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On strategic stability in discontinuous games

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ABSTRACT

We identify a class of discontinuous normal-form games whose members possess strategically stable sets, defined according to an infinite-game extension of Kohlberg and Mertens's (1986) equilibrium concept, and show that, generically, a set is stable if and only if it contains a single Nash equilibrium.

1. Introduction

There are several refinements of the Nash equilibrium concept for normal-form games with finite action spaces. Some authors have studied extensions of the standard refinements to normal-form games with infinitely many strategies (e.g., perfect equilibrium (Simon and Stinchcombe, 1995; Carbonell-Nicolau, forthcoming-a-b, 2011), strategic stability (Al-Najjar, 1995), and essential stability (e.g., Yu, 1999; Zhou et al., 2007; Carbonell-Nicolau, 2010a)).

This paper examines an infinite-game generalization of Kohlberg and Mertens's (1986) strategic stability. For this notion, Al-Najjar (1995) shows that metric, compact, and continuous games possess stable sets of mixed strategies. In this paper, we extend this existence result to a class of possibly discontinuous games with the property that for generic members of this class, stable sets reduce to equilibrium points.

2. Preliminaries

A ***normal-form game*** is a collection $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty action space for player i, and $u_i : X \to \mathbb{R}$, a bounded and Borel measurable map with domain $X := \times_{i=1}^N X_i$, represents player i's payoff function. When X_i is compact and metric for each $i \in \{1, \dots, N\}$, G is called a ***compact metric game***.

The ***mixed extension*** of a compact metric game G is the game

$$\overline{G} = (M_i, U_i)_{i=1}^N,$$

where, for each i, M_i represents the set of Borel probability measures on X_i, endowed with the weak* topology, and $U_i : M \to \mathbb{R}$ is defined by

$$U_i(\mu) := \int_X u_i \mathrm{d}\mu,$$

where $M := \times_{i=1}^N M_i$.

Given a compact, metric game $G = (X_i, u_i)_{i=1}^N$, the set M, together with the Prokhorov metric on M, can be viewed as a metric space.[1] The Prokhorov metric on M, $\varrho : M^2 \to \mathbb{R}$, is defined as

$$\varrho(\mu, \nu) := \inf\{\varepsilon > 0 : \mu(B) \le \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \le \mu(B^\varepsilon) + \varepsilon, \text{ for all } B\},$$

where

$$B^\varepsilon := \{x \in X : d(x, y) < \varepsilon \text{ for some } y \in B\},$$

and d denotes the metric associated with X.

A measure μ_i in M_i is said to be ***strictly positive*** if $\mu_i(O) > 0$ for every nonempty open subset O of X_i.

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[1] For compact metric games, the weak* topology on M coincides with the topology induced by the Prokhorov metric on M.

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For each i, let $\widehat{M}_i$ be the set of all strictly positive members of M_i, and define $\widehat{M} := \times_{i=1}^N \widehat{M}_i$. For $\nu = (\nu_1, \dots, \nu_N) \in \widehat{M}$ and $\delta = (\delta_1, \dots, \delta_N) \in [0, 1)^N$, define

$$M_i(\delta_i\nu_i) := \{\mu_i \in M_i : \mu_i \geq \delta_i\nu_i\}$$

and $M(\delta\nu) := \times_{i=1}^N M_i(\delta_i\nu_i)$. The game

$$\overline{G}_{\delta\nu} = (M_i(\delta_i\nu_i), U_i|_{M(\delta\nu)})_{i=1}^N$$

is called a ***Selten perturbation*** of G. When $\delta_1 = \cdots = \delta_N$, we slightly abuse notation and write $\overline{G}_{\delta\nu}$ with $\delta = \delta_1 = \cdots = \delta_N$.

The *graph* of G is the set

$$\Gamma_G := \{(x, \alpha) \in X \times \mathbb{R}^N : u_i(x) = \alpha_i, \text{ for all } i\}.$$

The closure of Γ_G is denoted by $\overline{\Gamma}_G$.

Definition 1. Given a game $G = (X_i, u_i)_{i=1}^N$, a strategy profile $x = (x_1, \dots, x_N) \in X$ is a ***Nash equilibrium*** of G if for each i, $u_i(x) \geq u_i(y_i, x_{-i})$ for every $y_i \in X_i$.

Given a game $G = (X_i, u_i)_{i=1}^N$, a Nash equilibrium of the mixed extension $\overline{G}$ is called a ***mixed-strategy Nash equilibrium*** of G. By a slight abuse of terminology, we sometimes refer to a mixed-strategy Nash equilibrium of G simply as a Nash equilibrium of G.

Definition 2. A mixed-strategy profile $\mu \in M$ is a ***trembling-hand perfect (thp) equilibrium*** of $G = (X_i, u_i)_{i=1}^N$ if there are sequences (δ^n), (ν^n), and (μ^n) such that $(0, 1)^N \ni \delta^n \to 0$ and $\nu^n \in \widehat{M}$ for each n, $\mu^n \to \mu$, and each μ^n is a Nash equilibrium of the perturbed game $\overline{G}_{\delta^n\nu^n}$.

Alternative definitions of trembling-hand perfection that are equivalent to Definition 2 can be found in Carbonell-Nicolau (forthcoming-b).

For $\emptyset \neq E \subseteq M$ and $\mu \in M$, define

$$\varrho(\mu, E) := \inf\{\varrho(\mu, \nu) : \nu \in E\}.$$

For $\varepsilon > 0$ and $\emptyset \neq E \subseteq M$, a profile $\mu \in M$ is said to be ***ε -close to*** E if $\varrho(\mu, E) < \varepsilon$.

Given a game $G = (X_i, u_i)_{i=1}^N$, let $\mathcal{S}_G$ be the family of all nonempty closed sets E of Nash equilibria of $\overline{G}$ with the following property: for each $\varepsilon > 0$ there exists $\alpha \in (0, 1]$ such that for each $\delta \in (0, \alpha)^N$ and every $\nu \in \widehat{M}$, the perturbed game $\overline{G}_{\delta\nu}$ has a Nash equilibrium ε-close to E.

Definition 3. A set of mixed-strategy profiles in M is a ***stable set*** of $G = (X_i, u_i)_{i=1}^N$ if it is a minimal element of the set $\mathcal{S}_G$ ordered by set inclusion.

3. Existence of stable sets

We adapt ideas from Carbonell-Nicolau (forthcoming-a), Carbonell-Nicolau (2011), and Carbonell-Nicolau (2010a) to derive the main results. Some arguments are omitted in the interest of brevity. The reader is referred to the working paper version, Carbonell-Nicolau (2010b), of the current manuscript for details.

The following definitions are taken from Reny (1999).

Definition 4. The game $G = (X_i, u_i)_{i=1}^N$ is ***better-reply secure*** if for every $(x, \alpha) \in \overline{\Gamma}_G$ such that x is not a Nash equilibrium of G, there exist i, $y_i \in X_i$, a neighborhood $O_{x_{-i}}$ of x_{-i}, and $\beta \in \mathbb{R}$ such that $u_i(y_i, y_{-i}) \geq \beta > \alpha_i$ for all $y_{-i} \in O_{x_{-i}}$.

Definition 5. The game $G = (X_i, u_i)_{i=1}^N$ is ***payoff secure*** if for each $\varepsilon > 0$, $x = (x_1, \dots, x_N) \in X$, and i, there exist $y_i \in X_i$ and a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) > u_i(x) - \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

The existence of stable sets in a game $G = (X_i, u_i)_{i=1}^N$ crucially relies on the existence of Nash equilibria in neighboring Selten perturbations of G.

Lemma 1. *Suppose that G is a compact, metric game. If $\overline{G}$ is better-reply secure and there exists $\alpha \in (0, 1)$ such that $\overline{G}_{\delta\mu}$ has a Nash equilibrium for every $(\delta, \mu) \in (0, \alpha] \times \widehat{M}$, then G possesses a stable set, and all stable sets of G contain only trembling-hand perfect equilibria, which are also Nash.*

Proof. The set of Nash equilibria in $\overline{G}$ belongs to $\mathcal{S}_G$. The set of Nash equilibria in $\overline{G}$ is nonempty and closed because $\overline{G}$ is compact, metric, and better-reply secure (Reny, 1999, Corollary 5.2 and Remark 3.1). The proof that for each $\varepsilon > 0$ there exists $\alpha \in (0, 1]$ such that for each $\delta \in (0, \alpha)^N$ and every $\nu \in \widehat{M}$, $\overline{G}_{\delta\nu}$ has a Nash equilibrium ε-close to the set of Nash equilibria in $\overline{G}$ is similar to the proof of Proposition 1 in Carbonell-Nicolau (forthcoming-a). We omit the details.

Next, every decreasing chain (E^α) in $\mathcal{S}_G$ (ordered by set inclusion) has a lower bound. In fact, $\bigcap_\alpha E^\alpha$ is a lower bound for (E^α). To see this, note first that the collection (E^α) has the finite intersection property, and therefore, since M is compact, $\bigcap_\alpha E^\alpha \neq \emptyset$. The set $\bigcap_\alpha E^\alpha$ is clearly a lower bound for (E^α) if $\bigcap_\alpha E^\alpha$ is a member of (E^α). We assume that $\bigcap_\alpha E^\alpha$ is a not member of (E^α) and derive a contradiction. Suppose that $\bigcap_\alpha E^\alpha$ is a not member of (E^α). Then, since $\bigcap_\alpha E^\alpha \neq \emptyset$, there exists $\varepsilon > 0$ such that for every $\alpha > 0$, there exist $\delta \in (0, \alpha)^N$ and $\nu \in \widehat{M}$ such that no Nash equilibrium of $\overline{G}_{\delta\nu}$ is ε-close to $\bigcap_\alpha E^\alpha$. Then, since each E^α is a member of $\mathcal{S}_G$, the Hausdorff distance (with respect to the Prokhorov metric) between $\bigcap_\alpha E^\alpha$ and E^β is bounded away from zero for every β (otherwise, for some β, the Hausdorff distance between E^β and $\bigcap_\alpha E^\alpha$ would lie below $\frac{\varepsilon}{2}$, and there would exist some $\alpha^* \in (0, 1]$ such that, for each $\delta \in (0, \alpha^*)^N$ and every $\nu \in \widehat{M}$, some Nash equilibrium of $\overline{G}_{\delta\nu}$ would be $\frac{\varepsilon}{2}$-close to E^β, and therefore ε-close to $\bigcap_\alpha E^\alpha$). Because (E^α) is a totally ordered subset of $\mathcal{S}_G$ (and hence a directed set), the map $\varsigma : (E^\alpha) \to (E^\alpha)$ defined by $\varsigma(E^\alpha) := E^\alpha$ is a net in $\mathcal{S}_G$. But since ς is a net of nonempty, closed subsets of the compact set M, and since the hyperspace of nonempty compact subsets of M is a compact, metric space relative to the Hausdorff metric, there is a subnet of ς, which we denote again by ς, that (Hausdorff) converges to some nonempty compact subset E of M. Hence, because $E = \bigcap_\alpha E^\alpha$, we see that ς (Hausdorff) converges to $E = \bigcap_\alpha E^\alpha$, which contradicts the fact that the Hausdorff distance between $\bigcap_\alpha E^\alpha$ and E^β is bounded away from zero for every β.

To see that $E = \bigcap_\alpha E^\alpha$, suppose first that there exists $\mu \in E \setminus (\bigcap_\alpha E^\alpha)$. Then there exists α such that $\mu \in E \setminus E^\alpha$. But $E \subseteq E^\alpha$, a contradiction. To see that $E \subseteq E^\alpha$, note that since E^α is closed, $\nu \in E \setminus E^\alpha$ implies that $\inf\{\varrho(\nu, p) : p \in E^\alpha\} > 0$, so (given $\nu \in E$) the Hausdorff distance between E and E^α is greater than or equal to $\inf\{\varrho(\nu, p) : p \in E^\alpha\}$. Hence, for all $E^\beta \subseteq E^\alpha$, the Hausdorff distance between E and E^β is greater than or equal to $\inf\{\varrho(\nu, p) : p \in E^\alpha\} > 0$, thereby contradicting that the net ς (Hausdorff) converges to E.

Next, suppose that there exists $\mu \in (\bigcap_\alpha E^\alpha) \setminus E$. Then, since E is closed, $\inf\{\varrho(\mu, \nu) : \nu \in E\} > 0$. Now, since ς (Hausdorff) converges to E, there exists β such that the Hausdorff distance between E^β and E, call it h, is less than $\inf\{\varrho(\mu, \nu) : \nu \in E\}$. But $\mu \in E^\beta$ (since $\mu \in \bigcap_\alpha E^\alpha$), and therefore

$$\inf\{\varrho(\mu, \nu) : \nu \in E\} \leq \max\{\sup_{p \in E^\beta} \inf_{\nu \in E} \varrho(p, \nu), \sup_{\nu \in E} \inf_{p \in E^\beta} \varrho(p, \nu)\} = h,$$

a contradiction.

Since $\bigcap_\alpha E^\alpha$ is a lower bound for (E^α) (and (E^α) was arbitrary), we conclude that every decreasing chain has a lower bound. Consequently, Zorn's lemma gives a minimal element E^* of $\mathcal{S}_G$, i.e., a stable set of G. Finally, it is easily seen that each element of E^* is a *thp* equilibrium of G. Hence, by Proposition 1 of Carbonell-Nicolau (forthcoming-a), the members of E^* are also Nash equilibria of G. □

In light of Lemma 1, we seek conditions on the data of a game $G = (X_i, u_i)_{i=1}^N$ that ensure the existence of Nash equilibria in neighboring Selten perturbations of G.

Condition (B). For each i and every $\varepsilon > 0$, there is a sequence (f_k) of Borel measurable maps $f_k : X_i \to X_i$ such that the following is satisfied:

(a) For each $(x_i, x_{-i}) \in X_i \times X_{-i}$ and each k, there is a neighborhood $O_{x_{-i}}$ of x_{-i} for which $u_i(f_k(x_i), y_{-i}) > u_i(x_i, x_{-i}) - \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

(b) For each $(x_i, x_{-i}) \in X_i \times X_{-i}$, there exists a real number $K_{(x_i,x_{-i})}$ such that for each $k \geq K_{(x_i,x_{-i})}$, there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(f_k(x_i), y_{-i}) < u_i(x_i, y_{-i}) + \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

We omit the proof of the following lemma, which is an adaptation of the argument used in Carbonell-Nicolau (forthcoming-a) to prove Lemma 4. The details appear in Carbonell-Nicolau (2010b).

Lemma 2. *Suppose that a compact, metric game G satisfies Condition (B). Then $\overline{G}_{\delta\mu}$ is payoff secure for every $(\delta, \mu) \in [0, 1) \times \widehat{M}$.*

We are now ready to state and prove the first main result.

Theorem 1. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is compact, metric, and satisfies Condition (B). Suppose further that $\sum_{i=1}^N u_i$ is upper semicontinuous. Then G has a stable set, and all stable sets of G contain only trembling-hand perfect equilibria, which are also Nash.*

Proof. Suppose that $G = (X_i, u_i)_{i=1}^N$ is compact, metric, and satisfies Condition (B). Suppose further that $\sum_{i=1}^N u_i$ is upper semicontinuous. By Lemma 2, $\overline{G}_{\delta\mu}$ is payoff secure for every $(\delta, \mu) \in [0, 1) \times \widehat{M}$. Further, since $\sum_{i=1}^N u_i$ is upper semicontinuous, so is $\sum_{i=1}^N U_i$. Consequently, by Proposition 3.2 in Reny (1999), $\overline{G}_{\delta\mu}$ is better-reply secure for every $(\delta, \mu) \in [0, 1) \times \widehat{M}$, and hence by Corollary 3.3 in Reny (1999), $\overline{G}_{\delta\mu}$ possesses a Nash equilibrium for every $(\delta, \mu) \in (0, 1) \times \widehat{M}$. Now apply Lemma 1. □

Remark 1. Carbonell-Nicolau (forthcoming-a) proves the existence of *thp* equilibria in a superset of the set of compact, metric games satisfying Condition (B) and upper semicontinuity of the sum of payoffs. It can be shown that this superset contains games G for which the following is true: given any $\mu \in \widehat{M}$ and $\delta \in (0, 1)^N$, there are many ρ arbitrarily close to μ such that $\overline{G}_{\delta\rho}$ fails payoff security and better-reply security. Consequently, proving that stable sets exist within the larger class considered in Carbonell-Nicolau (forthcoming-a) poses difficulties.

4. Generic games

This section provides conditions under which stable sets reduce to equilibrium points.

The following definition appears in Monteiro and Page (2007).

Definition 6. The game G is ***uniformly payoff secure*** if for each i, $\varepsilon > 0$, and $x_i \in X_i$, there exists $y_i \in X_i$ such that for every $x_{-i} \in X_{-i}$, there is a neighborhood $O_{x_{-i}}$ of x_{-i} such that $u_i(y_i, y_{-i}) > u_i(x_i, x_{-i}) - \varepsilon$ for all $y_{-i} \in O_{x_{-i}}$.

For fixed action spaces $X_1, \ldots, X_N$, and letting $X := \times_i X_i$, consider the following classes of games:

- The class $\mathfrak{g}_X^c$ of compact, metric games $(X_i, u_i)_{i=1}^N$ with u_i continuous for each i.
- The class $\mathfrak{g}_X^u$ of compact, metric, and uniformly payoff secure games $(X_i, u_i)_{i=1}^N$ with $\sum_{i=1}^N u_i$ upper semicontinuous.
- The class $\mathfrak{g}_X$ of compact, metric games $(X_i, u_i)_{i=1}^N$ satisfying Condition (B) and upper semicontinuity of $\sum_{i=1}^N u_i$.

It is clear that $\mathfrak{g}_X^c \subseteq \mathfrak{g}_X^u \supseteq \mathfrak{g}_X$. We view $\mathfrak{g}_X^c$, $\mathfrak{g}_X^u$, and $\mathfrak{g}_X$ as metric subspaces of the metric space $(B(X)^N, \rho_X)$, where $B(X)$ represents the set of bounded maps $f : X \to \mathbb{R}$, with associated metric $\rho_X : B(X)^N \times B(X)^N \to \mathbb{R}$ defined by

$$\rho_X((u_1, \ldots, u_N), (f_1, \ldots, f_N)) := \sum_{i=1}^N \sup_{x \in X} |u_i(x) - f_i(x)|.$$

Definition 7. Given a class of games $\mathfrak{g} \subseteq B(X)^N$ and $G \in \mathfrak{g}$, a Nash equilibrium μ of $\overline{G}$ is an ***essential equilibrium of** G **relative to*** $\mathfrak{g}$ if for every neighborhood O_μ of μ there is a neighborhood O_G of G such that for every $g \in O_G \cap \mathfrak{g}$, O_μ contains a mixed-strategy Nash equilibrium of g.

Theorem 2 (*Zhou et al., 2007, Theorem 1*)**.** *For any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X^c$, any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}_X^c$.*

It is easy to show, using Theorem 2, that for generic elements G of $\mathfrak{g}_X^c$ (i.e., for any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X^c$), any $\{\mu\}$ is stable for every mixed-strategy Nash equilibrium μ of $\overline{G}$.

In fact, given $\delta \in [0, 1)^N$ and $\mu \in \widehat{M}$, a carefully chosen perturbation of any G in $\mathfrak{g}_X^c$ has a mixed extension that "coincides" with $\overline{G}_{\delta\mu}$. To see this, consider the mixed extension of the game

$$G_{(\delta,\mu)} = (X_i, u_i^{(\delta,\mu)})_{i=1}^N,$$

where $u_i^{(\delta,\mu)} : X \to \mathbb{R}$ is defined by

$$u_i^{(\delta,\mu)}(x) := U_i((1-\delta_1)x_1 + \delta_1\mu_1, \ldots, (1-\delta_N)x_N + \delta_N\mu_N).$$

Here, $(1-\delta_i)x_i + \delta_i\mu_i$ is a member of M_i defined by $((1-\delta_i)x_i + \delta_i\mu_i)(B_i) := (1-\delta_i)\delta_{x_i}(B_i) + \delta_i\mu_i(B_i)$, where δ_{x_i} denotes the Dirac measure on X_i with support $\{x_i\}$. Observe that given a Nash equilibrium $(\sigma_1, \ldots, \sigma_N)$ in the mixed extension of $G_{(\delta,\mu)}$, $((1-\delta_1)\sigma_1 + \delta_1\mu_1, \ldots, (1-\delta_N)\sigma_N + \delta_N\mu_N)$ is a Nash equilibrium of $\overline{G}_{\delta\mu}$. Moreover, $G_{(\delta,\mu)}$ is a perturbation of G in $\mathfrak{g}_X^c$. Consequently, by Theorem 2, for a generic game G (in $\mathfrak{g}_X^c$) any singleton set of mixed-strategy Nash equilibria is stable. In addition, by Theorem 1, all stable sets of G contain only *thp* equilibria, which are also Nash.

In light of the following extension of Theorem 2 to the superset $\mathfrak{g}_X^u$ of $\mathfrak{g}_X^c$, it is natural to ask whether a similar result can be obtained for the class $\mathfrak{g}_X^u$.

Theorem 3 (*Carbonell-Nicolau, 2010a, Corollary 1*)**.** *For any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X^u$, any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}_X^u$.*

Unfortunately, given $G \in \mathfrak{g}_X^u \setminus \mathfrak{g}_X^c$, the perturbation $G_{(\delta,\mu)}$ need not lie in $\mathfrak{g}_X^u$ (Carbonell-Nicolau, forthcoming-a, Example 3), so that even an essential game in $\mathfrak{g}_X^u$ (i.e., a game whose mixed-strategy Nash equilibria are all essential) cannot be guaranteed to have stable singleton sets of mixed-strategy Nash equilibria via Theorem 3. Nevertheless, the genericity result extends to the class $\mathfrak{g}_X$. To see this, the following observations are essential (for their proofs, the reader is referred to Carbonell-Nicolau (2010b)).

Lemma 3. *Suppose that G is a compact, metric game satisfying Condition (B). Then, for every $(\delta, \mu) \in [0, 1)^N \times \widehat{M}$, $G_{(\delta,\mu)}$ is a compact, metric game satisfying Condition (B).*

Lemma 4. *Suppose that $\mathfrak{g} \subseteq \mathfrak{g}_X^u$ and $\mathfrak{g}$ is closed in $B(X)^N$. Then, for any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}$, any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}$.*

Lemma 5. *The set $\mathfrak{g}_X$ is closed in $B(X)^N$.*

Lemma 3 and upper semicontinuity of $\sum_{i=1}^N u_i$ imply that, given $G \in \mathfrak{g}_X$ and $(\delta, \mu) \in [0, 1)^N \times \widehat{M}$, we have $G_{(\delta,\mu)} \in \mathfrak{g}_X$ (upper semicontinuity of $\sum_{i=1}^N u_i^{(\delta,\mu)}$ is implied by that of $\sum_{i=1}^N U_i$, which, in turn, follows from upper semicontinuity of $\sum_{i=1}^N u_i$).

We are now ready to prove our second main result. Because $\mathfrak{g}_X \subseteq \mathfrak{g}_X^u$ is closed in $B(X)^N$ (Lemma 5), Lemma 4 implies that for any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X$, any mixed-strategy Nash equilibrium of G is essential relative to $\mathfrak{g}_X$. Therefore, since, given $\delta \in [0, 1)^N$ and $\mu \in \widehat{M}$, $G_{(\delta,\mu)}$ is a perturbation of G in $\mathfrak{g}_X$, and because $((1-\delta_1)\sigma_1 + \delta_1\mu_1, \ldots, (1-\delta_N)\sigma_N + \delta_N\mu_N)$ is a Nash equilibrium of $\overline{G}_{\delta\mu}$ whenever $(\sigma_1, \ldots, \sigma_N)$ is a Nash equilibrium of the mixed extension of $G_{(\delta,\mu)}$, for a generic game G in $\mathfrak{g}_X$ (i.e., for any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X$) any singleton set of mixed-strategy Nash equilibria is stable. In addition, by Theorem 1, all stable sets of G contain only *thp* equilibria, which are also Nash. Finally, it is easy to see that any stable set of G is a singleton set of mixed-strategy Nash equilibria.

Theorem 4. *For any G in a dense $\mathcal{G}_\delta$ subset of $\mathfrak{g}_X$, a set is stable if and only if it contains a single mixed-strategy Nash equilibrium of G.*

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