. Choose ε so that $0 < \varepsilon < \frac{1-x_1}{2}$ and $z_2 > z_1$ whenever $z \in B_\varepsilon(x)$. Next, we claim that $\alpha_2 = 0$ if $(x, \alpha) \in \overline{\Gamma}_H$. To see this, suppose that $(x^k, u(x^k)) \to (x, \alpha)$. Then $x_2^k > x_1^k$ for all sufficiently large k. If $x_2^k \ge 1$, then $u_2(x^k) = 0$. If $x_2^k < 1$, then $u_2(x^k) = 1 - x_2^k$. Therefore, $u_2(x^k) \to 0$. Now let $\overline{x}_2 = x_1 + \varepsilon$. Suppose that $z \in B_\varepsilon(x)$. Then $1 > x_1 + \varepsilon > z_1$. Therefore,

$$\begin{aligned} &z_2 \ge 1 \text{ and } x_1 + \varepsilon > z_1 \\ &\quad \Rightarrow u_2(z_1, x_1 + \varepsilon) - u_2(z_1, z_2) + H_2(z_1, z_2) = [1 - (x_1 + \varepsilon)] - 0 + 0 > \frac{1 - x_1}{2} \end{aligned}$$

and

$$\begin{aligned} &z_2 < 1 \text{ and } x_1 + \varepsilon > z_1 \\ &\quad \Rightarrow u_2(z_1, x_1 + \varepsilon) - u_2(z_1, z_2) + H_2(z_1, z_2) \end{aligned}$$

$$= [1 - (x_1 + \varepsilon)] - (1 - z_2) + (1 - z_2) = 1 - (x_1 + \varepsilon) > \frac{1 - x_1}{2},$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_2(2, \overline{x}_2) - u_1(z_1, z_2) + H(z_1, z_2)] > 0.$$

Case 4 $0 \le x_2 < 1$, $1 < x_1 \le 2$. Choose ε so that $0 < \varepsilon < \frac{1-x_2}{2}$ and $0 \le z_2 < 1$ and $1 < z_1 \le 2$ whenever $z \in B_\varepsilon(x)$. Note that $H(z) = 0$ for all $z \in B_\varepsilon(x)$ and that $(x, \alpha) \in \overline{\Gamma}_H$ implies that $\alpha = 0$. Suppose that $z \in B_\varepsilon(x)$. Then $z_2 < x_2 + \varepsilon$ and $z_2 < 1$. Consequently,

$$u_1(x_2 + \varepsilon, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) = [(1 - (x_2 + \varepsilon)] - 0 + 0 > \frac{1 - x_2}{2},$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2)] > 0.$$

Case 5 $0 \le x_2 < 1$, $x_1 = 1$. Choose ε so that $0 < \varepsilon < \frac{1-x_2}{2}$ and $z_1 > z_2$ whenever $z \in B_\varepsilon(x)$. Next, we claim that $\alpha_1 = 0$ if $(x, \alpha) \in \overline{\Gamma}_H$. To see this, suppose that $(x^k, u(x^k)) \to (x, \alpha)$. Then $x_1^k > x_2^k$ for all sufficiently large k. If $x_1^k \ge 1$, then $u_1(x^k) = 0$. If $x_1^k < 1$, then $u_1(x^k) = 1 - x_1^k$. Therefore, $u_1(x^k) \to 0$. Now let $\overline{x}_1 = x_2 + \varepsilon$ and note that $x_2 + \varepsilon < 1$. Suppose that $z \in B_\varepsilon(x)$. Then $x_2 + \varepsilon > z_2$. Consequently,

$$\begin{aligned} & z_1 \ge 1 \text{ and } x_2 + \varepsilon > z_2 \\ & \quad \Rightarrow u_1(x_2 + \varepsilon, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) = [1 - (x_2 + \varepsilon)] - 0 + 0 > \frac{1 - x_2}{2} \end{aligned}$$

and

$$\begin{aligned} & z_1 < 1 \text{ and } x_2 + \varepsilon > z_2 \\ & \quad \Rightarrow u_1(x_2 + \varepsilon, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2) \\ & \quad = [1 - (x_2 + \varepsilon)] - (1 - z_1) + (1 - z_1) > \frac{1 - x_2}{2}, \end{aligned}$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2)] > 0.$$

Case 6 $0 \le x_1 < 1, 0 \le x_2 < 1$. To begin, we claim that for each $(x, \alpha) \in \overline{\Gamma}_H$, there exist an i such that $\alpha_i = 0$. To see this, suppose that $(x^k, u(x^k)) \to (x, \alpha)$. If $x_1^k > x_2^k$ for all sufficiently large k, then $u_2(x^k) = 0$ for all sufficiently large k , implying that $\alpha_2 = 0$. Otherwise, there exists a subsequence $(x^{k_m}, u(x^{k_m}))$ with $x_1^{k_m} \le x_2^{k_m}$ for all m . Consequently, $u_1(x^{k_m}) = 0$ for all m implying that $\alpha_1 = 0$. So suppose that $\alpha_1 = 0$.

Choose ε so that $0 < \varepsilon < \frac{1-x_2}{2}$. Note that $H(z) = u(z)$ for all $z \in B_\varepsilon(x)$. Now let $\overline{x}_1 = x_2 + \varepsilon$ and note that $x_2 + \varepsilon < 1$. Suppose that $z \in B_\varepsilon(x)$. Then $x_2 + \varepsilon > z_2$. Consequently,

$$\begin{aligned} &u_1(x_2 + \varepsilon, z_2) - u_2(z_1, z_2) + H_2(z_1, z_2) \\ &\quad = [1 - (x_2 + \varepsilon)] - u_2(z_1, z_2) + u_2(z_1, z_2) > \frac{1 - x_2}{2}, \end{aligned}$$

implying that

$$\inf_{z \in B_\varepsilon(x)} [u_1(\overline{x}_1, z_2) - u_1(z_1, z_2) + H_1(z_1, z_2)] > 0.$$

A completely symmetric argument applies if $\alpha_2 = 0$, and the proof is complete.

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